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Chromatic Polynomials of Signed Book Graphs

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Cover Page Footnote

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Abstract

For $m \geq 3$ and $n \geq 1$, the m-cycle *book graph* $B(m,n)$ consists of n copies of the cycle C_m with one common edge. In this paper, we prove that (a) the number of switching non-isomorphic signed $B(m,n)$ is $n+1$, and (b) the chromatic number of a signed $B(m,n)$ is either 2 or 3. We also obtain explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

1 Introduction

A *signed graph* $\Sigma = (G, \sigma)$ consists of a graph $G = (V, E)$ and a sign function $\sigma : E \to \{1, -1\}$. The aim of this paper is to obtain formulas for the chromatic polynomials as well as the zero-free chromatic polynomials of switching non-isomorphic signed book graphs $B(m,n)$. In Section 2 of the paper, it is shown for given $m \geq 3$ and $n \geq 1$ that the number of switching non-isomorphic signed $B(m,n)$ is $n+1$. It is also shown that the chromatic number of a signed $B(m,n)$ is either 2 or 3. Finally, formulas for the chromatic polynomials and the zero-free chromatic polynomials of signed book graphs are presented in Section 3 and Section 4, respectively.

Let $\Sigma = (G, \sigma)$ be a signed graph. The graph G is called the *underlying* graph of $\Sigma = (G, \sigma)$. The set $\sigma^{-1}(-1) = \{e \in E(G) \mid \sigma(e) = -1\}$ is called the *signature* of Σ. That is, by the signature of a signed graph Σ we mean the set of negative edges in Σ. For convenience, we call σ itself a signature and write $|\sigma| = |\sigma^{-1}(-1)|$.

A cycle C in Σ is called *positive* if the product $\prod_{e \in E(C)} \sigma(e) = 1$ and *negative*, otherwise. A signed graph Σ is said to be *balanced* if every cycle of the graph is positive and *unbalanced*, otherwise. The notion of signed graphs and balance was introduced by Harary in [2].

In a signed graph, *switching* a vertex v changes the signs of all the edges incident to v. We say that a signed graph Σ_1 is *switching equivalent*, or simply *equivalent*, to another signed graph Σ_2 if one can be obtained from the other by a sequence of switchings. If we switch Σ by every vertex of a subset $X \subseteq V(\Sigma)$ then the resulting signed graph is denoted by Σ^X. It is easy to see that switching defines an equivalence relation on the set of all signed graphs with an underlying graph G. For a signed graph Σ, its switching equivalence class is denoted by $[\Sigma]$. As switching operation preserve the product of signs on each cycle, any property of signed graphs that depends only on the signs of the cycles is invariant for all signed graphs belonging to $[\Sigma]$.

One of the fundamental theorems in the theory of signed graphs is that the set of negative cycles uniquely determines the equivalence class to which a signed graph belongs. More precisely, we state the following theorem.

Theorem 1.1 ([4]). *Two signed graphs Σ_1 and Σ_2 are switching equivalent if and only if they have the same set of negative cycles.*

Two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (H, \sigma_2)$ are said to be *isomorphic*, denoted $\Sigma_1 \cong \Sigma_2$, if there exists a graph isomorphism $\psi : V(G) \to V(H)$ which preserve the edge signs. The signed graphs Σ_1 and Σ_2 are said to be *switching isomorphic*, denoted $\Sigma_1 \sim \Sigma_2$, if

Σ_1 is isomorphic to Σ_2^X for some $X \subseteq V(\Sigma_2)$. Two signed graphs having the same underlying graph are called *automorphic* to each other if they are isomorphic.

In [5], Zaslavsky initiated the study of vertex colorings in the context of signed graphs. A *coloring* of a signed graph $\Sigma = (G, \sigma)$ in $2k+1$ *signed colors* is defined to be a mapping $c : V(G) \to \{-k, \ldots, -1, 0, 1, \ldots, k\}$. Similarly, a *zero-free coloring* of a signed graph $\Sigma = (G, \sigma)$ in $2k$ *signed colors* is a mapping $c : V(G) \to \{-k, \ldots, -1, 1, \ldots, k\}$. A coloring c of Σ is said to be *proper* if $c(x) \neq \sigma(e)c(y)$ for every edge $e = xy$, where $\sigma(e)$ is the sign of the edge e. In other words, a coloring of a signed graph is proper if the colors of the vertices incident to a positive edge are not equal, while those incident to a negative edge are not equal in absolute values whenever they are of opposite signs or zero.

