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Rademacher-type formulas for restricted partition and overpartition functions

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Dedicated to George Andrews on the occasion of his seventieth birthday

Abstract A collection of Hardy-Ramanujan-Rademacher type formulas for restricted partition and overpartition functions is presented, framed by several biographical anecdotes.

Keywords partitions · circle method · Rogers-Ramanujan identities

Mathematics Subject Classification (2000) 11P82 · 11P85 · 05A19

1 Introduction

When George Andrews matriculated in the Ph.D. program at the University of Pennsylvania in the fall of 1961, his intention was to specialize in geometric number theory. He had been attracted to Penn's graduate program in part because the 1961–1962 academic year had been designated a special year in number theory there. The academic year culminated in a celebration of the seventieth birthday of Professor Hans Rademacher.

Rademacher taught Andrews in his analytic number theory class that year, and there Andrews was introduced to the theory of partitions. A partition λ of an integer n is a weakly decreasing finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ whose sum is n. Each λ_i is called a 'part' of the partition λ. The theory of integer partitions began with Euler [22], who introduced generating functions to study $p(n)$, the number of partitions of n, and found that the generating function for $p(n)$ was representable as an elegant infinite product:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{m\geqq 1} \frac{1}{1-x^m}. \tag{1.1}$$

The "circle method" was created by Hardy and Ramanujan and later improved by Rademacher, in connection with the study of the function $p(n)$, the number of partitions of the integer n.

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The circle method has proved to be one of the most useful tools in the history of analytic number theory. Expositions of the circle method may be found in [2, 9, 53, 54, 56].

Rademacher's formula for $p(n)$ is given by

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i n h/k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right), \tag{1.2}$$

where $\omega(h,k)$ is a $24k$th root of unity that frequently occurs in the study of modular forms and is given by

$$\omega(h,k) = \begin{cases} \left(\frac{-k}{h}\right) \exp\left(-\pi i \left\{\frac{1}{4}(2 - hk - h) + \frac{1}{12}(k - \frac{1}{k})(2h - H + h^2 H)\right\}\right), & \text{if } 2 \nmid h, \\ \left(\frac{-h}{k}\right) \exp\left(-\pi i \left\{\frac{1}{4}(k-1) + \frac{1}{12}(k - \frac{1}{k})(2h - H + h^2 H)\right\}\right), & \text{if } 2 \nmid k, \end{cases}$$

$\left(\frac{a}{b}\right)$ is the Legendre-Jacobi symbol, and H is any solution of the congruence

$$hH \equiv -1 \pmod{k}.$$

Andrews reports [66] that the formula for $p(n)$

> . . . was a revolutionary and surprising achievement. The form of this formula is even more stunning. It involves transcendental numbers and expressions that seem to be totally unrelated that might be appropriate, say, in a course on engineering or theoretical physics, but for actually counting how many ways you can add up sums to get a particular number, they seem absolutely incredible. In fact, I was *stunned* the first time I saw this formula. I could not *believe* it, and the experience of seeing it explained and understanding how it took shape really, I think, convinced me that this was the area of mathematics that I wanted to pursue.

Many practitioners, including a number of Ph.D. students and postdocs who worked under Rademacher, have used the circle method to study various restricted partition functions, often associated with sets of partitions enumerated in famous theorems. These practitioners included Grosswald [24, 25], Haberzetle [26], Hagis [27–35], Hua [39], Iseki [40–42], Lehner [43], Livingood [44], Niven [52], and Subramanyasastri [65].

Let us consider several examples.

Theorem 1 (Euler, 1748) *Let $q(n)$ denote the number of partitions of n into odd parts. Let $r(n)$ denote the number of partitions of n into distinct parts. Then $q(n) = r(n)$ for all integers n.*

Theorem 2 (Hagis, 1963)

$$q(n) = \frac{\pi}{\sqrt{24n+1}} \sum_{\substack{k \ge 1 \\ 2 \nmid k}} \frac{1}{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{-2\pi n h/k} \frac{\omega(h,k)}{\omega(2h,k)} I_1\left(\frac{\pi\sqrt{24n+1}}{6\sqrt{2}k}\right), \tag{1.3}$$

where

$$I_\nu(z) := \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)} \tag{1.4}$$

is the Bessel function of purely imaginary argument.

