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Alpha Labeling of Amalgamated Cycles

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Abstract

A graceful labeling of a bipartite graph is an $\alpha$-labeling if it has the property that the labels assigned to the vertices of one stable set of the graph are smaller than the labels assigned to the vertices of the other stable set. A concatenation of cycles is a connected graph formed by a collection of cycles, where each cycle shares at most either two vertices or two edges with other cycles in the collection. In this work we investigate the existence of $\alpha$-labelings for this kind of graphs, exploring the concepts of vertex amalgamation to produce a family of Eulerian graphs, and edge amalgamation to generate a family of outerplanar graphs. In addition, we determine the number of graphs obtained with $k$ copies of the cycle $C_n$, for both types of amalgamations.

1 Introduction

Since the introduction of the graceful graph concept, different types of studies related to this idea have been published, some of these works present structural properties of this kind of graph, or focus on an enumerative aspect, although the two most common topics correspond to new families of graceful graphs and new alternatives to combine existing graceful graphs to create new varieties of graceful graphs. In this work we study two methods where certain even cycles are combined to generate Eulerian and outerplanar graphs. In this work we determine the number of vertex amalgamation to produce a family of Eulerian graphs, and edge amalgamation to generate a family of outerplanar graphs. In addition, we determine the number of graphs obtained with $k$ copies of the cycle $C_n$, for both types of amalgamations.

A difference vertex labeling of a graph $G$ of size $n$ is an injective mapping $f$ from $V(G)$ into a set $N$ of nonnegative integers, such that every edge $uv$ of $G$ has associated a weight defined by $|f(u) - f(v)|$. The labeling $f$ is called graceful when $N = \{0, 1, \ldots, n\}$ and the set of induced weights is $\{1, 2, \ldots, n\}$. When a graph admits such a labeling it is called graceful. Let $G$ be a bipartite graph and $\{A, B\}$ be the natural bipartition of $V(G)$, we say that $A$ and $B$ are the stable sets of $V(G)$ and assume that $|A| = a$ and $|B| = b$. A bipartite labeling of $G$ is an injection $f : V(G) \to \{0, 1, \ldots, s\}$ for which there is an integer $\lambda$, named the boundary value of $f$, such that $f(u) \leq \lambda < f(v)$ for every $(u, v) \in A \times B$, that induces $n$ different weights. This is an extension of the original definition given by Rosa and Širáň [9]. From the definition we may conclude that $s \geq |E(G)|$; furthermore, the labels assigned by $f$ on the vertices of $A$ and $B$ are in the sets $\{0, 1, \ldots, \lambda\}$ and $\{\lambda + 1, \lambda + 2, \ldots, s\}$, respectively.

Through this entire work, we may refer to the elements of $A$ as the black vertices, while the elements of $B$ are the white vertices. If $s = n$, the function $f$ is an $\alpha$-labeling and $G$ is an $\alpha$-graph. If $f$ is an $\alpha$-labeling of a tree and $f^{-1}(0) \in A$, then its boundary value is $\lambda = a - 1$.

Suppose that $f : V(G) \to \{0, 1, \ldots, n\}$ is a graceful labeling of a graph $G$ of size $n$:

- $\overline{f} : V(G) \to \{0, 1, \ldots, n\}$, defined for every $v \in V(G)$ as $\overline{f}(v) = n - f(v)$, is the complementary labeling of $f$.

- $g : V(G) \to \{c, c+1, \ldots, c+n\}$, defined for every $v \in V(G)$ and $c \in \mathbb{N}$ as $g(v) = c + f(v)$, is the shifting of $f$ in $c$ units.

Note that both, $\overline{f}$ and $g$ preserve the weights induced by $f$. Suppose now that $f$ is an $\alpha$-labeling of $G$ with boundary value $\lambda$. 

• $f_r : V(G) \to \{0, 1, \ldots, n\}$, defined for every $v \in V(G)$ as $f_r(v) = \lambda - f(v)$ if $f(v) \leq \lambda$, and $f_r(v) = n + \lambda + 1 - f(v)$ if $f(v) > \lambda$, is the reverse labeling of $f$. This function is also an $\alpha$-labeling with boundary value $\lambda$.

• $g : V(G) \to \mathbb{N}$, defined for every $v \in V(G)$ and any positive integer $d$ as $g(v) = f(v)$ if $f(v) \leq \lambda$ and $g(v) = f(v) + d - 1$ if $f(v) > \lambda$, is the $d$-graceful labeling of $G$ obtained from $f$. The labels assigned by $g$ on the stable sets of $V(G)$ are in the intervals $[0, \lambda]$ and $[\lambda + d, n + d - 1]$ and the set of induced weights is $\{d, d + 1, \ldots, n + d - 1\}$.

Therefore, if $f$ is an $\alpha$-labeling with boundary value $\lambda$ of a graph $G$ of size $n$, for each $\omega \in \{1, 2, \ldots, n\}$, there exists $uw \in E(G)$, where $u \in A$ and $v \in B$, such that $f(v) - f(u) = \omega$. Moreover, the weight of $uv$ under the complementary labeling $\overline{f}$ is exactly the same, but $u$ and $v$ change their colors. Since

$$f_r(v) - f_r(u) = n + \lambda + 1 - f(v) - (\lambda - f(u))$$

$$= n + \lambda + 1 - f(v) - \lambda + f(u)$$

$$= n + 1 - (f(v) - f(u))$$

$$= n + 1 - \omega,$$

this implies, for example, that the edges of weight 1 and $n$ under $f$, have weights $n$ and 1 under $f_r$, but $u$ and $v$ have the same color under both labelings. Consequently, the edge $uv$ of weight $\omega$ under $f$ has weight $n + 1 - \omega$ under $\overline{f}_r$ and both $u$ and $v$ have different colors under these two labelings. In Figure 1 we show these properties by exhibiting the labelings $f$, $\overline{f}$, $f_r$, and $\overline{f}_r$ for a tree of size 9, highlighting the stable sets and the edges with the extreme weights.