The *chromatic polynomial* $\chi_\Sigma(\lambda)$ of a signed graph Σ is the function whose value, for odd positive arguments $\lambda = 2k+1$, equals the number of proper colorings of Σ in $2k+1$ signed colors. The *zero-free chromatic polynomial* $\chi_\Sigma^b(\lambda)$ of a signed graph Σ is the function, where $\chi_\Sigma^b(2k)$ counts the zero-free proper colorings in $2k$ signed colors. The chromatic polynomial of a graph G is denoted by $\chi_G(\lambda)$. The *chromatic number* $\chi(\Sigma)$ of Σ is defined to be the minimum number of the set

$$\left\{2k+1 \;:\; \chi_\Sigma(2k+1) > 0\right\} \cup \left\{2r \;:\; \chi_\Sigma^b(2r) > 0\right\}.$$

There is a strong conclusion if Σ is balanced. More precisely, we have the following lemma.

Lemma 1.2 ([5]). *If $\Sigma = (G, \sigma)$ is balanced, then $\chi_G(\lambda) = \chi_\Sigma(\lambda) = \chi_\Sigma^b(\lambda)$.*

Let the signed graphs Σ and Σ' be switching equivalent. It is clear that a coloring c of Σ is proper if and only if the coloring c' of Σ' is proper, where c' is obtained from c after negating the colors of the vertices by which Σ is switched. Thus the chromatic number of a signed graph remains invariant under switching operation.

In [5], Zaslavsky proved that the chromatic number and the chromatic polynomials of a signed graph are invariant under switching operation. In perfect analogy to ordinary graph coloring theory, we have the following theorem.

Theorem 1.3 ([5]). *If Σ is a signed graph on n vertices, then $\chi_\Sigma(\lambda)$ and $\chi_\Sigma^b(\lambda)$ are monic polynomial functions of λ of degree n.*

Let e be a positive edge in $\Sigma = (G, \sigma)$. The *edge-contraction* Σ/e is obtained by identifying the end vertices of e and deleting e. We also have the signed analogue of edge deletion-contraction formula for the chromatic polynomial of simple graph.

Theorem 1.4 ([5]). *Let Σ be a signed graph and e be a positive edge in Σ. Then*

$$\chi_\Sigma(\lambda) = \chi_{\Sigma\setminus e}(\lambda) - \chi_{\Sigma/e}(\lambda) \text{ and } \chi_\Sigma^b(\lambda) = \chi_{\Sigma\setminus e}^b(\lambda) - \chi_{\Sigma/e}^b(\lambda).$$

Chromatic polynomials of signed graphs has been less studied. However, in order to compute both the chromatic polynomials (*i.e.,* the chromatic polynomial as well as the zero-free chromatic polynomial) of a signed graph, a remarkable work was done by Mathias Beck and his team. In [1], they published a SAGE code which produces both the chromatic polynomials as output when a signed graph is given as input. Using that code they have

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presented explicit formulas for both kind of chromatic polynomials of six switching non-isomorphic signed Petersen graphs.

For $m \geq 3$ and $n \geq 1$, the m-cycle *book graph* $B(m,n)$ consists of n copies of the cycle C_m with one common edge. The copies of the cycle C_m are called the *pages* of the book graph $B(m,n)$. Let $V(B(m,n)) = \{u,v\} \cup \{u_j^i \mid 1 \leq i \leq n, \;\; 1 \leq j \leq m-2\}$, and let uv be the common edge to the cycles C_m^i, where $C_m^i = uu_1^i u_2^i u_3^i ... u_{m-3}^i u_{m-2}^i vu$, for $1 \leq i \leq n$. For example, the cycle C_4^1 in $B(4,3)$ is the cycle $uu_1^1 u_2^1 vu$, where the graph $B(4,3)$ is shown in Figure 1.

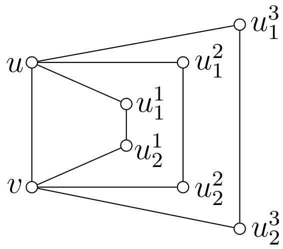


Figure 1: The book graph $B(4,3)$

It is clear that any permutation of the n copies of the cycle C_m or any permutation of u and v determines an automorphism of $B(m,n)$, and vice-versa. Thus, if $n > 1$ then an automorphism of $B(m,n)$ can only permute the vertices u and v, and permute its n pages.

2 The chromatic number of signed book graphs

Note that every signed cycle is switching equivalent to a signed cycle whose signature is either empty or of size one. We will use this fact in the proof of the following theorem.