Theorem 3 **(Schur, 1926)** *Let $s(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Let $t(n)$ denote the number of partitions λ of n where $\lambda_i - \lambda_{i+1} \geqq 3$ and $\lambda_i - \lambda_{i+1} > 3$ if $3 \mid \lambda_i$. Then $s(n) = t(n)$ for all n.*

Theorem 4 **(Niven, 1940)**

$$s(n) = \frac{\pi}{\sqrt{36n-3}} \sum_{d|6} \sqrt{(d-2)(d-3)} \sum_{\substack{k \geqq 1 \\ (k,6)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi nh/k} \frac{\omega(h,k)\omega(6h/d,k/d)}{\omega(\frac{2h}{(d,2)},\frac{k}{(d,2)})\omega(\frac{3h}{(d,3)},\frac{k}{(d,3)})} I_1\left(\frac{\pi\sqrt{d\,(12n-1)}}{3\sqrt{6}k}\right). \quad (1.5)$$

Recently, the author found [61]

$$\overline{p}(n) = \frac{1}{2\pi} \sum_{\substack{k \geqq 1 \\ 2 \nmid k}} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-2\pi inh/k} \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{n}}{k}\right)}{\sqrt{n}}\right) \quad (1.6)$$

and

$$pod(n) = \frac{2}{\pi\sqrt{6}} \sum_{d|4} \sqrt{(d-2)(5d-17)} \sum_{\substack{k \geqq 1 \\ (k,4)=d}} \sqrt{k}$$
$$\times \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \frac{\omega(h,k)\ \omega\,(4h/d,k/d)}{\omega\left(\frac{2h}{(d,2)},\frac{k}{(d,2)}\right)} e^{-2\pi inh/k} \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{d(8n-1)}}{4k}\right)}{\sqrt{8n-1}}\right), \quad (1.7)$$

where $pod(n)$ denotes the number of partitions of n where no odd part is repeated, and $\overline{p}(n)$ denotes the number of overpartitions of n. An *overpartition* of n is a finite weakly decreasing sequence of positive integers where the last occurrence of a given part may or may not be overlined. Thus the eight overpartitions of 3 are (3), $(\overline{3})$, $(2,1)$, $(\overline{2},1)$, $(2,\overline{1})$, $(\overline{2},\overline{1})$, $(1,1,1)$, $(1,1,\overline{1})$. Overpartitions were introduced by S. Corteel and J. Lovejoy in [19] and have been studied extensively by them and others including Bringmann, Chen, Fu, Goh, Hirschhorn, Hitczenko, Lascoux, Mahlburg, Robbins, Rødseth, Sellers, Yee, and Zho [11,16–21,23,37,38,45–51,57,58].

Recently, Bringmann and Ono [12] have given exact formulas for the coefficients of all harmonic Maass forms of weight $\leqq \frac{1}{2}$. All of the generating functions considered herein are weakly holomorphic modular forms of weight either 0 or $-\frac{1}{2}$, and thus they are harmonic Maass forms of weight $\leqq \frac{1}{2}$. Accordingly, all of the exacxt formulas for restricted partition and overpartition functions presented here could be derived from the general theorem in [12].

In this article, we will present several anecdotes from the professional life of George Andrews, and present some new Rademacher type formulas related to the events described.

2 Identities in the Lost Notebook

Certainly one of the most exciting incidents of George Andrews' professional life was his unearthing of Ramanujan's lost notebook at the Wren Library at Trinity College, Cambridge University in 1976 (see [4, pp. 5–6, §1.5] and [6, p. 1 ff] for a full account). As is now well known, the lost notebook contains many identities of the Rogers-Ramanujan type. Many of the infinite products appearing in these identities are easily identified as generating functions for certain restricted classes of partitions or overpartitions. The methods of Rademacher may be applied to find explicit formulas for the coefficients appearing in the series expansions of these generating functions.

Below is a list of some of the Rogers-Ramanujan type identities which appear in the lost notebook. Some of these identities also appear in Slater [59]. Specifically, Eq. (2.1) is Slater's (6); Eq. (2.3) is Slater's (12); Eq. (2.5) is Slater's (22); Eq. (2.6) is Slater's (25); Eq. (2.7) is Slater's (28); Eq. (2.8) is Slater's (29); and Eq. (2.11) is Slater's (50).