![Figure 1: Four related $\alpha$-labelings of the same graph](https://digitalcommons.georgiasouthern.edu/tag/vol9/iss2/11)

We must observe that depending on the structure of the graph and the specific characteristics of the labeling, it may occur that $f = f_r$ or $f = \overline{f}_r$. This is the case of the $\alpha$-labeling of the path $P_n$ given by Rosa in [7], where $f = f_r$ when $n$ is odd and $f = \overline{f}_r$ when $n$ is even; e.g., for $P_9$ consider $f = (0, 8, 1, 7, 2, 6, 3, 5, 4)$. In [8], Rosa presented the following $\alpha$-labeling of the same graph: $g = (1, 6, 2, 8, 0, 7, 4, 5, 3)$. Since the edges of weights 1 and $n = 8$ are in different positions within the path, neither $g_r$ nor $\overline{g}_r$ is the same as $g$.

A $d$-graceful labeling of a graph $G$ on size $n$ is an injection $g : V(G) \to \{0, 1, \ldots, n + d - 1\}$ such that the set of induced weights is $\{d, d + 1, \ldots, n + d - 1\}$. This definition was introduced
in 1980 by Maheo and Thuillier [5] and Slater [10]. There is a method that transforms an α-labeling \( f \) of \( G \) into a \( d \)-graceful labeling for each integer \( d > 1 \). Assuming that \( f \) has boundary value \( \lambda \), the function \( g \) is defined every \( v \in V(G) \) as \( g(v) = f(v) \) if \( f(v) \leq \lambda \) and \( g(v) = f(v) + d - 1 \) if \( f(v) > \lambda \).

This work is organized in the following form. In Section 2 we study the use of vertex amalgamations of \( \alpha \)-cycles to produce a family of Eulerian graphs that admit an \( \alpha \)-labeling. Edge amalgamations of \( \alpha \)-cycles are used in Section 3 to generate a family of outerplanar graphs that can be \( \alpha \)-labeled as well. We close this work in Section 4, where we determine the number of members in both families, when \( k \) copies of the same cycle are used in the corresponding amalgamations.

The graphs considered in this work are simple, that is, finite with no loops nor multiple edges. All terms not defined in this work are taken from [3] and/or [4].

## 2 Vertex Amalgamation and Eulerian Graphs

Several of the best known constructions of graceful graphs use vertex amalgamations of graphs with special characteristics. In this case, we use \( \alpha \)-labeled graphs to perform the amalgamations. For \( i = 1, 2 \), let \( G_i \) be a graph of order \( m_i \) and size \( n_i \). A graph \( G \) of order \( m_1 + m_2 - 1 \) and size \( n_1 + n_2 \) is said to be a vertex amalgamation of \( G_1 \) and \( G_2 \) if \( E(G) = E(G_1) \cup E(G_2) \) and a vertex of \( G_1 \) is merged with a vertex of \( G_2 \). The following result is used to amalgamate \( \alpha \)-labeled graphs, where the merged vertices of \( G_1 \) and \( G_2 \) are those labeled \( \lambda \) and 0 or \( \lambda + 1 \) and \( n \), respectively. We can trace its origins to the work of Stanton and Zarnke [11]; we include its proof here for the sake of completeness.

**Theorem 2.1.** Suppose that for \( i \in \{1, 2\} \), \( f_i \) is an \( \alpha \)-labeling with boundary value \( \lambda_i \) of a graph \( G_i \) of size \( n_i \). If the vertex of \( G_1 \) labeled \( \lambda_1 \) (resp. \( \lambda_1 + 1 \)) is amalgamated with the vertex of \( G_2 \) labeled 0 (resp. \( n_2 \)), then the graph \( G \) that results of this amalgamation is an \( \alpha \)-graph.

**Proof.** Since \( G \) is built identifying a vertex of \( G_1 \) with a vertex of \( G_2 \) and the edges are not touched in any way, the graph \( G \) has size \( n_1 + n_2 \).

We start transforming \( f_1 \) into a \((n_2 + 1)\)-graceful labeling, adding the constant \( n_2 \) to every label greater than \( \lambda_1 \). Then, the labels used on \( G_1 \) are in the set \( \{0, 1, \ldots, \lambda_1\} \cup \{\lambda_1 + 1 + n_2, \lambda_1 + 2 + n_2, \ldots, n_1 + n_2\} \) and the induced weights are \( 1 + n_2, 2 + n_2, \ldots, n_1 + n_2 \). Since the new labeling of \( G_1 \) is the result of a partial shifting of the labels assigned by \( f_1 \), it is also an injective function.

Suppose first that the vertices, originally labeled, \( \lambda_1 \) and 0 are selected to be amalgamated. The labels assigned by \( f_2 \) to the vertices of \( G_2 \) are shifted \( \lambda_1 \); so, the new labeling of \( G_2 \) is injective as well and assigns labels from \( \{\lambda_1, \lambda_1 + 1, \ldots, \lambda_1 + n_2\} \) to induce the weights \( 1, 2, \ldots, n_2 \). The vertex of \( G_2 \) originally labeled 0 is now labeled \( \lambda_1 \). Therefore, if this vertex is amalgamated with the vertex of \( G_1 \) labeled \( \lambda_1 \), we obtain the graph \( G \) with a labeling that assigns labels from \( \{0, 1, \ldots, n_1 + n_2\} \) to induce the weights \( 1, 2, \ldots, n_1 + n_2 \). Considering the fact that the stable set of \( G \) that has the vertex labeled \( \lambda_1 \) also contains all the vertices with labels in \( \{0, 1, \ldots, \lambda_1 + \lambda_2\} \), and \( \lambda_1 + \lambda_2 \) is smaller than the smallest label in the other
stable set, we conclude that the final labeling of $G$ is, indeed, an $\alpha$-labeling which boundary value is $\lambda_1 + \lambda_2$.

