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Dedicated to George Andrews on the occasion of his 70th birthday

1 Introduction

A partition of an integer n is a representation of n as an unordered sum of positive integers. In a recent paper [1], Andrews introduced the notion of a "signed partition," that is, a representation of a positive integer as an unordered sum of integers, some possibly negative.

Consider the following q-series identity:

Theorem 1 (Ramanujan and Slater)**.** *For* $|q| < 1$*,*

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\cdots(1+q^{2j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} = \prod_{\substack{m \geqq 1 \\ m \equiv 1,4,7 \pmod 8}} \frac{1}{1-q^m}. \tag{1.1}$$

An identity equivalent to (1.1) was recorded by Ramanujan in his lost notebook [2, Entry 1.7.11]. The first proof of (1.1) was given by Slater [5, Eq. (36)].

Identity (1.1) became well known after B. Gordon [4] showed that it is equivalent to the following partition identity, which had been discovered independently by H. Göllnitz [3]:

Theorem 2 (Göllnitz and Gordon). *Let $A(n)$ denote the number of partitions of n into parts which are distinct, nonconsecutive integers where no consecutive even integers appear. Let $B(n)$ denote the number of partitions of n into parts congruent to 1, 4, or 7 modulo 8. Then $A(n) = B(n)$ for all integers n.*

Andrews [1, p. 569, Theorem 8] provided the following alternate combinatorial interpretation of (1.1).

Theorem 3 (Andrews). *Let $C(n)$ denote the number of signed partitions of n where the negative parts are distinct, odd, and smaller in magnitude than twice the number of positive parts, and the positive parts are even and have magnitude at least twice the number of positive parts. Let $B(n)$ be as in Theorem 2. Then $C(n) = B(n)$ for all n.*

Proof. The result follows immediately after rewriting the left hand side of (1.1) as

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q^{-1})(1+q^{-3})\cdots(1+q^{-(2j-1)})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})}.$$

See [1, p. 569] for more details. $\square$

The purpose of this paper is to provide a bijection between the set of ordinary Göllnitz-Gordon partitions (those enumerated by $A(n)$ in Theorem 2) and Andrews' "signed Göllnitz-Gordon partitions" enumerated by $C(n)$ in Theorem 3.

2 Definitions and Notations

A *partition* λ of an integer n with j parts is a j-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_j)$ where each $\lambda_i \in \mathbb{Z}$,

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_j \geqq 1$$

and

$$\sum_{k=1}^{j} \lambda_k = n.$$

Each λ_i is called a *part* of λ. The *weight* of λ is $n = \sum_{k=1}^{j} \lambda_k$ and is denoted $|\lambda|$. The number of parts in λ is also called the *length* of λ and is denoted $\ell(\lambda)$.

Sometimes it is more convenient to denote a partition by

$$\langle 1^{f_1} 2^{f_2} 3^{f_3} \cdots \rangle$$

meaning that the partition is comprised of f_1 ones, f_2 twos, f_3 threes, etc.

When generalizing the notion of partitions to Andrews' "signed partitions," i.e. partitions where some of the parts are allowed to be negative, it will be convenient to segregate the positive parts from the negative parts. Thus we define a *signed partition* σ of an integer n as a pair of (ordinary) partitions (π, ν) where $n = |\pi| - |\nu|$. The parts of π are the *positive parts* of σ and the parts of ν are the *negative parts* of σ. We may also refer to π (resp. ν) as the *positive (resp. negative) subpartition* of σ.

Let us denote the *parity function* by

$$P(k) := \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Let $\mathcal{G}_{n,j}$ denote the set of partitions

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_j)$$

of weight n and length j, where for $1 \leqq i \leqq j-1$,

$$\gamma_i - \gamma_{i+1} \geqq 2 \tag{2.1}$$

$$\gamma_i - \gamma_{i+1} > 2 \quad \text{if } \gamma_i \text{ is even.} \tag{2.2}$$

Thus $\mathcal{G}_{n,j}$ is the set of those partitions enumerated by $A(n)$ in Theorem 2 which have length j.