The standard abbreviations

$$(a;b)_j = \prod_{i=0}^{j-1}(1-ab^i), \qquad (a)_j := (a;x)_j$$

will be used. Note that $(a;b)_0 = 1$. Here and throughout, we assume $|x| < 1$ to guarantee convergence.

$$\sum_{n=0}^{\infty} \frac{x^{n^2}(-1)_n}{(x)_n(x;x^2)_n} = \prod_{m=1}^{\infty} \frac{(1+x^{3m-2})(1+x^{3m-1})}{(1-x^{3m-2})(1-x^{3m-1})} \qquad \text{[7, Ent 4.2.8]} \qquad (2.1)$$

$$\sum_{n=0}^{\infty} \frac{x^{n^2}(-x)_n}{(x)_n(x;x^2)_{n+1}} = \prod_{m=1}^{\infty} \frac{(1+x^{3m-2})(1+x^{3m-1})}{(1-x^{3m-2})(1-x^{3m-1})} \qquad \text{[7, Ent 4.2.9]} \qquad (2.2)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+1)/2}(-1)_n}{(x)_n} = \prod_{m=1}^{\infty} \frac{1+x^{2m-1}}{1-x^{2m-1}} \qquad \text{[7, Ent 1.7.14]} \qquad (2.3)$$

$$\sum_{n=0}^{\infty} \frac{x^{n^2}(-x^2;x^2)_n}{(x)_{2n+1}} = \prod_{m=1}^{\infty} \frac{1+x^{2m-1}}{1-x^{2m-1}} \qquad \text{[7, Ent 1.7.13]} \qquad (2.4)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+1)}(-x)_n}{(x;x^2)_{n+1}(x)_n} = \prod_{m=1}^{\infty} \frac{(1-x^{6m})(1-x^{6m-1})(1-x^{6m-5})}{(1-x^m)(1-x^{2m-1})} \qquad \text{[7, Ent 4.2.12]} \qquad (2.5)$$

$$\sum_{n=0}^{\infty} \frac{x^{n^2}(-x;x^2)_n}{(x^4;x^4)_n} = \prod_{m=1}^{\infty} \frac{(1-x^{3m})(1-x^{12m})}{(1-x^{6m-5})(1-x^{6m-1})(1-x^{4m})} \qquad \text{[7, Ent 4.2.7]} \qquad (2.6)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+1)}(-x^2;x^2)_n}{(x)_{2n+1}} = \prod_{m=1}^{\infty} \frac{(1-x^{12m})(1-x^{12m-9})(1-x^{12m-3})}{1-x^m} \qquad \text{[7, Ent 4.3.12]} \qquad (2.7)$$

$$\sum_{n=0}^{\infty} \frac{x^{n^2}(-x;q^2)_n}{(x)_{2n}} = \prod_{m=1}^{\infty} \frac{(1-x^{6m})(1-x^{12m-6})}{1-x^m} \qquad [7, \text{Ent. } 5.2.3] \quad (2.8)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+1)/2}(-x^2;x^2)_n}{(x)_n(x;x^2)_{n+1}} = \prod_{m=1}^{\infty} \frac{1+x^m}{(1-x^{2m-1})(1-x^{8m-4})} \qquad [7, \text{Ent. } 1.7.5] \quad (2.9)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+1)/2}(-1;x^2)_n}{(x)_n(x;x^2)_n} = \prod_{m=1}^{\infty} \frac{(1-x^{4m})(1-x^{8m-4})(1+x^m)}{1-x^m} \qquad [7, \text{Ent. } 1.7.4] \quad (2.10)$$

$$\sum_{n=0}^{\infty} \frac{x^{n(n+2)}(-x;x^2)_n}{(x)_{2n+1}} = \prod_{m=1}^{\infty} \frac{(1-x^{12m})(1-x^{12m-10})(1-x^{12m-9})}{1-x^m} \qquad [7, \text{Ent. } 3.4.4] \quad (2.11)$$

Let us denote the cöefficient of x^n in the power series expansion of equation (j) above by $R_j(n)$. The following combinatorial interpretations are then immediate:

- $R_{2.1}(n) = R_{2.2}(n) =$ the number of overpartitions of n into nonmultiples of 3.
- $R_{2.3}(n) = R_{2.4}(n) =$ the number of overpartitions of n with only odd parts.
- $R_{2.5}(n) =$ the number of overpartitions of n where nonoverlined parts are congruent to $\pm 2, 3 \pmod 6$.
- $R_{2.7}(n) =$ the number of partitions of n into parts not congruent to $0, \pm 3 \pmod{12}$.
- $R_{2.9}(n) =$ the number of overpartitions of n where the nonoverlined parts are odd or congruent to $4 \pmod 8$
- $R_{2.11}(n) =$ the number of partitions of n into parts not congruent to $0, \pm 2 \pmod{12}$.

