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Combinatorics of Ramanujan-Slater Type Identities

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Abstract

We provide the missing member of a family of four q-series identities related to the modulus 36, the other members having been found by Ramanujan and Slater. We examine combinatorial implications of the identities in this family, and of some of the identities we considered in "Identities of the Ramanujan-Slater type related to the moduli 18 and 24," [*J. Math. Anal. Appl.* **344**/2 (2008) 765–777].

1. Introduction

The Rogers-Ramanujan identities,

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} = \prod_{\substack{k\geqq 1 \\ k\equiv\pm 1 (\mathrm{mod}\ 5)}} \frac{1}{1-q^k} \tag{1.1}$$

and

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j} = \prod_{\substack{k\geqq 1 \\ k\equiv\pm 2 (\mathrm{mod}\ 5)}} \frac{1}{1-q^k}, \tag{1.2}$$

where

$$(a;q)_j := \prod_{k=0}^{j-1} (1-aq^k)$$

were first proved by L.J. Rogers [13] in 1894 and later independently rediscovered (without proof) by S. Ramanujan [12, Vol II, p. 33]. Many additional "q-series = infinite product" identities were found by Ramanujan and recorded in his lost notebook [5], [6]. A large collection of such identities was produced by L.J. Slater [17].

Just as the Rogers-Ramanujan identities (1.1), (1.2) are a family of two similar identities where the infinite products are related to the modulus 5, most Rogers-Ramanujan type identities exist in a family of several similar identities where the sum sides are similar and the product sides involve some common modulus.

In most cases Ramanujan and Slater found all members a given family, but in a few cases they found just one or two members of a family of four or five identities. In [11], we found some "missing" members of families of identities related to the moduli 18 and 24 where Ramanujan and/or Slater had found one or two of the family members, as well as two new complete families.

In this paper, we find the missing member in a family of four identities related to the modulus 36. We examine combinatorial implications of the identities in this family, and of some of the identities we considered in [11].

2. Combinatorial Definitions

Informally, a partition of an integer n is a representation of n as a sum of positive integers where the order of the summands is considered irrelevant. Thus the five partitions of 4 are 4 itself, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. The summands are called the "parts" of the partition, and since the order of the parts is irrelevant, $2+1+1$, $1+2+1$, and $1+1+2$ are all considered to be the same partition of 4. It is often convenient to impose a canonical ordering for the parts and to separate parts with commas instead of plus signs, and so we make the following definitions:

A partition λ of an integer n into ℓ parts is an ℓ-tuple of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ where

$$\lambda_i \geqq \lambda_{i+1} \text{ for } 1 \leqq i \leqq \ell - 1, \text{ and}$$

$$\sum_{i=1}^{\ell} \lambda_i = n.$$

The number of parts $\ell = \ell(\lambda)$ of λ is also called the *length* of λ. The sum of the parts of λ is called the *weight* of λ and is denoted $|\lambda|$.

Thus in this notation, the five partitions of 4 are (4), $(3,1)$, $(2,1,1)$, and $(1,1,1,1)$.

In [4], G. Andrews considers some of the implications of generalizing the notion of partition to include the possibility of some negative integers as parts. We may formalize this idea with the following definitions:

A *signed partition* σ of an integer n is a partition pair (π, ν) where

$$n = |\pi| - |\nu|.$$

We may call π (resp. ν) the *positive (resp. negative) subpartition of* σ and $\pi_1, \pi_2, \ldots, \pi_{\ell(\pi)}$ (resp. $\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}$) the *positive (resp. negative) parts* of σ.

Thus $((6,3,3,1),(4,2,1,1))$, which represents $6+3+3+1-1-1-2-4$, is an example of a signed partition of 5. Of course, there are infinitely many unrestricted signed partitions of any integer, but when we place restrictions on how parts may appear, signed partitions arise naturally in the study of certain q-series.

Remark 2.1. Notice that the way we have defined signed partitions, the "negative parts" are positive numbers (which count negatively toward the weight of the signed partition), much as the "imaginary part" of a complex number is real.