Suppose now that the vertices, originally labeled, $\lambda_1 + 1$ and $n_2$ are used in the amalgamation. In this case the labels assigned by $f_2$, to the vertices of $G_2$, are shifted $\lambda_1 + 1$ units. As in the previous case, the new labeling of $G_2$ is injective and assigns labels from $\{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + 1 + n_2\}$ to induce the weights $1, 2, \ldots, n_2$. The vertex of $G_2$ originally labeled $n_2$ is now labeled $\lambda_1 + 1 + n_2$. Thus, when the vertex of $G_1$ labeled $\lambda_1 + 1 + n_2$ is amalgamated with it, an $\alpha$-graph $G$ is obtained; its $\alpha$-labeling has boundary value $\lambda_1 + \lambda_2 + 1$. \qed

If either $f_1$ or $f_2$ is replaced by its complementary labeling, its reverse, or the complementary of its reverse, several graphs can be constructed with $G_1$ and $G_2$ via vertex amalgamation. In the next theorem we explore a family of graphs that can be obtained using vertex amalgamations of some $\alpha$-cycles.

A $kC_n$-snake is a connected graph in which the $k \geq 2$ blocks are isomorphic to the cycle $C_n$ and the block-cutpoint graph is a path. In other terms, a $kC_n$-snake is built with $k$ copies of the cycle $C_n$ in such a way that for each $i < k$, a vertex of the $i$-th copy is amalgamated with a vertex of the $(i + 1)$th copy, the degree of every vertex is either 2 or 4, and every copy of $C_n$ has exactly two vertices of degree 4 except the first and the last copies, which only have one vertex of degree 4. Thus, the vertices of degree 4 are the cut-vertices of the snake. Suppose that for each $i \in \{2, 3, \ldots, k - 1\}$, where $k \geq 3$, $u_i$ and $v_i$ are the vertices of degree 4 in the $i$th copy of $C_n$, and $d_i = \text{dist}(u_i, v_i)$. Then, the $kC_n$-snake is associated with the string $d_2, d_3, \ldots, d_{k-1}$, where $d_i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$. In Figure 2 we show an example of the $6C_8$-snake associated to the string $3, 2, 4, 1$.

Cyclic snakes were introduced by Rosa [8]; a triangular cactus is a connected graph where all the blocks are triangles and the block-cutpoint is a tree. A triangular snake (or $kC_3$-snake) is a triangular cactus whose block-cutpoint graph is a path. Rosa conjectured that all triangular cacti with $k$ blocks are graceful when $k \equiv 0, 1 \pmod{4}$. Moulton [6] proved this conjecture for the case of all $kC_3$-snakes. Barrientos [1] proved that all $kC_4$-snakes admit an $\alpha$-labeling. In the next theorem we prove that all $kC_n$-snakes are $\alpha$-graphs when $n = 8, 12, 16$.

Suppose that $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle $C_n$. For $n = 4, 8, 12, 16$, the following $\alpha$-labelings of $C_n$ allow us to prove the existence of an $\alpha$-labeling for any $kC_n$-snake.

For $C_4$ : $g_1 = (0, 4, 1, 2)$

For $C_8$ : $g_1 = (0, 8, 1, 7, 3, 6, 4, 5)$
$g_2 = (0, 8, 3, 6, 4, 5, 1, 7)$

For $C_{12}$ : $g_1 = (0, 12, 1, 11, 2, 10, 3, 8, 4, 7, 5, 6)$
$g_2 = (0, 12, 2, 10, 1, 8, 4, 9, 6, 7, 5, 11)$
$g_3 = (0, 12, 4, 8, 2, 9, 6, 7, 5, 10, 1, 11)$
For $C_{16}$:  
\[g_1 = (0, 16, 1, 15, 2, 14, 3, 13, 4, 11, 5, 10, 6, 9, 7, 8)\]
\[g_2 = (0, 16, 2, 11, 3, 14, 1, 13, 6, 10, 4, 9, 7, 8, 5, 15)\]
\[g_3 = (0, 16, 4, 11, 6, 10, 2, 13, 3, 9, 7, 8, 5, 14, 1, 15)\]
\[g_4 = (0, 16, 2, 11, 4, 10, 6, 9, 7, 8, 3, 14, 1, 13, 5, 15)\]

Within the proof of the next result we use these labelings together with some of their complementary labelings. The $\alpha$-graph $G$ constructed in Theorem 2.1 is the result of the amalgamation of the vertices labeled $\lambda$ in $G_1$ and 0 in $G_2$, or $\lambda+1$ in $G_1$ and $n_2$ in $G_2$. In the following diagram we summarize the distances between this type of vertices for each of the labelings given above, the number within parenthesis is the distance using the corresponding complementary labeling.

<table>
<thead>
<tr>
<th></th>
<th>$d_i$</th>
<th>$d_i$</th>
<th>$d_i$</th>
<th>$d_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 and $\lambda$</td>
<td>0 and $\lambda + 1$</td>
<td>$n$ and $\lambda$</td>
<td>$n$ and $\lambda + 1$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$g_1$ (or $\overline{g_1}$)</td>
<td>2 (2)</td>
<td>1 (1)</td>
<td>1 (1)</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$g_1$ (or $\overline{g_1}$)</td>
<td>2 (2)</td>
<td>1 (3)</td>
<td>3 (1)</td>
</tr>
<tr>
<td></td>
<td>$g_2$ (or $\overline{g_2}$)</td>
<td>4 (4)</td>
<td>3 (3)</td>
<td>3 (3)</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>$g_1$ (or $\overline{g_1}$)</td>
<td>2 (2)</td>
<td>1 (3)</td>
<td>3 (1)</td>
</tr>
<tr>
<td></td>
<td>$g_2$ (or $\overline{g_2}$)</td>
<td>4 (4)</td>
<td>3 (5)</td>
<td>5 (3)</td>
</tr>
<tr>
<td></td>
<td>$g_3$ (or $\overline{g_3}$)</td>
<td>6 (6)</td>
<td>5 (5)</td>
<td>5 (5)</td>
</tr>
<tr>
<td>$C_{16}$</td>
<td>$g_1$ (or $\overline{g_1}$)</td>
<td>2 (2)</td>
<td>1 (3)</td>
<td>3 (1)</td>
</tr>
<tr>
<td></td>
<td>$g_2$ (or $\overline{g_2}$)</td>
<td>4 (4)</td>
<td>3 (5)</td>
<td>5 (3)</td>
</tr>
<tr>
<td></td>
<td>$g_3$ (or $\overline{g_3}$)</td>
<td>6 (6)</td>
<td>5 (7)</td>
<td>7 (5)</td>
</tr>
<tr>
<td></td>
<td>$g_4$ (or $\overline{g_4}$)</td>
<td>8 (8)</td>
<td>7 (7)</td>
<td>7 (7)</td>
</tr>
</tbody>
</table>