The circle method yields the following formulas, which are believed to be new to the literature. It could be argued that a number of them capture much of the elegance of the formula for $p(n)$. They were found with the aid of *Mathematica* program written by the author. For a discussion of the automation of certain key steps of the circle method, along with additional examples of Rademacher type formulas for restricted partition and overpartition functions, please see [62]. As noted earlier, they could also be derived using the results of Bringmann and Ono [12].

$$R_{2.1}(n) = \frac{\pi}{3\sqrt{2n}} \sum_{\substack{k \geqq 1 \\ 2\nmid k, 3\nmid k}} \frac{1}{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(6h,k)}{\omega(2h,k)\omega(3h,k)^2} I_1\left(\frac{\pi\sqrt{2n}}{k\sqrt{3}}\right) \quad (2.12)$$

$$R_{2.3}(n) = \frac{\pi}{4\sqrt{n}} \sum_{\substack{k \geqq 1 \\ 2\nmid k}} \frac{1}{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(4h,k)}{\omega(2h,k)^3} I_1\left(\frac{\pi\sqrt{n}}{k\sqrt{2}}\right). \quad (2.13)$$

$$R_{2.5}(n) = \frac{\pi}{2\sqrt{18n+6}} \sum_{\substack{k \geqq 1 \\ 2\nmid k}} \frac{\sqrt{(k,6)}}{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\,\omega\left(\frac{3h}{(k,3)}, \frac{k}{(k,3)}\right)}{\omega\left(\frac{6h}{(k,6)}, \frac{k}{(k,6)}\right)} I_1\left(\frac{\pi\sqrt{6n+2}}{3k}\right) \quad (2.14)$$

$$R_{2.6}(n) = \frac{\pi}{3\sqrt{264n-33}} \sum_{d\in\{1,4,12\}} \sqrt{d^2+83d+48} \sum_{\substack{k\geqq 1\\(k,12)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega\left(\frac{4h}{(d,4)},\frac{k}{(d,4)}\right)\omega\left(\frac{6h}{(d,6)},\frac{k}{(d,6)}\right)}{\omega\left(\frac{3h}{(d,3)},\frac{k}{(d,3)}\right)^2\omega\left(\frac{2h}{(d,2)},\frac{k}{(d,2)}\right)}$$
$$\times I_1\left(\pi\frac{\sqrt{(16d-d^2-12)(8n-1)}}{12k}\right) \tag{2.15}$$

$$R_{2.7}(n) = \frac{\pi}{4\sqrt{90n+30}} \sum_{d|6} \sqrt{(d-3)(9d^2-52d+28)} \sum_{\substack{k\geqq 1\\(k,12)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi ihn/k} \frac{\omega(h,k)\omega\left(\frac{6h}{(d,6)},\frac{k}{(d,6)}\right)}{\omega\left(\frac{3h}{(d,3)},\frac{k}{(d,3)}\right)\omega\left(\frac{12h}{(d,12)},\frac{k}{(d,12)}\right)} I_1\left(\frac{\pi\sqrt{(8+8d-d^2)(3n+1)}}{3k\sqrt{10}}\right) \tag{2.16}$$

$$R_{2.8}(n) = \frac{\pi}{3\sqrt{264n-11}} \sum_{d\in\{1,4,12\}} \sqrt{2d^2+d+96} \sum_{\substack{k\geqq 1\\(k,12)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi ihn/k} \frac{\omega(h,k)\omega\left(\frac{12h}{d},\frac{k}{d}\right)}{\omega\left(\frac{6h}{(d,6)},\frac{k}{(d,6)}\right)^2} I_1\left(\frac{\pi\sqrt{(84+16d-d^2)(24n-1)}}{12k\sqrt{33}}\right) \tag{2.17}$$