3. Partitions and q-series Identities of Ramanujan and Slater

Using ideas that originated with Euler, MacMahon [12, vol. II, ch. III] and Schur [14] independently realized that (1.1) and (1.2) imply the following partition identities:

Theorem 3.2 (First Rogers-Ramanujan identity—combinatorial version)**.** *For all integers n, the number of partitions λ of n where*

$$\lambda_i - \lambda_{i+1} \geqq 2 \textit{ for } 1 \leqq i \leqq \ell(\lambda) - 1 \tag{3.3}$$

equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

Theorem 3.3 (Second Rogers-Ramanujan identity—combinatorial version)**.** *For all integers n, the number of partitions λ of n where*

$$\lambda_i - \lambda_{i+1} \geqq 2 \textit{ for } 1 \leqq i \leqq \ell(\lambda) - 1 \tag{3.4}$$

and

$$\lambda_{\ell(\lambda)} > 1, \tag{3.5}$$

equals the number of partitions of n into parts congruent to $\pm 2 \pmod 5$.

When studying sets of partitions where the appearance or exclusion of parts is governed by difference conditions such as (3.3), it is often useful to introduce a second parameter a. The exponent on a indicates the length of a partition being enumerated, while the exponent on q indicates the weight of the partition.

For example, it is standard to generalize (1.2) and (1.1) as follows:

$$F_1(a,q) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(q;q)_j} = \frac{1}{(aq;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j+3)/2} (a;q)_j (1-aq^{2j+1})}{(q;q)_j} \tag{3.6}$$

$$F_2(a,q) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(q;q)_j} = \frac{1}{(aq;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j-1)/2} (a;q)_j (1-aq^{2j})}{(1-a)(q;q)_j}, \tag{3.7}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k).$$

It is then easily seen that $F_1(a,q)$ and $F_2(a,q)$ satisfy the following system of q-difference equations:

$$F_1(a,q) = F_2(aq,q) \tag{3.8}$$

$$F_2(a,q) = F_1(a,q) + aqF_1(aq,q). \tag{3.9}$$

Notice that there are straightforward combinatorial interpretations to (3.8) and (3.9). Equation (3.8) states that if we start with the collection of partitions satisfying (3.3) and add 1 to each part (i.e. replace a by aq), then we obtain the set of partitions that satisfy (3.4) and (3.5); the difference condition is maintained, but the new partitions will have no ones. The left hand side of (3.9) generates partitions that satisfy (3.3) while the right hand side segregates these partitions into two classes: those where no ones appear (generated by $F_1(a,q)$) and those where a unique one appears (generated by $aqF_1(aq,q)$).

Remark 3.4. It may seem awkward to have the a-generalization of the first (resp. second) Rogers-Ramanujan identity labeled $F_2(a,q)$ (resp. $F_1(a,q)$), but this is actually standard practice (see, e.g. Andrews [2, Ch. 7]). Here and in certain generalizations, the subscript on F corresponds to one more than the maximum number of ones which can appear in the partitions enumerated by the function.

Remark 3.5. While the a-generalizations are useful for studying the relevant partitions, the price paid for generalizing (1.1) to (3.7) and (1.2) to (3.6) is that the a-generalizations no longer have infinite product representations; only in the $a = 1$ cases will Jacobi's triple product identity [10, p. 15, Eq. (1.6.1)] allow the right hand sides of (1.2) and (1.1) to be transformed into infinite products.