**Theorem 2.2.** For $n = 8, 12, 16$ and $k \geq 2$, all $kC_n$-snakes are $\alpha$-graphs.

**Proof.** Let $G$ be a $kC_n$-snake, where $n$ is either 8, 12, or 16. Denote by $C^1, C^2, \ldots, C^k$ the consecutive copies of $C_n$ in $G$; thus, for each $2 \leq i \leq k - 1$, $C^i$ has two cut-vertices of $G$ and $D_i$ is the distance between these cut-vertices. Assume that $C^i$ has an initial labeling that corresponds to one of the $\alpha$-labelings given above, in particular, the labeling where the distance between the vertex labeled 0 (or $n$) and the vertex labeled $\lambda$ or $\lambda + 1$ is $d_i$.

In order to prove that $G$ is an $\alpha$-graph we proceed by induction on $k$. If $k = 2$, then $G$ is the one-point union of two $\alpha$-cycles; thus, by Theorem 2.1 we know that $G$ is indeed an $\alpha$-graph.

Let $G'$ be the subgraph of $G$ formed by $C^1, C^2, \ldots, C^{k-1}$. Suppose that $G'$ is an $\alpha$-graph where the labeling of each block is a $d$-graceful labeling obtained from one of the $\alpha$-labelings given above or their variations, i.e., $f$, $\overline{f}$, $f_r$, or $\overline{f_r}$. The $\alpha$-labeling of $G'$ has been obtained by using recursively Theorem 2.1, but this theorem uses any $\alpha$-labeling of the graph $G_2$. In the case of $G'$, the labeling of the copy $C^{k-1}$ is chosen in such a way that its cut-vertices have the appropriate labels, in particular, if $v$ is the cut-vertex of $G$ that is not a cut-vertex of $G'$, then its label is $\lambda$ or $\lambda + 1$. Either way, to obtain an $\alpha$-labeling of $G$ we apply Theorem 2.1 with $G_1 = G'$ and $G_2 = C^k$. Therefore, $G$ is an $\alpha$-graph. \qed
In Figure 2 we show an example of this method for a $6C_8$-snake, where the distances between consecutive cut vertices are $d_2 = 3$, $d_3 = 2$, $d_4 = 4$ and $d_5 = 1$. In this cyclic snake, the first copy of $C_8$ is the one containing the vertex labeled 0.

![Figure 2: An $\alpha$-labeling of a $6C_8$-snake](image)

A wider range of $\alpha$-labeled cyclic snakes can be built employing the labelings given above. A *blended cyclic snake* is any cyclic snake where not all the blocks are isomorphic. The fact that any blended cyclic snake, formed with the cycles $C_4$, $C_8$, $C_{12}$, and $C_{16}$, is an $\alpha$-graph can be proved by induction as we did with Theorem 2.2. In Figure 3 we show an $\alpha$-labeled blended cyclic snake described by the sequence $C_{16}, C_4, C_8, C_{12}, C_4, C_8$ with associated string $d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 1$.

**Theorem 2.3.** Let $G$ be a blended cyclic snake composed of cycles $C^1, C^2, \ldots, C^k$, where each $C^i$ is one of $C_4, C_8, C_{12}, C_{16}$. An $\alpha$-labeling of $G$ is obtained amalgamating the vertex labeled $\lambda$ in $C^i$ with the vertex labeled 0 in $C^{i+1}$.

### 3 Edge Amalgamation and Outerplanar Graphs

Let $G_1$ and $G_2$ be two graphs of positive size. The graph $G$ obtained by identifying an edge of $G_1$ with an edge of $G_2$ is called an *edge amalgamation* of $G_1$ and $G_2$. The order of $G$ is $|V(G_1)| + |V(G_2)| - 2$ and its size is $|E(G_1)| + |E(G_2)| - 1$. 

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In [2], Barrientos and Minion proved that when $G_1$ and $G_2$ are $\alpha$-graphs, the edge amalgamation of them, obtained identifying the edge of weight 1 in $G_1$ with the edge of maximum weight in $G_2$, is an $\alpha$-graph. For the sake of completeness, we present its proof again. This result is used later to construct three families of $\alpha$-labeled outerplanar graphs. Given the simplicity of the argument used to in its proof and the similarity with Theorem 2.1 we omit its proof, that can be found in [2].

**Theorem 3.1.** If $G_1$ and $G_2$ are two $\alpha$-graphs, then there is an edge amalgamation $G$ of $G_1$ and $G_2$ that is an $\alpha$-graph.

Two important properties of $\alpha$-graphs and $\alpha$-labelings, that we use in the rest of this section, are:

- Suppose that $G$ is a graph of size $n > 1$ and $f$ is an $\alpha$-labeling of $G$ which boundary value is $\lambda$. Assuming that the vertices of $G$ have been labeled by $f$, then the edge of weight $n$ has end-vertices labeled 0 and $n$; the edge of weight 1 has end-vertices labeled $\lambda$ and $\lambda + 1$. If these two edges are incident, then either $0 = \lambda$ or $n = \lambda + 1$. Both cases imply that $G$ is the star $S_n = K_{1,n}$. Thus, if $G$ is an $\alpha$-graph, other than the star, the extreme weights, i.e., 1 and $n$, are induced on two non-incident edges, regardless of the $\alpha$-labeling that induced these extreme weights.