$$R_{2.9}(n) = \frac{\pi\sqrt{3}}{4\sqrt{8n+2}} \sum_{\substack{k\geqq 1\\2\nmid k}} \frac{1}{k} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi ihn/k} \frac{\omega(h,k)^2\omega(4h,k)}{\omega(2h,k)^2\omega(8h,k)} I_1\left(\frac{\pi\sqrt{12n+3}}{4k}\right) \tag{2.18}$$

$$R_{2.10}(n) = \frac{\pi\sqrt{3}}{8\sqrt{n}} \sum_{\substack{k\geqq 1\\2\nmid k}} \frac{1}{k} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi ihn/k} \frac{\omega(h,k)^2\omega(8h,k)}{\omega(4h,k)^2\omega(2h,k)} I_1\left(\frac{\pi\sqrt{3n}}{2k}\right) \tag{2.19}$$

$$R_{2.11}(n) = \frac{\pi}{6\sqrt{24n+15}} \sum_{d=1}^{4} \sqrt{(2-d)(7d^2-46d+48)} \sum_{\substack{k\geqq 1\\(k,12)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi ihn/k} \frac{\omega(h,k)\omega\left(\frac{4h}{(d,4)},\frac{k}{(d,4)}\right)\omega\left(\frac{6h}{(d,6)},\frac{k}{(d,6)}\right)}{\omega\left(\frac{2h}{(d,2)},\frac{k}{(d,2)}\right)\omega\left(\frac{12h}{d},\frac{k}{d}\right)^2} I_1\left(\frac{\pi\sqrt{(8n+5)(d^2-4d+12)}}{12k}\right) \tag{2.20}$$

3 Capparelli's Conjecture

The year 1992 marked the one hundredth anniversary of the birth of Rademacher, and on July 21–25 of that year a conference honoring the memory of Rademacher was held at Penn State, and George Andrews was of course one of the conference organizers. On the first day of the conference, James Lepowsky of Rutgers gave a talk in which he mentioned that his student Stefano Capparelli had conjectured the following partition identity [13] as a result of his studies of the standard level 3 modules associated with the Lie algebra $A_2^{(2)}$:

Theorem 5 (Capparelli)

- *Let $C(n)$ denote the number of partitions of n into parts $\equiv \pm 2, \pm 3 \pmod{12}$.*
- *Let $D(n)$ denote the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n such that*
  - $\lambda_j - \lambda_{j+1} \geqq 2$,
  - $\lambda_j - \lambda_{j+1} = 2$ *only if* $\lambda_j \equiv 1 \pmod 3$, *and*
  - $\lambda_j - \lambda_{j+1} = 3$ *only if* λ_j *is a multiple of* 3.
- *Then $C(n) = D(n)$ for all n.*

This identity is clearly similar in the spirit of those in the classical literature such as Schur's identity (our Theorem 3), yet was new. Needless to say, Andrews and others at the conference were quite intrigued by the conjecture. Andrews worked intently for the next several evenings, and was able to find a proof [5] of the identity in time to present it as his talk on the last day of the conference. Of this proof, Andrews wrote [1, p. 505], "In my proof of Capparelli's conjecture, I was completely guided by the Wilf-Zeilberger method, even if I didn't use Doron's program explicitly. I couldn't have produced my proof without knowing the principle behind 'WZ.' " Although Andrews' WZ-inspired proof (see [64, 67–70]) was the first proof of the Capparelli conjecture, Lie theoretic proofs were later found by Tamba and Xie [63] and Capparelli himself [14].

The generating function for the partitions enumerated by the $C(n)$ in Capparelli's identity is

$$\sum_{n=0}^{\infty} C(n)x^n = \prod_{m \geqq 1} \frac{1}{(1-x^{12m-10})(1-x^{12m-9})(1-x^{12m-3})(1-x^{12m-2})},$$

and indeed the Rademacher method may be applied to find an explicit formula for $C(n)$.