An exception to Remark 3.5 may be found in one of the identities in Ramanujan's lost notebook [6, Entry 5.3.9]; cf. [11, Eq. (1.16)]:

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(q^3;q^6)_j}{(q;q^2)_j^2 (q^4;q^4)_j} = \prod_{\substack{j \geqq 1 \\ j\equiv 1 (\mathrm{mod}\ 2) \text{ or } j \equiv \pm 2 (\mathrm{mod}\ 12)}} \frac{1}{1-q^j}. \tag{3.10}$$

Equation (3.10) admits an a-generalization with an infinite product:

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2} (q^3;q^6)_j}{(q;q^2)_j (aq;q^2)_j (q^4;q^4)_j} = \prod_{j\geqq 1} \frac{1+aq^{4j-2}+a^2q^{8j-4}}{1-aq^{2j-1}}. \tag{3.11}$$

Notice that the right hand side of (3.11) is easily seen to be equal to

$$\sum_{n,\ell\geqq 0} s(\ell,n) a^\ell q^n,$$

where $s(\ell,n)$ denotes the number of partitions of n into exactly ℓ parts where no even part appears more than twice nor is divisible by 4. Note also that the right hand side of (3.10) generates partitions where parts may appear as in Schur's 1926 partition theorem [15] (i.e. partitions into parts congruent to $\pm 1 \pmod 6$), dilated by a factor of 2, along with unrestricted appearances of odd parts. It is a fairly common phenomenon for a Rogers-Ramanujan type identity to generate partitions whose parts are restricted according to a well-known partition theorem, dilated by a factor of m, and where nonmultiples of m may appear without restriction. See, e.g., Connor [8] and Sills [16].

A partner to (3.10) was found by Slater [17, p. 164, Eq. (110), corrected], cf. [11, Eq. (1.19)]:

$$\sum_{j=0}^{\infty} \frac{q^{j^2+2j} (q^3;q^6)_j}{(q;q^2)_j (q;q^2)_{j+1} (q^4;q^4)_j} = \prod_{\substack{j\geqq 1 \\ j\equiv 1 \pmod 2 \text{ or } j\equiv \pm 4 \pmod{12}}} \frac{1}{1-q^j}. \tag{3.12}$$

An a-generalization of (3.12) is

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2+2j} (q^3;q^6)_j}{(q;q^2)_j (aq;q^2)_{j+1} (q^4;q^4)_j} = \prod_{j\geqq 1} \frac{1+aq^{4j}+a^2q^{8j}}{1-aq^{2j-1}} = \sum_{n,\ell\geqq 0} t(\ell,n) a^\ell q^n, \tag{3.13}$$

where $t(\ell,n)$ denotes the number of partitions of n into ℓ parts where even parts appear at most twice and are divisible by 4.

Remark 3.6. An explanation as to why (3.10) and (3.12) admit a-generalizations which include infinite products and (1.1) and (1.2) do not, may be found in the theory of basic hypergeometric series. The Rogers-Ramanujan identities (1.1) and (1.2) arise as limiting cases of Watson's q-analog of Whipple's theorem [18],[10, p. 43, Eq. (2.5.1)]; see [10, pp. 44–45, §2.7]. In contrast, (3.11) and (3.13) are special cases of Andrews's q-analog of Bailey's ${}_2F_1(\frac{1}{2})$ sum [1, p. 526, Eq. (1.9)],[10, p. 354, (Eq. II.10)].

Remark 3.7. S. Corteel and J. Lovejoy interpreted (3.10) and (3.12) combinatorially using overpartitions in [9].

4. A Family of Ramanujan and Slater

4.1. A long-lost relative

Let us define

$$Q(w,x) := (-wx^{-1}, -x, w; w)_\infty (wx^{-2}, wx^2; w^2)_\infty,$$

where

$$(a_1, a_2, \ldots, a_r; w)_\infty := \prod_{k=1}^{r} (a_k; w)_\infty.$$

Then it is clear that an identity is missing from the family

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+2)}(q^3;q^6)_j}{(q^2;q^2)_{2j+1}(q;q^2)_j} = \frac{Q(q^{18},q^7)}{(q^2;q^2)_\infty} \quad \text{(Slater [17, Eq. (125)])} \tag{4.14}$$

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j}{(q^2;q^2)_{2j+1}(q;q^2)_j} = \frac{Q(q^{18},q^5)}{(q^2;q^2)_\infty} \quad \text{(Slater [17, Eq. (124)])} \tag{4.15}$$