- If $f$ is an $\alpha$-labeling $f$ of a graph, then there are other three $\alpha$-labelings that can be easily obtained: $\overline{f}$, $f_r$, and $f_r$. This implies that if $e = uv$ is the edge of $G$ which weight under $f$ is $\omega$, with $u \in A$ and $v \in B$, then under $\overline{f}$ its weight is still the same but $u \in B$ and $v \in A$; under $f_r$ its weight is $n + 1 - \omega$ with $u \in A$ and $v \in B$, consequently, under $\overline{f_r}$ its weight is also $n + 1 - \omega$ but $u \in B$ and $v \in A$. Hence, if one of these four labelings is known, the remaining three are automatically known.
Let $G$ be an $\alpha$-graph of size $n$ and $e_1, e_2$ be any pair of non-incident edges of $G$. Suppose that for each $i \in \{1, 2, \ldots, t\}$, $f_i$ is an $\alpha$-labeling of $G$. We say that $L = \{f_1, f_2, \ldots, f_t\}$ is a complete set of $\alpha$-labelings of $G$ if there exists a unique $i \in \{1, 2, \ldots, t\}$ such that $f_i$ induces the weights 1 and $n$ on the edges $e_1$ and $e_2$. In order to determine the cardinality of $L$ we must take under consideration the size of the graph, automorphisms, and the fact that $e_1$ and $e_2$ are non-incident; in Figure 4 we show two examples of this type of set, for two unicyclic graphs of size 7; if we analyze the first graph on the top row, its group of automorphisms has order 12, given by the permutations of the vertices labeled 3, 4, and 7, and the permutations of the vertices labeled 2 and 6; since the vertices labeled 3, 4, and 7 are equivalent as well as the vertices 2 and 6, there are only two essentially different sets of non-incident edges, which are represented with blue and red lines.

Suppose that $G_1, G_2, \ldots, G_k$ are copies of a graph $G$ of order $m$ and size $n$; for each $i \in \{1, 2, \ldots, k\}$, let $e^i_1$ and $e^i_2$ be two non-incident edges of $G_i$. The family $\mathcal{G}_k$ is formed for all those graphs of order $k(m-2) + 2$ and size $k(n-1) + 1$ built using edge amalgamation of $G_1, G_2, \ldots, G_k$ in such a way that for each $i \in \{2, 3, \ldots, k-1\}$, the edge $e^i_1$ is amalgamated with $e^{i-1}_2$ and $e^i_2$ is amalgamated with $e^{i+1}_1$. Figure 5 exhibits an example of a member of $\mathcal{G}_5$ where $G \cong C_{12}$. In the next theorem we prove that when a graph $G$ has a complete set of $\alpha$-labelings, then any member of $\mathcal{G}_k$ is an $\alpha$-graph.

**Theorem 3.2.** Let $G$ be an $\alpha$-graph of order $m$ and size $n$. If $G$ has a complete set of $\alpha$-labelings, then any member of $\mathcal{G}_k$ admits an $\alpha$-labeling for any positive integer $k$.

**Proof.** Let $H \in \mathcal{G}_k$ and $G_1, G_2, \ldots, G_k$ be the copies of $G$ used to build $H$. The colors of the stable sets of $H$ are extended to the stable sets of each $G_i$. For each $i \in \{2, 3, \ldots, k-1\}$, let $e^i_1$ and $e^i_2$ be the non-incident edges of $G_i$ used to amalgamate $G_i$ to $G_{i-1}$ and $G_{i+1}$,
respectively. Since $G$ has a complete set of $\alpha$-labelings, we know that there exists an $\alpha$-labeling of $G_i$, denoted by $f_i$, such that the weights induced by $f_i$ on $e_1^i$ and $e_2^i$ are $n$ and 1, respectively. Let $\lambda_i$ be the boundary value of $f_i$. The $\alpha$-labelings of $G_1$ and $G_k$, are chosen in such a way that $e_1^1$ has weight 1 and $e_2^k$ has weight $n$. All these labelings are selected in such a way that they are also consistent with the colors of the stable sets of $H$.

Now that the $\alpha$-labeling of each copy of $G$ has been identified, we modify them to produce the final $\alpha$-labeling of the graph $H$. The labeling $f_i$ of $G_i$ is transformed into a $d_i$-graceful labeling, where $d_i = (n-1)(k-i)+1$. In this way, the weights on the edges of $G_i$ form the set $W_i = \{(n-1)(k-i)+1, (n-1)(k-i)+2, \ldots, (n-1)(k-i)+n\}$. Note that $\cup_{i=1}^k W_i = \{1, 2, \ldots, k(n-1)+1\}$, where $k(n-1)+1$ is the size of the graph $H$.

Moreover, since $\max(W_{i+1}) = (n-1)(k-(i+1)) + n = (n-1)(k-i) + 1$, we conclude that $\min(W_i) = \max(W_{i+1})$; in other terms, there is only one weight repeated between $W_i$ and $W_{i+1}$, that weight corresponds to the original weight 1 in $G_i$ and the original weight $n$ in $G_{i+1}$. The labels on the end-vertices of $e_1^i$ are 0 and $d_i+n-1$, the labels on the end-vertices of $e_2^i$ are $\lambda_i$ and $\lambda_i+d_i$.

In order to proceed with the edge amalgamation, we need to conveniently shift the labels of the copies of $G$; in general, the shifting of the labeling of $G_i$ is decided by the final labeling of $G_{i-1}$. The final labeling of $G_1$ is the $d_1$-graceful labeling obtained from $f_1$. Thus, the labels on the end-vertices of the edge $e_1^1$ are $\lambda_1$ and $\lambda_1+d_1$. For each $i \in \{2, 3, \ldots, k\}$, the final labeling of $G_i$ is a shifting in $c_i = \sum_{j=1}^{i-1} \lambda_j$ units of the $d_i$-graceful labeling obtained from $f_i$. In this way, the weights on $G_i$ remain the same and the labels on the end-vertices of $e_1^i$ are $c_i$ and $c_i+d_i+n-1$; the labels on the end-vertices of $e_2^i$ are $\lambda_i$ and $\lambda_i+d_i$.

Recall that the edges $e_2^i$ and $e_1^{i+1}$ will be amalgamated, which implies that the labels on the end-vertices must match. Indeed, since

$$c_{i+1} = \sum_{j=1}^{i} \lambda_j = \lambda_i + \sum_{j=1}^{i-1} \lambda_j = \lambda_i + c_i$$

and

$$c_i + d_i + n - 1 = \lambda_i + c_i + (n-1)(k-i-1) + 1 + n - 1 = \lambda_i + c_i + (n-1)(k-i) + 1 = c_i + \lambda_i + d_i,$$

we conclude that the labels on the end-vertices of these two edges actually match.