Theorem 2.1. *Let $m \geq 3, n \geq 1$ be fixed and $(B(m,n),\sigma)$ a signed book graph. Then $(B(m,n),\sigma)$ is equivalent to $(B(m,n),\tau)$, where $\tau \subseteq \{uu_1^1, \ldots, uu_1^n\}$. Further, the number of switching non-isomorphic signed $B(m,n)$ is $n+1$.*

Proof. Let $(B(m,n),\sigma)$ be a signed book graph. By suitable switchings, if needed, we can make each negative edge of $B(m,n)$ incident to u. If the edge uv is negative, switching u will make it positive. Thus we get a signature τ equivalent to σ such that $\tau \subseteq \{uu_1^1, \ldots, uu_1^n\}$.

Note that if $\sigma, \tau \subseteq \{uu_1^1, \ldots, uu_1^n\}$ and $|\sigma| = |\tau|$, then an one-one correspondence between σ and τ determines an isomorphism between $(B(m,n),\sigma)$ and $(B(m,n),\tau)$. However, if $|\sigma| \neq |\tau|$, then $(B(m,n),\sigma)$ cannot be switching isomorphic to $(B(m,n),\tau)$ because the number of negative cycles C_m are different. Therefore, the number of switching non-isomorphic signed $B(m,n)$ is $n+1$. □

Let $\sigma_0 = \emptyset$. For each $1 \leq l \leq n$, let $\sigma_l = \{uu_1^1, uu_1^2, \cdots, uu_1^l\}$. By the preceding theorem, $\sigma_0, \sigma_1, \ldots, \sigma_n$ are switching non-isomorphic signatures of $B(m,n)$.

It is proved in [3, Proposition 2.4] that $\chi_{(C_m,\sigma)} \leq 3$, where (C_m,σ) is a signed cycle. Therefore we have $\chi_{(B(m,1),\sigma)} \leq 3$. In the following theorem, we assume that $n \geq 2$.

Theorem 2.2. *Let $m \geq 3, n \geq 2$ and $0 \leq l \leq n$. Then*

$$\chi(B(m,n),\sigma_l) = \begin{cases} 2 & \text{if } m \text{ is odd and } l = n, \\ 2 & \text{if } m \text{ is even and } l = 0, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. In $(B(2k+1,n),\sigma_n)$, assign colors 1 and -1 to u and v, respectively. For each $1 \leq r \leq n$, assign colors $1,-1,1,...,-1,1$ to the vertices $u_1^r, u_2^r, u_3^r, ..., u_{2k-2}^r, u_{2k-1}^r$, respectively. This gives a proper 2-coloring of $(B(2k+1,n),\sigma_n)$.

In $(B(2k,n),\sigma_0)$, assign the colors 1 and -1 to the vertices u and v, respectively. For each $1 \leq r \leq n$, the vertices $u_1^r, u_2^r, u_3^r, u_4^r, ..., u_{2k-3}^r, u_{2k-2}^r$ are colored with $-1,1,-1,1,...,-1,1$, respectively. This gives a proper 2-coloring of $(B(2k,n),\sigma_0)$.

Note that the odd cycle $C_{2k+1}^n = uu_1^n u_2^n u_3^n ... u_{2k-2}^n u_{2k-1}^n vu$ is positive in $(B(2k+1,n),\sigma_l)$, where $0 \leq l \leq n-1$. It is well known that at least three colors are needed to properly color the vertices of a positive odd cycle. Therefore, $\chi(B(2k+1,n),\sigma_l) \geq 3$. We now give a proper 3-coloring of $(B(2k+1,n),\sigma_l)$. Assign colors 0 and -1 to u and v, respectively. For each $1 \leq r \leq n$, the vertices $u_1^r, u_2^r, u_3^r, ..., u_{2k-2}^r, u_{2k-1}^r$ are colored with $1,-1,1,...,-1,1$, respectively. This assignment is a proper 3-coloring of $(B(2k+1,n),\sigma_l)$, where $0 \leq l \leq n-1$.

Finally, $(B(2k,n),\sigma_l)$ contains a negative $2k$-cycle for each $1 \leq l \leq n$. The fact that an unbalanced even cycle needs at least 3 colors to have a proper coloring implies that $\chi(B(2k,n),\sigma_l) \geq 3$. To complete the proof it suffices to give a proper 3-coloring of $(B(2k,n),\sigma_l)$. Assign colors 0 and 1 to u and v, respectively. For each $1 \leq r \leq n$, the vertices $u_1^r, u_2^r, u_3^r, ..., u_{2k-2}^r, u_{2k-1}^r$ are colored with $1,-1,1,...,1,-1$, respectively. This assignment is a proper 3-coloring of $(B(2k,n),\sigma_l)$. □

As the chromatic number of a signed graph is invariant under switching operation, by Theorem 2.2, we find that the chromatic number of a signed book graph is either 2 or 3.