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(q^3;q^6)_j}{(q^2;q^2)_{2j}(q;q^2)_j} = \frac{Q(q^{18},q^3)}{(q^2;q^2)_\infty} \quad \text{(Ramanujan [6, Entry 5.3.4]).} \tag{4.16}$$

The following identity completes the above family:

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j}{(q^2;q^2)_{2j}(q;q^2)_{j+1}} = \frac{Q(q^{18},q)}{(q^2;q^2)_\infty}. \tag{4.17}$$

Theorem 4.8. *Identity* (4.17) *is valid.*

Proof. We show that (4.15)+$q\times$(4.14) = (4.17). For the series side,

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j}{(q^2;q^2)_{2j+1}(q;q^2)_j} + q\sum_{j=0}^{\infty} \frac{q^{2j(j+2)}(q^3;q^6)_j}{(q^2;q^2)_{2j+1}(q;q^2)_j} &= \sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j(1+q^{2j+1})}{(q^2;q^2)_{2j+1}(q;q^2)_j} \\ &= \sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j}{(q^2;q^2)_{2j}(q;q^2)_{j+1}}.\end{aligned}$$

For the product side, we make use of the quintuple product identity:

$$Q(w,x) = (wx^3, w^2x^{-3}, w^3; w^3)_\infty + x(wx^{-3}, w^2x^3, w^3; w^3)_\infty.$$

Hence

$$\begin{aligned}Q(q^{18},q^5) + qQ(q^{18},q^7) &= (q^{33},q^{21},q^{54};q^{54})_\infty + q^5(q^3,q^{51},q^{54};q^{54})_\infty \\ &\quad + q((q^{39},q^{15},q^{54};q^{54})_\infty + q^7(q^{-3},q^{57},q^{54};q^{54})_\infty) \\ &= (q^{33},q^{21},q^{54};q^{54})_\infty + q^5(q^3,q^{51},q^{54};q^{54})_\infty \\ &\quad + q((q^{39},q^{15},q^{54};q^{54})_\infty - q^4(q^{51},q^3,q^{54};q^{54})_\infty) \\ &= (q^{21},q^{33},q^{54};q^{54})_\infty + q(q^{15},q^{39},q^{54};q^{54})_\infty = Q(q^{18},q).\end{aligned}$$

The result now follows. □

4.2. Combinatorial Interpretations

We interpret (4.16) combinatorially.

Theorem 4.9. *The number of signed partitions $\sigma = (\pi, \nu)$ of n, where*

- *$\ell(\pi)$ is even, and each positive part is even and $\geqq \ell(\pi)$, and*
- *the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice*

equals the number of (ordinary) partitions of n into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$.

Proof. Starting with the left hand side of (4.16), we find

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{q^{2j^2}(q^3;q^6)_j}{(q^2;q^2)_{2j}(q;q^2)_j} &= \sum_{j=0}^{\infty} \frac{q^{2j^2}\prod_{k=1}^{j}(1+q^{2k-1}+q^{4k-2})}{(q^2;q^2)_{2j}} \\
&= \sum_{j=0}^{\infty} \frac{q^{2j^2}\prod_{k=1}^{j} q^{4k-2}(1+q^{-(2k-1)}+q^{-(4k-2)})}{(q^2;q^2)_{2j}} \\
&= \sum_{j=0}^{\infty} \frac{q^{2j^2+4(1+2+\cdots+j)-2j}\prod_{k=1}^{j}(1+q^{-(2k-1)}+q^{-(4k-2)})}{(q^2;q^2)_{2j}} \\
&= \sum_{j=0}^{\infty} \frac{q^{4j^2}}{(q^2;q^2)_{2j}} \times \prod_{k=1}^{j}(1+q^{-(2k-1)}+q^{-(4k-2)})
\end{aligned}$$