Once the edge $e_2^i$ has been amalgamated with the edge $e_1^{i+1}$ for all $i \in \{1, 2, \ldots, k-1\}$, we obtain the graph $H$, which has been $\alpha$-labeled.

The process of edge amalgamation presented within the proof of the last theorem can be extended even further. If each $G_i$ is a graph that has a complete set of $\alpha$-labelings, any graph obtained by edge amalgamation of $G_1, G_2, \ldots, G_k$ admits an $\alpha$-labeling provided that there exists an $\alpha$-labeling of $G_i$, on the edges $e_1^i$ and $e_2^i$ that connect $G_i$ with $G_{i-1}$ and $G_{i+1}$. We must observe that the proofs of these last two theorems can also be done by induction on $k$. 

\quad \square
Theorem 3.3. For $i \in \{1, 2, \ldots, k\}$, let $G_i$ be a graph of size $n_i$ and $e_1^i, e_2^i$ be any pair of non-incident edges of $G_i$. If $G_i$ has a complete set of $\alpha$-labelings, then an $\alpha$-graph is obtained when $e_1^i$ is amalgamated with $e_2^{i-1}$ for each $i \geq 2$.

Recall that $\mathcal{G}_k$ is the family of all graphs obtained via edge amalgamation of $k$ copies of a graph $G$, where the copy $G_i$ shares exactly one edge with $G_{i-1}$, one edge with $G_{i+1}$, and these two edges are non-incident. The labeling $f = (4, 0, 2, 1)$ of $G = C_4$ constitutes, by itself, a complete set of $\alpha$-labelings, but in this case, for a fixed value of $k$, $\mathcal{G}_k$ has only one member that is the ladder $L_{k+1} = P_{k+1} \times P_2$ that is a well-known family of $\alpha$-graphs. When $G = C_8$, there is no complete set of $\alpha$-labelings of $G$, we searched all the $\alpha$-labelings of this cycle and found that between the distinguished edges $e_1$ and $e_2$, there is always an odd number of edges, as in the following two examples: $f_1 = (8, 0, 5, 4, 6, 3, 7, 1)$ and $f_2 = (8, 0, 7, 1, 5, 4, 6, 3)$. This means that an $\alpha$-labeling (if exists) of the members of $\mathcal{G}_k$, with $G \cong C_8$ must be found using a different process, not by the introduction of a complete set of $\alpha$-labelings of $C_8$; however, we strongly believe that the members of this family are $\alpha$-graphs and that the existence of $\alpha$-labeling of any member of the family can be obtained via edge amalgamation of graphs that result of the edge amalgamation of two copies of $C_8$, for which a complete set of $\alpha$-labelings exists. In the next result we prove that when $G \cong C_{12}$ or $G \cong C_{16}$, any member of $\mathcal{G}_k$ is an $\alpha$-graph. The result of these edge amalgamations is an outerplanar graph where the maximum degree is $\Delta = 3$ and each induced cycle is isomorphic to $G$.

Theorem 3.4. If $G \cong C_{12}$ or $G \cong C_{16}$, then any graph $H$ in $\mathcal{G}_k$ is an $\alpha$-graph.

Proof. Based on Theorem 3.2, we just need to show that for both $C_{12}$ and $C_{16}$, there exists a complete set of $\alpha$-labelings.

For $C_{12}$, consider the following labelings, where the end-vertices of $e_1$ and $e_2$ are in red and blue, respectively:

\[
\begin{align*}
    f_1 &= (12, 0, 6, 5, 7, 4, 8, 3, 10, 2, 11, 1), \\
    f_2 &= (12, 0, 11, 5, 6, 4, 7, 3, 8, 1, 10, 2), \\
    f_3 &= (12, 0, 11, 1, 6, 5, 7, 4, 8, 2, 10, 3), \\
    f_4 &= (12, 0, 7, 3, 8, 5, 6, 4, 10, 2, 11, 1), \\
    f_5 &= (12, 0, 11, 3, 7, 4, 6, 5, 10, 1, 8, 2).
\end{align*}
\]

Thus, in the labeling $f_i$ the distance between $e_1$ and $e_2$ is exactly $i$. It is not complicated to check that all these are $\alpha$-labelings of $C_{12}$. Therefore, $\{f_1, f_2, f_3, f_4, f_5\}$ is a complete set of $\alpha$-labelings.

Similarly, for $C_{16}$, consider the following labelings, where the end-vertices of $e_1$ and $e_2$
are in red and blue, respectively:

\[ f_1 = (16, 0, 8, 7, 9, 6, 10, 5, 11, 4, 13, 3, 14, 2, 15, 1), \]
\[ f_2 = (16, 0, 10, 7, 8, 6, 13, 4, 9, 5, 11, 3, 14, 2, 15, 1), \]
\[ f_3 = (16, 0, 13, 7, 9, 8, 11, 6, 10, 2, 12, 5, 14, 3, 15, 1), \]
\[ f_4 = (16, 0, 13, 5, 11, 8, 9, 7, 12, 2, 14, 3, 10, 6, 15, 1), \]
\[ f_5 = (16, 0, 14, 6, 10, 7, 9, 8, 13, 3, 15, 2, 11, 5, 12, 1), \]
\[ f_6 = (16, 0, 15, 5, 11, 6, 9, 7, 8, 4, 13, 1, 14, 3, 10, 2), \]
\[ f_7 = (16, 0, 15, 4, 14, 1, 13, 5, 8, 7, 9, 3, 10, 6, 11, 2). \]

Thus, in the labeling \( f_i \) the distance between \( e_1 \) and \( e_2 \) is exactly \( i \). It is not complicated to check that all these are \( \alpha \)-labelings of \( C_{16} \). Therefore, \( \{f_1, f_2, f_3, f_4, f_5, f_6, f_7\} \) is a complete set of \( \alpha \)-labelings.

In Figure 5 we show an example for a graph \( H \) in \( G_5 \) where \( G \cong C_{12} \). Note that \( H \) is an outerplanar graph, where the chords are "parallel", that is, an outerplanar where any given vertex has degree 2 or 3 and every induced cycle, other than the outercycle, is isomorphic to \( C_{12} \).