$$C(n) = \frac{\pi}{\sqrt{24n-1}} \sum_{d \in \{1,2,3,12\}} \sqrt{12+308d+12d^2-2d^3} \sum_{\substack{k \geqq 1 \\ (k,12)=d}} \frac{1}{k} \times \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi nh/k} \frac{\omega(\frac{12h}{d}, \frac{k}{d})\,\omega(\frac{3h}{(d,3)}, \frac{k}{(d,3)})\,\omega(\frac{2h}{(d,2)}, \frac{k}{(d,2)})}{\omega(\frac{6h}{(d,6)}, \frac{k}{(d,6)})^2\,\omega(\frac{4h}{(d,4)}, \frac{k}{(d,4)})} \times I_1\left(\frac{\pi\sqrt{(24n-1)(201-231d+91d^2-6d^3)}}{6\sqrt{165}k}\right). \quad (3.1)$$

4 The Bailey Chain

Of course, Andrews has contributed a large number of important and useful discoveries to the body of mathematical knowledge. One of this author's favorites is the *Bailey chain*, i.e. the realization that the Bailey lemma is self-replicating and therefore any Bailey pair implies infinitely many others. In particular, every Rogers-Ramanujan type identity is automatically part of an infinite family (see [3,4]).

The Bailey chain provides an explanation and a context for many infinite family q-series identities and their combinatorial counterparts. For example, David Bressoud's identity [10, p. 15, Eq. (3.4) with $k=r$]

$$\sum_{n_1,n_2,\dots,n_{r-1}\geqq 0} \frac{x^{N_1^2+N_2^2+\cdots+N_{r-1}^2}}{(x)_{n_1}(x)_{n_2}\cdots(x)_{n_{r-2}}(x^2;x^2)_{n_{r-1}}} = \prod_{m=1}^{\infty} \frac{(1-x^{2rm-r})(1-x^{rm})}{1-x^m}, \tag{4.1}$$

where $N_j := n_j + n_{j+1} + \cdots + n_{r-1}$ and $r \geqq 2$, follows from inserting the Bailey pair

$$\alpha_n(a,x) = \frac{(-1)^n x^{n^2}(1-ax^{2n})(a^2;x^2)_n}{(1-a)(x^2;x^2)_n}, \qquad \beta_n(a,x) = \frac{1}{(x^2;x^2)_n}$$

into a certain limiting case of the Bailey chain [4, p. 30, Theorem 3.5], setting $a=1$, and then applying Jacobi's triple product identity [4, p. 63, Eq. (7.1)]. Although Bressoud's combinatorial counterpart to [10, p. 15, Eq. (3.4)] excludes the special case with $k=r$ (our (4.1) above), the author provided a combinatorial interpretation [60, p. 315, Theorem 6.9], which we recall here:

Theorem 6 *For $r \geqq 2$, let $B_r(n)$ denote the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n such that*

- 1 *appears as a part less than r times,*
- $\lambda_j - \lambda_{j+r-1} \geqq 2$, *and*
- *if $\lambda_j - \lambda_{j+r-2} \leqq 1$, then $\sum_{h=0}^{r-2} \lambda_{j+h} \equiv (r-1) \pmod 2$.*

For $r \geqq 3$, let $A_r(n)$ denote the number of partitions of n such that

- *no part is a multiple of r,*
- *for any nonnegative integer j, either $rj+1$ or $r(j+1)-1$, but not both, may appear as parts,*

and let $A_2(n)$ denote the number of partitions of n into distinct odd parts. Then $A_r(n) = B_r(n)$ for all integers n.

Remark 1 The combinatorial interpretation of the $A_r(n)$ was facilitated by ideas advanced by Andrews and Lewis [8].

We conclude with a Rademacher-type formula for the $A_r(n)$:

$$A_r(n) = \frac{2\pi\sqrt{2}}{\sqrt{24n-1}} \sum_{d|r} \frac{(d,r)}{\sqrt{dr}} \chi\left(2r + d^2 > 4(d,r)^2\right) \sum_{\substack{k\geqq 1 \\ (h,k)=d}} k^{-1}$$
$$\times \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2rh/d,k/d)}{\omega\left(\frac{rh}{(d,r)}, \frac{k}{(d,r)}\right)} I_1\left(\frac{\pi}{6k}\sqrt{\frac{(24n-1)\left(2r+d^2-4(d,r)^2\right)}{2r}}\right), \tag{4.2}$$

where

$$\chi(P) = \begin{cases} 1 & \text{if } P \text{ is true, and} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

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