Notice that

$$\frac{1}{(q;q)_{2j}}$$

is the generating function for partitions into at most $2j$ parts, thus

$$q^{2j^2}\frac{1}{(q;q)_{2j}} = \frac{q^{\overbrace{j+j+j+\cdots+j}^{2j \text{ terms}}}}{(q;q)_{2j}}$$

is the generating function for partitions into exactly $2j$ parts, where each part is at least j. Thus

$$q^{4j^2}\frac{1}{(q^2;q^2)_{2j}}$$

is the generating function for partitions into exactly $2j$ parts, each of which is even and at least $2j$. Also, $\prod_{k=1}^{j}(1+q^{-(2k-1)}+q^{-(4k-2)})$ is the generating function for signed partitions into odd negative parts $< 2j$ and appearing at most twice each. Summing over all j, we find that the left hand side of (4.16) is the generating function for signed partitions $\sigma = (\pi, \nu)$ of n, where $\ell(\pi)$ is even, and each positive part is even and $\geqq \ell(\pi)$, and the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice.

Now the RHS of (4.16) is

$$\frac{Q(q^{18}, q^3)}{(q^2;q^2)_\infty} = \frac{(-q^3, -q^{15}; q^{18}; q^{18})_\infty (q^{12}, q^{24}; q^{36})_\infty}{(q^2;q^2)_\infty} = \prod_{\substack{i \geqq 1 \\ i \equiv \pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}}} \frac{1}{1-q^i},$$

which is clearly the generating function for partitions into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8$ (mod 18). $\square$

Remark 4.10. Andrews provided a different combinatorial interpretation of (4.16) in [3, p. 175, Theorem 2].

Following the ideas in the proof of Theorem 4.9, the analogous combinatorial interpretation of Identity (4.15) is as follows.

Theorem 4.11. *The number of signed partitions $\sigma = (\pi, \nu)$ of n, where*

- *$\ell(\pi)$ is odd, and each positive part is even and $\geqq \ell(\pi) - 1$, and*
- *the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice*

equals the number of (ordinary) partitions of n into parts congruent to $\pm 2, \pm 4, \pm 5, \pm 6$ (mod 18).

We next interpret (4.14) combinatorially. Note that the theorem equates the number in a certain class of signed partitions of $n + 1$ with the number in a certain class of regular partitions of n.

Theorem 4.12. *The number of signed partitions $\sigma = (\pi, \nu)$ of $n + 1$, where*

- *$\ell(\pi)$ is odd, and each positive part is odd and $\geqq \ell(\pi)$, and*
- *the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice*

equals the number of (ordinary) partitions of n into parts congruent to $\pm 2, \pm 6, \pm 7, \pm 8$ (mod 18).

Proof. The proof is similar to that of Theorem 4.9, except that

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+2)}(q^3;q^6)_j}{(q^2;q^2)_{2j+1}(q;q^2)_j} = \frac{1}{q}\sum_{j=0}^{\infty} \frac{q^{4j^2+4j+1}}{(q^2;q^2)_{2j+1}} \times \prod_{k=1}^{j}(1+q^{-(2k-1)}+q^{-(4k-2)}),$$

and since

$$4j^2+4j+1 = \underbrace{(2j+1)+(2j+1)+\cdots+(2j+1)}_{2j+1 \text{ terms}},$$

it follows that

$$\frac{q^{4j^2+4j+1}}{(q^2;q^2)_{2j+1}}$$

is the generating function for partitions into exactly $2j+1$ parts, each of which is odd and at least $2j+1$. $\square$

Lastly, we give a combinatorial interpretation of (4.17).

Theorem 4.13. *The number of signed partitions $\sigma = (\pi, \nu)$ of $n+1$, where*

- *π contains an odd positive part m (which may be repeated), exactly $m-1$ positive even parts, all $\geqq m-1$, and*
- *negative parts are all odd, $< m$, and appear at most twice,*

equals the number of (ordinary) partitions of n into parts congruent to $\pm 1, \pm 4, \pm 6, \pm 8 \pmod{18}$.