![Figure 5: An \( \alpha \)-labeling of a member of \( G_5 \) where \( G \cong C_{12} \)](image-url)
4 Enumerating Concatenated Cycles

Let $G$ be any $kC_n$-snake. Recall that for each $i \in \{2, 3, \ldots, k-1\}$, where $k \geq 3$, $u_i$ and $v_i$ are the vertices of degree 4 in the $i$th copy of $C_n$, and $d_i = \text{dist}(u_i, v_i)$. Then, every $kC_n$-snake is associated with the string $d_2, d_3, \ldots, d_{k-1}$, where $d_i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We must note that the strings $d_2, d_3, \ldots, d_{k-1}$ and $d_{k-1}, d_3, d_2$ correspond to the same snake, they depend on which of the two extreme blocks of the snake is considered the first one. So, in order to determine the number $T(n, k)$ of nonisomorphic $kC_n$-snakes, we must find first the number of reversible strings, that is, the number of those strings of the form $d_2, d_3, \ldots, d_{k-1}$ such that $d_i = d_{k+1-i}$.

When $k$ is odd, a reversible string of length $k - 2$ has the form
\[
d_2, d_3, \ldots, d_{\frac{k-1}{2}}, d_{\frac{k+1}{2}}, d_{k-1}, \ldots, d_3, d_2.
\]
Hence, $\lfloor \frac{n}{2} \rfloor^{\frac{k-1}{2}}$ is the number of reversible strings when $k$ is odd and $\frac{1}{2}(\lfloor \frac{n}{2} \rfloor^{k-2} + \lfloor \frac{n}{2} \rfloor^{\frac{k-1}{2}})$ is the number of nonisomorphic $kC_n$-snakes.

Similarly, when $k$ is even, a reversible string of length $k - 2$ has the form
\[
d_2, d_3, \ldots, d_{\frac{k}{2}}, d_{\frac{k}{2}}, d_{k-1}, \ldots, d_3, d_2.
\]
Then, $\lfloor \frac{n}{2} \rfloor^{\frac{k-2}{2}}$ is the number of reversible strings when $k$ is even and $\frac{1}{2}(\lfloor \frac{n}{2} \rfloor^{k-2} + \lfloor \frac{n}{2} \rfloor^{\frac{k-2}{2}})$ is the number of nonisomorphic $kC_n$-snakes.

Thus, we have proven the following theorem.

**Theorem 4.1.** For $n \geq 3$ and $k \geq 2$, the number of nonisomorphic $kC_n$-snakes is
\[
T(n, k) = \frac{1}{2}(\lfloor \frac{n}{2} \rfloor^{k-2} + \lfloor \frac{n}{2} \rfloor^{\frac{k-1}{2}}).
\]

Note that for each odd value on $n \geq 5$, $T(n, k) = T(n - 1, k)$. In Table 1 we show the first values of $T(n, k)$. When the entries of this table are read by anti-diagonals, they form the sequence A308203 in OEIS. Several other sequences in OEIS can be found within the sequence formed by the values of $T(n, k)$; for instance, from $T(n, 4)$ until $T(n, 12)$ we get: A000217, A002411, A037270, A168178, A071232, A168194, A071231, A168372, A071236; and from the even values of $n \geq 4$ we get A005418, A032120, A032121, A032122, A056308.

A $k$-cell polygonal chain is an outerplanar graph whose vertices have either degree 2 or 3 and any of the $k$ polygons, established by the $k - 1$ chords, shares at most two edges (the chords) with other polygons. In particular, a $k$-cell $C_n$-chain is a connected graph formed with $k$ copies of the cycle $C_n$, denoted by $C^1, C^2, \ldots, C^k$, in such a way that for every $i \in \{2, 3, \ldots, k-1\}$, $C^i$ shares two non-incident edges, one with $C^{i-1}$ and the other one with $C^{i+1}$. Thus, if $G$ is a $k$-cell $C_n$-chain, then $|V(G)| = n + (k-1)(n-2) = k(n-2) + 2$ and $|E(G)| = kn - (k - 1) = k(n-1) + 1$.

An edge shared by two copies of $C_n$ is called link. We denote by $u_i$ and $v_i$ the endvertices of the link between $C^i$ and $C^{i+1}$. Since links are non-incident edges, for each $i \in \{2, 3, \ldots, k-1\}$, $C^i$ has $n - 3$ edges that can be selected to be the link with $C^{i+1}$. In order to characterize and count this type of polygonal chain, we use some strings of numbers that can be associated to them.
Table 1: Number on non isomorphic $kC_n$-snakes

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Suppose that $G$ is a $k$-cell $C_n$-chain. There is a $u_i$-$v_i$ path that only uses edges of the outer cycle (i.e., no links are used) and includes all vertices of degree 3. Within this path, the vertices of degree 3 appear in the sequence $u_1, u_2, \ldots, u_{k-2}, u_{k-1}, v_{k-1}, v_{k-2}, \ldots, v_2, v_1$. For each $i \in \{1,2,\ldots,k-2\}$, $D_i$ denotes the number of vertices of degree 2 between $u_i$ and $u_{i+1}$. Thus, $d_1, d_2, \ldots, d_{k-2}$ is a string of numbers where $d_i \in \{0,1,\ldots,n-4\}$. Clearly, every string of length $k-2$, whose entries are in $\{0,1,\ldots,n-4\}$, is associated with a unique $k$-cell $C_n$-chain. In the opposite direction, the situation is different, because for any given $k$-cell $C_n$-chain, the associated string depends on the selection of the first cell and the vertex $u_i$. Thus, the graph $G$ may be associated to four different strings, or maybe two or one, depending on its group of automorphisms.

Let $D = \{0,1,\ldots,n-4\}$ and $d = d_1, d_2, \ldots, d_{k-2}$ be a string of length $k-2$ where each $d_i \in D$. There are $(n-3)^{k-2}$ such strings. Suppose that $a = a_1, a_2, \ldots, a_{k-2}$ and $b = b_1, b_2, \ldots, b_{k-2}$ are two of these strings. We say that $a$ and $b$ are equivalent if, for each $i \in \{1,2,\ldots,k-2\}$, one of the following conditions holds:

1. $b_i = a_i$,  
2. $b_i = a_{k-1-i}$,  
3. $b_i = n - 4 - a_{k-1-i}$,  
4. $b_i = n - 4 - a_i$.