3 Chromatic polynomials of signed book graphs

For convenience, we write $B_l(m,n)$ to denote the signed book graph $(B(m,n),\sigma_l)$. Also, we write $B^{uv}(m,n)$ to denote $(B(m,n),\{uv\})$. It is clear that $B_n(m,n) \sim B^{uv}(m,n)$. Since the chromatic polynomials of a signed graph are switching invariant, it is sufficient to the determine the chromatic polynomials of $B^{uv}(m,n)$ and $B_l(m,n)$ for $0 \leq l \leq n-1$.

Let an unbalanced cycle on n vertices be denoted by C_n^-. For instance, C_2^- is shown in Figure 2.

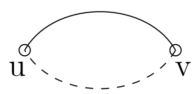


Figure 2: An unbalanced cycle of length two

Example 3.1. Let the colors $-k,\ldots,-1,0,1\ldots,k$ be available. The number of proper colorings of C_2^- with these $2k+1$ colors, in which one of u or v is colored with 0, is $4k$. Else, the number of proper colorings of C_2^- is $2k(2k-2)$. Thus $\chi_{C_2^-}(2k+1) = 4k + 2k(2k-2) = (2k)^2$.

From Example 3.1, we find $\chi_{C_2^-}(\lambda) = (\lambda-1)^2$. In [1], it is proved that $\chi_{C_3^-}(\lambda) = (\lambda-1)^3$. Now we give the formula of $\chi_{C_n^-}(\lambda)$ for all n.

Lemma 3.2. *Let C_n^- be an unbalanced cycle, where $n \geq 2$. Then $\chi_{C_n^-}(\lambda) = (\lambda-1)^n$.*

Proof. We prove this lemma by induction on n. If $n = 2$, the result is true by Example 3.1. Let us assume that the result holds for all $n \leq r-1$, where $r \geq 3$. We shall prove that the result is also true for $n = r$. If e is a positive edge of C_r^- then by edge deletion-contraction formula, we get

$$\chi_{C_r^-}(\lambda) = \chi_{P_r}(\lambda) - \chi_{C_{r-1}^-}(\lambda). \tag{1}$$

It is well known that $\chi_{P_r}(\lambda) = \lambda(\lambda-1)^{r-1}$, and by induction hypothesis $\chi_{C_{r-1}^-}(\lambda) = (\lambda-1)^{r-1}$. Therefore Equation (1) gives

$$\chi_{C_r^-}(\lambda) = \lambda(\lambda-1)^{r-1} - (\lambda-1)^{r-1} = (\lambda-1)^r.$$

Thus the proof follows by induction. □

It is well known that $\chi_{C_m}(\lambda) = (\lambda-1)^m + (-1)^m(\lambda-1)$. Using $\chi_{C_m}(\lambda)$, a polynomial function γ_m of degree $m-2$ is defined as

$$\gamma_m = \frac{\chi_{C_m}(\lambda)}{\lambda(\lambda-1)} = \frac{(\lambda-1)^{m-1} + (-1)^m}{\lambda}. \tag{2}$$

The function γ_m can also be expressed as a finite geometric series. More precisely, we have

$$\gamma_m = \frac{(\lambda-1)^{m-1} + (-1)^m}{\lambda} = \sum_{i=0}^{m-2} (-1)^i (\lambda-1)^{(m-2)-i}. \tag{3}$$

We will use the function γ_m in the expression of formulas of both kind of chromatic polynomials of signed book graphs.

Let Σ be a given signed graph. Construct the signed graph Σ_{t+1} by attaching the path $P_{t+1} := uu_1u_2\ldots u_t$ to a vertex u of Σ. If the chromatic polynomial of Σ is known then we can compute the chromatic polynomial of Σ_{t+1} using the following lemma.

Lemma 3.3. *Let Σ be a signed graph. Then $\chi_{\Sigma_{t+1}}(\lambda) = (\lambda-1)^t \chi_\Sigma(\lambda)$.*

Proof. We prove this lemma by induction on t. Note that $\chi_{P_{t+1}}(\lambda) = \lambda(\lambda-1)^t$. Let $t = 1$, and $e_1 = uu_1$ be attached to a vertex u of Σ. Using edge deletion-contraction formula on e_1, we get

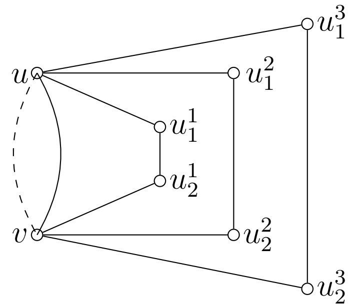
$$\chi_{\Sigma_2}(\lambda) = \lambda\chi_\Sigma(\lambda) - \chi_\Sigma(\lambda) = (\lambda-1)\chi_\Sigma(\lambda).$$