Proof. This time

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+1)}(q^3;q^6)_j}{(q^2;q^2)_{2j}(q;q^2)_{j+1}} = \frac{1}{q}\sum_{j=0}^{\infty} \frac{q^{4j^2}}{(q^2;q^2)_{2j}} \frac{q^{2j+1}}{1-q^{2j+1}} \times \prod_{k=1}^{j}(1+q^{-(2k-1)}+q^{-(4k-2)}),$$

so that, as before,

$$\frac{q^{4j^2}}{(q^2;q^2)_{2j}}$$

is the generating function for partitions into exactly $2j$ parts, each of which is even and at least $2j$, and

$$\frac{q^{2j+1}}{1-q^{2j+1}}$$

generates partitions consisting of the part $2j+1$ and containing at least one such part. $\square$

5. Combinatorial Interpretations of a Family of Mod 18 Identities

In [11], we presented the following family of Rogers-Ramanujan-Slater type identities related to the modulus 18:

$$\sum_{j=0}^{\infty} \frac{q^{j(j+1)}(-1;q^3)_j}{(-1;q)_j(q;q)_{2j}} = \frac{(q,q^8,q^9;q^9)_\infty(q^7,q^{11};q^{18})_\infty}{(q;q)_\infty} \tag{5.18}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(-1;q^3)_j}{(-1;q)_j(q;q)_{2j}} = \frac{(q^2,q^7,q^9;q^9)_\infty(q^5,q^{13};q^{18})_\infty}{(q;q)_\infty} \tag{5.19}$$

$$\sum_{j=0}^{\infty} \frac{q^{j(j+1)}(-q^3;q^3)_j}{(-q;q)_j(q;q)_{2j+1}} = \frac{(q^3,q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty} \tag{5.20}$$

$$\sum_{j=0}^{\infty} \frac{q^{j(j+2)}(-q^3;q^3)_j}{(q^2;q^2)_j(q^{j+2};q)_{j+1}} = \frac{(q^4,q^5,q^9;q^9)_\infty(q,q^{17};q^{18})_\infty}{(q;q)_\infty}. \tag{5.21}$$

We give a combinatorial interpretation of (5.21).

Theorem 5.14. *The number of signed partitions $\sigma = (\pi,\nu)$ of $n+2$, wherein*

- π_1*, the largest positive part, is even,*
- *the integers $1,2,\dots,\frac{\pi_1}{2}-1$ all appear an even number of times and at least twice,*
- *the integer $\frac{\pi_1}{2}$ does not appear,*
- *the integers $\frac{\pi_1}{2}+1, \frac{\pi_1}{2}+2,\dots,\pi_1$ all appear at least once, and*
- *there are exactly $\frac{\pi_1}{2}-1$ negative parts, each $\equiv 1 \pmod 3$ and $\leqq \frac{3\pi_1}{2}-2$, with the parts greater than 1 occurring at most once*

equals the number of (ordinary) partitions of n into parts congruent to $\pm2,\pm3,\pm6,\pm7,\pm8$ $\pmod{18}$.

Proof. We consider the general term on the left side of (5.21).

$$\begin{aligned}\frac{q^{j(j+2)}(-q^3;q^3)_j}{(q^2;q^2)_j(q^{j+2};q)_{j+1}} &= \frac{q^{j^2+2j}}{(q^2;q^2)_j}\frac{q^{(3j^2+3j)/2}}{\prod_{k=0}^{j}(1-q^{j+2+k})}\prod_{k=1}^{j}(1+q^{-3k})\\ &= \frac{1}{q^2}\frac{q^{j^2+j}}{(q^2;q^2)_j}\frac{q^{(3j^2+7j+4)/2}}{\prod_{k=0}^{j}(1-q^{j+2+k})}q^{-j}\prod_{k=1}^{j}(1+q^{-3k})\end{aligned}$$

The factors

$$\frac{q^{j^2+j}}{(q^2;q^2)_j} = \frac{q^{2+4+6+\cdots+2j}}{(q^2;q^2)_j}$$

generates parts in $\{2,4,6,\ldots,2j\}$ where each part appears at least once. Then by mapping each even part $2r$ to $r+r$, we have parts $\{1,2,3,\ldots,j\}$ where each part appears an even number of times and at least twice.