It is straightforward to see that this is an equivalence relation on the set of all strings of length $k-2$ with elements of $D$. In addition, there is a bijection between the set of all $k$-cell $C_n$-chains and the set of equivalence classes determined by this equivalence relation. Therefore, instead of counting non-isomorphic polygonal chains, we count equivalence classes.

**Theorem 4.2.** For every $n \geq 4$ and $k \geq 2$, the number $S(n,k)$ of non-isomorphic $k$-cell $C_n$-chains is:
\[ \frac{1}{4}((n - 3)^{k-2} + 2(n - 3)^{\frac{k-1}{2}} + 1) \] when \( n \) is even and \( k \) is even,

\[ \frac{1}{4}((n - 3)^{k-2} + (n - 3)^{\frac{k+1}{2}} + (n - 3)^{\frac{k-3}{2}} + 1) \] when \( n \) is even and \( k \) is odd,

\[ \frac{1}{4}((n - 3)^{k-2} + 2(n - 3)^{\frac{k+1}{2}}) \] when \( n \) is odd and \( k \) is even,

\[ \frac{1}{4}((n - 3)^{k-2} + (n - 3)^{\frac{k+1}{2}}) \] when \( n \) is odd and \( k \) is odd,

**Proof.** Let \( A \) be the set of all strings of length \( k-2 \) which entries are in \( D = \{0, 1, \ldots, n-4\} \). Thus, \( |A| = (n - 3)^{k-2} \). Assume that \( d = d_1, d_2, \ldots, d_{k-2} \) is one of these strings and \([d]\) is the equivalence class of \( d \) induced by the equivalence relation given above. Suppose that for each \( i \in \{1, 2, \ldots, k-2\} \), one of the following conditions holds:

(i) \( d_i = d_{k-1-i} \),

(ii) \( d_i = n - 4 - d_{k-1-i} \).

Then, \([d]\) = \( \{d, d^*\} \), where \( d^* = n - 4 - d_1, n - 4 - d_2, \ldots, n - 4 - d_{k-2} \) because \( d \) and \( d^* \) satisfy condition (3).

If conditions (i) and (ii) are satisfied simultaneously, then \([d]\) = \( d \), because \( d_i = n - 4 - d_i \), which is equivalent to say that \( d_i = \frac{n-4}{2} \). But this is only possible when \( n \) is even.

If none of (i) and (ii) is satisfied, then \([d]\) = \( \{d, d_r, d_c, d^c\} \), where \( d_r = d_{k-2}, d_{k-3}, \ldots, d_1 \) (condition (2)), \( d_c = n - 4 - d_{k-2}, n - 4 - d_{k-3}, \ldots, n - 4 - d_1 \) (condition (3)), and \( d^c = n - 4 - d_1, n - 4 - d_2, \ldots, n - 4 - d_{k-2} \) (condition (4)).

We define \( S_1 \) to be the subset of \( A \) containing all the strings that satisfy condition (i); similarly, \( S_2 \) contains those strings satisfying (ii), and \( S_3 \) consists of all the strings complying with (i) and (ii) simultaneously. Thus, every element of either \( S_1 \) or \( S_2 \) is also in \( A \), and every element in \( S_3 \) is in \( A \), \( S_1 \), and \( S_2 \). Hence,

\[ S(n, k) = \frac{1}{4}(|A| + |S_1| + |S_2| + |S_3|). \]

So, in order to determine the exact value of \( S(n, k) \), we need to calculate the cardinality of each of these sets.

If \( d = d_1, d_2, \ldots, d_{k-2} \) is in \( S_1 \), then for each \( 1 \leq i \leq \left\lfloor \frac{k-2}{2} \right\rfloor \), \( d_i = d_{k-1-i} \). Since \( d_i \in D \), there are \((n - 3)^{\left\lfloor \frac{k-2}{2} \right\rfloor}\) possibilities for \( d \). Hence, \(|S_1| = (n - 3)^{\frac{k-1}{2}}\) when \( k \) is even and \(|S_1| = (n - 3)^{\frac{k-1}{2}}\) when \( k \) is odd.

If \( d \in S_2 \), then for each \( 1 \leq i \leq \left\lfloor \frac{k-2}{2} \right\rfloor \), \( d_i = n - 4 - d_{k-1-i} \). This implies that when \( k \) is odd, the central entry of \( d \), that is, \( d_{k-1} \), must be self-complementary, i.e., \( d_{k-1} = \frac{n-4}{2} \), which on its own implies that \( n \) must be even. Thus, when \( n \) and \( k \) are odd, \(|S_2| = 0\); when \( n \) is even and \( k \) is odd, \(|S_2| = (n - 3)^{\frac{k-1}{2}}\). If \( k \) is even, \(|S_2| = (n - 3)^{\frac{k-1}{2}}\) regardless the parity of \( n \).

If \( d \in S_3 \), then \(|S_3| = 0\) when \( n \) is odd, because for each \( 1 \leq i \leq \left\lfloor \frac{k-2}{2} \right\rfloor \), \( d_i \neq n - 4 - d_{k-1-i} \), which implies that \( d \neq d_r \) and condition (i) is not satisfied. When \( n \) is even, the only number in \( D \) that is self-complementary is \( \frac{n-2}{2} \); so every entry of \( d \) must equal this value. In other terms, \(|S_3| = 1|\).

Analyzing, independently, the four possible cases we get:
Table 2: Number on non-isomorphic $k$-cell $C_n$-chains

- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{k-2} + (n-3)^{k-2} + 1)$ when both $n$ and $k$ are even.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{k-1} + (n-3)^{k-1} + 1)$ when $n$ is even and $k$ is odd.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{k-2} + (n-3)^{k-2} + 1)$ when $n$ is odd and $k$ is even.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{k-1})$ when both $n$ and $k$ are odd.

This concludes the proof. □

In Table 2 we show the first values of $S(n,k)$, for $4 \leq n \leq 20$ and $2 \leq k \leq 12$.

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## References