Now assume that the result holds for $t = r-1$, that is, $\chi_{\Sigma_r}(\lambda) = (\lambda-1)^{r-1}\chi_\Sigma(\lambda)$, where $r \geq 3$. Now let $t = r$ and let $e = uu_1$. Using edge deletion-contraction formula on the edge e of Σ_{r+1}, we get $\chi_{\Sigma_{r+1}}(\lambda) = \chi_{P_r \cup \Sigma}(\lambda) - \chi_{\Sigma_r}(\lambda)$, where P_r and Σ are disjoint. Thus we have

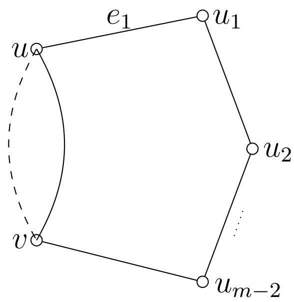
$$\begin{aligned}
\chi_{\Sigma_{r+1}}(\lambda) &= \lambda(\lambda-1)^{r-1}\chi_\Sigma(\lambda) - \chi_{\Sigma_r}(\lambda) \\
&= \lambda(\lambda-1)^{r-1}\chi_\Sigma(\lambda) - (\lambda-1)^{r-1}\chi_\Sigma(\lambda) \\
&= (\lambda-1)^r\chi_\Sigma(\lambda).
\end{aligned}$$

Hence the proof follows by induction. □

Replace the edge uv of $B(m,n)$ by an unbalanced cycle of length two, and denote the graph so obtained by B_m^n. For example, B_4^3 and B_m^1 are shown in Figure 3. As a convention, write $B_2^1 = C_2^-$ so that $\chi_{B_2^1}(\lambda) = (\lambda-1)^2$.



(a) The signed graph B_4^3



(b) The signed graph B_m^1

Figure 3: The signed graphs B_4^3 and B_m^1

Lemma 3.4. *For $m \geq 2$, the chromatic polynomial of B_m^1 is given by*

$$\chi_{B_m^1}(\lambda) = (\lambda-1)^2\gamma_m.$$

Proof. We prove this lemma by induction on m. Consider B_m^1 as given in Figure 3(b). For $m = 2$, the result is true by Example 3.1 since $\gamma_2 = 1$. Assume that the result is true for $m = r-1$, where $r \geq 3$. That is,

$$\chi_{B_{r-1}^1}(\lambda) = (\lambda-1)^2\gamma_{r-1}. \tag{4}$$

An application of edge deletion-contraction on the edge $e_1 = uu_1$ of B_r^1 is shown in Figure 4. Using Lemma 3.3, the chromatic polynomial of the graph in the middle of Figure 4 can be computed easily, since $\chi_{C_2^-}(\lambda)$ is known. The chromatic polynomial of the third graph of Figure 4 is given in Equation (4). Therefore

$$\begin{aligned}
\chi_{B_r^1}(\lambda) &= (\lambda-1)^{r-2}\chi_{C_2^-}(\lambda) - (\lambda-1)^2\gamma_{r-1} \\
&= (\lambda-1)^r - (\lambda-1)^2\frac{(\lambda-1)^{r-2} + (-1)^{r-1}}{\lambda} \\
&= (\lambda-1)^2\frac{(\lambda-1)^{r-1} + (-1)^r}{\lambda} \\
&= (\lambda-1)^2\gamma_r.
\end{aligned}$$

This completes the proof. □

Now we give the formula for the chromatic polynomial of the signed graph B_m^n, where $n \geq 2$. To do so we use Theorem 1.4, Lemma 3.3, and Lemma 3.4.

Proposition 3.1. *For $n \geq 2$, the chromatic polynomial of the signed graph B_m^n is given by*

$$\chi_{B_m^n}(\lambda) = (\lambda-1)^2\gamma_m^n.$$

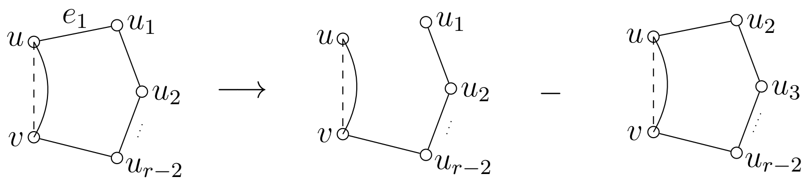


Figure 4: An application of edge deletion-contraction on B_r^1

Proof. We sequentially use edge deletion-contraction formula on the edges uu_1^n, uu_2^n, uu_3^n, and so on. This process gives

$$\begin{aligned}\chi_{B_m^n}(\lambda) &= \left[\sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}\right]\chi_{B_m^{n-1}} \\ &= \gamma_m \chi_{B_m^{n-1}}(\lambda).\end{aligned}$$