The factors
$$\frac{q^{(3j^2+7j+4)/2}}{\prod_{k=0}^{j}(1-q^{j+2+k})}=\frac{q^{(j+2)+(j+3)+\cdots+(2j+2)}}{\prod_{k=0}^{j}(1-q^{j+2+k})}$$
generates partitions from the parts $\{j+2,j+3,\ldots,2j+1,2j+2\}$ and where each part appears at least once. Lastly,
$$q^{-j}\prod_{k=1}^{j}(1+q^{-3k})$$
is the generating function for signed partitions with negative parts that are congruent to 1 modulo 3, $\leq 3j+1$, the parts greater than 1 occur at most once, and the total number of parts is j (the number of 1's being j minus the number of other parts).

Upon summing over $j\geq 0$, we get that
$$\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q^2;q^2)_j}\frac{q^{(3j^2+7j+4)/2}}{\prod_{k=0}^{j}(1-q^{j+2+k})}q^{-j}\prod_{k=1}^{j}(1+q^{-3k})$$
is the generating function for signed partitions with the properties itemized in the statement of the theorem.

The right side of (5.21) is
$$\frac{(q^4,q^5,q^9;q^9)_\infty(q,q^{17};q^{18})_\infty}{(q;q)_\infty}=\prod_{\substack{i\geqq 1\\ i\equiv\pm2,\pm3,\pm6,\pm7,\pm8 \pmod{18}}}\frac{1}{1-q^i},$$
which is the generating function for partitions into parts congruent to ±2, $\pm3,\pm6,\pm7,\pm8$ (mod 18). □

The corresponding combinatorial interpretation of (5.19) is given by the following theorem.

Theorem 5.15. *The number of signed partitions $\sigma=(\pi,\nu)$ of n, where*

- *π_1, the largest positive part, is even,*
- *the integers $1,2,\ldots,\frac{\pi_1}{2}-1$ all appear an even number of times and at least twice,*
- *the integers $\frac{\pi_1}{2},\frac{\pi_1}{2}+1,\ldots,\pi_1$ all appear at least once, and*
- *there are exactly $\frac{\pi_1}{2}-1$ negative parts, each $\equiv 2 \pmod 3$ and $\leqq\frac{3\pi_1}{2}-1$, with the parts greater than 2 occurring at most once,*

equals the number of (ordinary) partitions of n *into parts congruent to* $\pm 1, \pm 3, \pm 4, \pm 6, \pm 8 \pmod{18}$.

Proof. The proof is similar to that of Theorem 5.14, except we rewrite the general term on the left side of (5.19) as follows

$$\begin{aligned}\frac{q^{j^2}(-1;q^3)_j}{(-1;q)_j(q;q)_{2j}} &= \frac{q^{j^2}}{(q^2;q^2)_{j-1}} \frac{q^{(3j^2-3j)/2}}{\prod_{k=0}^{j}(1-q^{j+k})} \prod_{k=1}^{j-1}(1+q^{-3k}) \\ &= \frac{q^{j^2-j}}{(q^2;q^2)_{j-1}} \frac{q^{(3j^2+3j)/2}}{\prod_{k=0}^{j}(1-q^{j+k})} q^{-2j} \prod_{k=1}^{j-1}(1+q^{-3k}).\end{aligned}$$

□

The identity at (5.18) may be interpreted combinatorially as follows.

Theorem 5.16. *The number of signed partitions* $\sigma = (\pi, \nu)$ *of* n*, where*

- π_1*, the largest positive part, is even,*
- *the integers* $1, 2, \ldots, \frac{\pi_1}{2} - 1$ *all appear an even number of times and at least twice,*
- *the integers* $\frac{\pi_1}{2}, \frac{\pi_1}{2} + 1, \ldots, \pi_1$ *all appear at least once, and*
- *there are exactly* $\frac{\pi_1}{2} - 1$ *negative parts, each* $\equiv 1 \pmod 3$ *and* $\leqq \frac{3\pi_1}{2} - 2$*, with the parts greater than 1 occurring at most once,*

equals the number of (ordinary) partitions of n *into parts congruent to* $\pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \pmod{18}$.