Since $\chi_{B_m^1}(\lambda)$ is known from Lemma 3.4, after $n-2$ recursions, we find that

$$\chi_{B_m^n}(\lambda) = (\lambda-1)^2\gamma_m^n.$$

This completes the proof. □

It is well known that if G and H are two simple graphs such that $G \cap H$ is a complete graph, then the chromatic polynomial of $G \cup H$ is given by

$$\chi_{G\cup H}(\lambda) = \frac{\chi_G(\lambda)\cdot\chi_H(\lambda)}{\chi_{G\cap H}(\lambda)}. \tag{5}$$

Using Equation (5), we now determine the chromatic polynomial of an unsigned book graph.

Theorem 3.5. *The chromatic polynomial of $B(m,n)$ is given by*

$$\chi_{B(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

Proof. Clearly $B(m,1) = C_m$. By the definition of γ_m we have

$$\chi_{B(m,1)}(\lambda) = \chi_{C_m}(\lambda) = \lambda(\lambda-1)\gamma_m. \tag{6}$$

Now repeated use of Equations (5) and (6) give that

$$\chi_{B(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

This completes the proof. □

From Lemma 1.2 and Theorem 3.5, it follows that

$$\chi_{B_0(m,n)}(\lambda) = \chi_{B(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

It is clear that $B_1(m,1) \cong C_m^-$. Thus by Lemma 3.2, we have $\chi_{B_1(m,1)}(\lambda) = (\lambda-1)^m$. We now compute the chromatic polynomials of the signed book graphs $B_1(m,n)$.

Theorem 3.6. *The chromatic polynomial of $B_1(m,n)$ is given by*

$$\chi_{B_1(m,n)}(\lambda) = (\lambda-1)^m \gamma_m^{n-1}.$$

Proof. Recall that $B_1(m,n)$ is the signed book graph with signature $\{uu_1^1\}$. For $n = 1$, result holds true due to Lemma 3.2. For $n \geq 2$, consider $e_1 = uu_1^n$. Using edge deletion-contraction formula on e_1, we get

$$\chi_{B_1(m,n)}(\lambda) = \chi_{B_1'(m,n-1)}(\lambda) - \chi_{B_1''(m,n)}(\lambda),$$

where $B_1'(m,n-1)$ denotes the graph obtained from $B_1(m,n-1)$ by attaching a path P_{m-1} at vertex v and $B_1''(m,n)$ denotes the graph which is almost same as $B_1(m,n)$ but one of its pages is a cycle of length $m-1$ and the vertex obtained by contraction of e_1 is denoted by u again.

Using Lemma 3.3, the chromatic polynomial of the signed graph $B_1'(m,n-1)$ can be obtained in terms of $\chi_{B_1(m,n-1)}(\lambda)$. For $B_1''(m,n)$, we apply the edge deletion-contraction formula again on the edge $e_2 = uu_2^n$. Repeated use of edge deletion-contraction formula and Lemma 3.3 give the chromatic polynomial of $B_1(m,n)$ as

$$\begin{aligned}\chi_{B_1(m,n)}(\lambda) =&(\lambda-1)^{m-2}\chi_{B_1(m,n-1)}(\lambda) - (\lambda-1)^{m-3}\chi_{B_1(m,n-1)}(\lambda) + \cdots\cdots \\ &+ (-1)^{m-2}(\lambda-1)^{(m-2)-(m-2)}\chi_{B_1(m,n-1)}(\lambda) \\ =&\left[\sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}\right]\chi_{B_1(m,n-1)}(\lambda) \\ =&\gamma_m \chi_{B_1(m,n-1)}(\lambda).\end{aligned}$$

Since $\chi_{B_1(m,1)}(\lambda) = (\lambda-1)^m$, after $n-2$ recursions, we find that

$$\chi_{B_1(m,n)}(\lambda) = (\lambda-1)^m \gamma_m^{n-1}.$$

This completes the proof. □

Theorem 3.7. *For $n \geq 2$, the chromatic polynomial of $B^{uv}(m,n)$ is given by*

$$\begin{aligned}\chi_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)\gamma_{m-1}\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^m(\lambda-1)^2\gamma_m^{n-1} \\ &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^m(\lambda-1)^2\gamma_m \frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1} - \gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1} - \gamma_m}.\end{aligned}$$

Proof. Recall that $B^{uv}(m,n)$ is the signed book graph with signature $\{uv\}$. Consider the edge $e_1 = uu_1^n$. Using edge deletion-contraction formula on e_1, we have

$$\chi_{B^{uv}(m,n)}(\lambda) = \chi_{\widetilde{B^{uv}}(m,n-1)}(\lambda) - \chi_{\widetilde{\widetilde{B^{uv}}}(m,n)}(\lambda),$$

where $\widetilde{B^{uv}}(m,n-1)$ denotes the graph obtained from $B^{uv}(m,n-1)$ by attaching a path P_{m-1} at the vertex v and $\widetilde{\widetilde{B^{uv}}}(m,n)$ denotes the graph which is almost same as $B^{uv}(m,n)$ but one of its pages is a cycle of length $m-1$.

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Using Lemma 3.3, the chromatic polynomial of the signed graph $\widetilde{B^{uv}}(m,n-1)$ can be expressed in terms of $\chi_{B^{uv}(m,n-1)}(\lambda)$. For the graph $\widetilde{\widetilde{B^{uv}}}(m,n)$, we apply the edge deletion-contraction formula again on the edge $e_2 = uu_2^n$.

Applying edge deletion-contraction formula repeatedly and using Lemma 3.3, we obtain the chromatic polynomial of $B^{uv}(m,n)$ as

$$\begin{aligned}\chi_{B^{uv}(m,n)}(\lambda) =& (\lambda-1)^{m-2}\chi_{B^{uv}(m,n-1)}(\lambda) - (\lambda-1)^{m-3}\chi_{B^{uv}(m,n-1)}(\lambda) + \cdots\cdots \\ &+ (-1)^{m-3}(\lambda-1)^{(m-2)-(m-3)}\chi_{B^{uv}(m,n-1)}(\lambda) \\ &+ (-1)^{m-2}(\lambda-1)^{(m-2)-(m-2)}\chi_{B_m^{n-1}}(\lambda) \\ =& \Big[\sum_{i=0}^{m-3}(-1)^i(\lambda-1)^{(m-2)-i}\Big]\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^m\chi_{B_m^{n-1}}(\lambda).\end{aligned}$$

Note that, in the last step of edge deletion-contraction formula, the resulting graph is nothing but the signed graph B_m^{n-1}. We know by Proposition 3.1 that $\chi_{B_m^{n-1}}(\lambda) = (\lambda-1)^2\gamma_m^{n-1}$. Thus we have

$$\chi_{B^{uv}(m,n)}(\lambda) = (\lambda-1)\gamma_{m-1}\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^m(\lambda-1)^2\gamma_m^{n-1}.$$

After $n-2$ recursions, we have

$$\begin{aligned}\chi_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^m\frac{\gamma_m^{n+1}}{\gamma_{m-1}^2}\sum_{i=2}^{n}\left(\frac{(\lambda-1)\gamma_{m-1}}{\gamma_m}\right)^i \\ &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^m(\lambda-1)^2\gamma_m\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}-\gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1}-\gamma_m}.\end{aligned}$$

This completes the proof. □

We now give a formula for the chromatic polynomial of $B_l(m,n)$ for $2 \leq l \leq n-1$.

Theorem 3.8. *Let $n \geq 3$ and $2 \leq l \leq n-1$. The chromatic polynomial of $B_l(m,n)$ is given by*

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B^{uv}(m,l)}(\lambda).$$

Proof. Repeated use of edge deletion-contraction formula and Lemma 3.3 give that

$$\chi_{B_l(m,n)}(\lambda) = \frac{(\lambda-1)^{m-1}+(-1)^m}{\lambda}\chi_{B_l(m,n-1)}(\lambda) = \gamma_m\chi_{B_l(m,n-1)}(\lambda). \tag{7}$$

Applying the recursion of Equation (7) repeatedly, we have

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B_l(m,l)}(\lambda).$$

Note that $B_l(m,l)$ is switching equivalent to $B^{uv}(m,l)$. Therefore we have

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B^{uv}(m,l)}(\lambda).$$

This completes the proof. □

4 Zero-free chromatic polynomials of signed book graphs

In [5], the author explained that the chromatic polynomial and the zero-free chromatic polynomial of Σ are different unless Σ is balanced. We reformulate Lemma 3.3 in terms of the zero-free chromatic polynomial.

Lemma 4.1. *Let Σ be a signed graph. Then*

$$\chi^b_{\Sigma^{t+1}}(\lambda) = (\lambda-1)^t \chi^b_{\Sigma}(\lambda).$$

The zero-free chromatic polynomial of an unbalanced cycle of length two can be easily obtained as

$$\chi^b_{C_2^-}(\lambda) = \lambda^2 - 2\lambda = \lambda\gamma_3.$$

Indeed, we have the following.