Proof. The general term on the left side of (5.18) may be written as

$$\begin{aligned}\frac{q^{j^2+j}(-1;q^3)_j}{(-1;q)_j(q;q)_{2j}} &= \frac{q^{j^2+j}}{(q^2;q^2)_{j-1}} \frac{q^{(3j^2-3j)/2}}{\prod_{k=0}^{j}(1-q^{j+k})} \prod_{k=1}^{j-1}(1+q^{-3k}) \\ &= \frac{q^{j^2-j}}{(q^2;q^2)_{j-1}} \frac{q^{(3j^2+3j)/2}}{\prod_{k=0}^{j}(1-q^{j+k})} q^{-j} \prod_{k=1}^{j-1}(1+q^{-3k}).\end{aligned}$$

□

Finally, we provide a combinatorial interpretation of (5.20)

Theorem 5.17. *The number of signed partitions* $\sigma = (\pi, \nu)$ *of* $n + 1$*, wherein*

- π_1, *the largest positive part, is odd,*
- *the integers* $1, 2, \ldots, \frac{\pi_1-1}{2}$ *all appear an even number of times and at least twice,*
- *the integers* $\frac{\pi_1-1}{2}+1, \frac{\pi_1-1}{2}+2, \ldots, \pi_1$ *all appear at least once, and*
- *there are exactly* $\frac{\pi_1-1}{2}$ *negative parts, each* $\equiv 1 \pmod 3$ *and* $\leqq \frac{3\pi_1}{2}-2$, *with the parts greater than 1 occurring at most once,*

equals the number of (ordinary) partitions of n *into parts congruent to* 1, 2, 4, 5, 7 *or* 8 *modulo* 9, *such that for any nonnegative integer* j, $9j+1$ *and* $9j+2$ *do not both appear, and for any nonnegative integer* k, $9k+7$ *and* $9k+8$ *do not both appear.*

Proof. The general term on the left side of (5.20) may be written as

$$\begin{aligned}\frac{q^{j(j+1)}(-q^3;q^3)_j}{(-q;q)_j(q;q)_{2j+1}} &= \frac{q^{j^2+j}}{(q^2;q^2)_j}\frac{q^{(3j^2+3j)/2}}{(q^{j+1};q)_{j+1}}\prod_{k=1}^{j}(1+q^{-3k})\\ &= \frac{1}{q}\frac{q^{j^2+j}}{(q^2;q^2)_j}\frac{q^{(3j^2+5j+2)/2}}{(q^{j+1};q)_{j+1}}q^{-j}\prod_{k=1}^{j}(1+q^{-3k}),\end{aligned}$$

thus the interpretation of the left side is similar to that of the previous identities.

The right side of (5.20) provides a challenge because of the double occurrence of the factors $(q^3;q^{18})_\infty$ and $(q^{15};q^{18})_\infty$ in the numerator. Accordingly, we turn to a partition enumeration technique introduced by Andrews and Lewis. In [7, p. 79, Eq. (2.2) with $k=9$], they show that

$$\frac{(q^{a+b};q^{18})_\infty}{(q^a,q^b;q^9)_\infty} \tag{5.22}$$

is the generating function for partitions of n into parts congruent to a or b modulo 9 such that for any k, $9k+a$ and $9k+b$ do not both appear as parts, where $0<a<b<9$.

With this in mind, we immediately see that

$$\frac{(q^3,q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty} = \frac{1}{(q^4,q^5;q^9)_\infty}\times\frac{(q^3;q^{18})_\infty}{(q,q^2;q^9)_\infty}\times\frac{(q^{15};q^{18})_\infty}{(q^7,q^8;q^9)_\infty}$$

generates the partitions stated in our theorem. □

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