Theorem 4.2. *For each $n \geq 2$, the zero-free chromatic polynomial of C_n^- is given by*

$$\chi^b_{C_n^-}(\lambda) = (\lambda-1)^n - (-1)^n = \lambda\gamma_{n+1}.$$

Proof. The proof follows by induction on n. □

Now we present analogue of the results of Section 3 in the context of the zero-free chromatic polynomials. The proofs are similar to the corresponding proofs presented for chromatic polynomials.

Lemma 4.3. *For $m \geq 2$, the zero-free chromatic polynomial of B_m^1 is given by*

$$\chi^b_{B_m^1}(\lambda) = \lambda(\lambda-2)\gamma_m.$$

Proposition 4.1. *For $n \geq 2$, the zero-free chromatic polynomial of B_m^n is given by*

$$\chi^b_{B_m^n}(\lambda) = \lambda(\lambda-2)\gamma_m^n.$$

From Lemma 1.2 and Theorem 3.5, it follows that

$$\chi^b_{B_0(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

Theorem 4.4. *For $n \geq 1$, the zero-free chromatic polynomial of $B_1(m,n)$ is given by*

$$\chi^b_{B_1(m,n)}(\lambda) = \lambda\gamma_m^{n-1}\gamma_{m+1}.$$

Theorem 4.5. *For $n \geq 2$, the zero-free chromatic polynomial of $B^{uv}(m,n)$ is given by*

$$\begin{aligned}\chi^b_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)\gamma_{m-1}\chi^b_{B^{uv}(m,n-1)}(\lambda) + (-1)^m\lambda(\lambda-2)\gamma_m^{n-1} \\ &= \lambda(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}\gamma_{m+1} + (-1)^m\lambda(\lambda-2)\gamma_m\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1} - \gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1} - \gamma_m}.\end{aligned}$$

Theorem 4.6. *For $n \geq 2$ and $2 \leq l \leq n-1$, the zero-free chromatic polynomial of $B_l(m,n)$ is given by*

$$\chi^b_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi^b_{B^{uv}(m,l)}(\lambda).$$

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5 Conclusion and future directions

In this paper, we have determined explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

In ordinary graph theory, it is well known that the chromatic polynomial of union $G \cup H$ of two graphs G and H, where $G \cap H$ is a complete graph, is given by

$$\chi_{G \cup H}(\lambda) = \frac{\chi_G(\lambda) \cdot \chi_H(\lambda)}{\chi_{G \cap H}(\lambda)}.$$

In the context of signed graph theory, we pose the following problem.

Problem 5.1. Let Σ_1 and Σ_2 be two signed graphs such that $\Sigma_1 \cap \Sigma_2$ is a signed complete graph and that $\chi_{\Sigma_1}(\lambda), \chi_{\Sigma_2}(\lambda)$ and $\chi_{\Sigma_1 \cap \Sigma_2}(\lambda)$ be known. What is the formula for $\chi_{\Sigma_1 \cup \Sigma_2}(\lambda)$?

We pose the same problem for the zero-free chromatic polynomial of the union of two signed graphs.

Problem 5.2. Let Σ_1 and Σ_2 be two signed graphs such that $\Sigma_1 \cap \Sigma_2$ is a signed complete graph and that $\chi^b_{\Sigma_1}(\lambda), \chi^b_{\Sigma_2}(\lambda)$ and $\chi^b_{\Sigma_1 \cap \Sigma_2}(\lambda)$ be known. What is the formula for $\chi^b_{\Sigma_1 \cup \Sigma_2}(\lambda)$?

In Problem 5.1 and Problem 5.2, it is implicitly assumed that the signatures of Σ_1 and Σ_2 agree on their intersection so that $\Sigma_1 \cup \Sigma_2$ is well defined. Thus, if the answers of Problem 5.1 and Problem 5.2 are known, then both kind of chromatic polynomials of $(B(m,n), \sigma)$ can be obtained easily, as $(B(m,n), \sigma)$ is the union of a signed $B(m, n-1)$ and a signed m-cycle, where their intersection is a signed complete graph on two vertices.

References

[1] M. Beck, E. Meza, B. Nevarez, A. Shine, and M. Young. The chromatic polynomials of signed Petersen graphs. *Involve*, 8:825–831, 2015.

[2] F. Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2:143–146, 1954.

[3] E. Máčajová, A. Raspaud, and M. Škoviera. The chromatic number of a signed graph. *The Electronic J. Comb.*, 23(1):P1.14, 2016.

[4] T. Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 4(1):47–74, 1982.

[5] T. Zaslavsky. Signed graph coloring. *Discrete Math.*, 39:215–228, 1982.