Let $\mathcal{S}_{n,j}$ denote the set of signed partitions $\sigma = (\pi, \nu)$ of n such that

$$\ell(\pi) = j \tag{2.3}$$

$$\ell(\nu) \leqq j \tag{2.4}$$

$$\pi_i \text{ is even for all } i = 1, 2, \dots, j \tag{2.5}$$

$$\pi_i \geqq 2j \text{ for all } i = 1, 2, \dots j \tag{2.6}$$

$$\nu_i \text{ is odd for all } i = 1, 2, \dots, \ell(\nu) \tag{2.7}$$

$$\nu_i \leqq 2j - 1 \text{ for all } i = 1, 2, \dots, \ell(\nu) \tag{2.8}$$

$$\nu_i - \nu_{i+1} \geqq 2 \text{ for all } i = 1, 2, \dots, \ell(\nu) - 1, \tag{2.9}$$

i.e. the positive subpartition is a partition into j even parts, all at least $2j$, and the negative subpartition is a partition into distinct odd parts, all less than $2j$. Thus $\mathcal{S}_{n,j}$ is the set of those signed partitions enumerated by $C(n)$ in Theorem 3 which have exactly j parts.

3 A bijection between ordinary and signed Göllnitz-Gordon partitions

Theorem 4. *The map*

$$g : \mathcal{G}_{n,j} \to \mathcal{S}_{n,j}$$

given by

$$(\gamma_1, \gamma_2, \ldots, \gamma_j) \overset{g}{\mapsto} \Big((\pi_1, \pi_2, \ldots, \pi_j), \langle 1^{f_1} 3^{f_3} \cdots (2j-1)^{f_{2j-1}} \rangle \Big)$$

where

$$\pi_k = \gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2 \sum_{i=k+1}^{j} P(\gamma_i)$$

and

$$f_{2k-1} = P(\gamma_k)$$

is a bijection.

Proof. Suppose that $\gamma \in \mathcal{G}_{n,j}$ and that the image of γ under g is the signed partition $\sigma = (\pi, \nu)$.

Claim 1. $|\sigma| = |\pi| - |\nu| = n$.

Proof of Claim 1.

$$\begin{aligned}
|\pi| - |\nu| &= \sum_{k=1}^{j} \left(\gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2 \sum_{i=k+1}^{j} P(\gamma_i) \right) \\
&\quad - \left(\sum_{h=1}^{j} (2h-1) P(\gamma_h) \right) \\
&= \left(\sum_{k=1}^{j} \gamma_k \right) + 4 \frac{j(j+1)}{2} - 2j^2 - 2j + \sum_{k=1}^{j} P(\gamma_k) \\
&\quad + 2 \sum_{k=1}^{j} \sum_{i=k+1}^{j} P(\gamma_i) - \left(\sum_{h=1}^{j} (2h-1) P(\gamma_h) \right) \\
&= n - \sum_{h=1}^{j} (2h-2) P(\gamma_h) + 2 \sum_{i=1}^{j} (h-1) P(\gamma_h) \\
&= n
\end{aligned}$$

$\square$

Claim 2. $\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_j$.

Proof of Claim 2. Fix k with $1 \leqq k < j$.

$$\begin{aligned}\pi_k - \pi_{k+1} &= \gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2\sum_{i=k+1}^{j} P(\gamma_i) \\ &\quad - \left(\gamma_{k+1} + 4(k+1) - 2j - 2 + P(\gamma_{k+1}) + 2\sum_{i=k+2}^{j} P(\gamma_i)\right) \\ &= \gamma_k - \gamma_{k+1} + P(\gamma_k) - P(\gamma_{k+1}) - 4.\end{aligned}$$

The minimum value of $\gamma_k - \gamma_{k+1}$ varies depending on the parities of γ_k and γ_{k+1}.

- If $\gamma_k \equiv \gamma_{k+1} \equiv 0 \pmod 2$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) - P(\gamma_{k+1}) - 4 \geqq 4 + 0 + 0 - 4 = 0.$$

- If $\gamma_k \equiv 1 \pmod 2$ and $\gamma_{k+1} \equiv 0 \pmod 2$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) - P(\gamma_{k+1}) - 4 \geqq 3 + 1 + 0 - 4 = 0.$$

- If $\gamma_k \equiv 0 \pmod 2$ and $\gamma_{k+1} \equiv 1 \pmod 2$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) - P(\gamma_{k+1}) - 4 \geqq 3 + 0 + 01 - 4 = 0.$$

- If $\gamma_k \equiv \gamma_{k+1} \equiv 1 \pmod 2$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) - P(\gamma_{k+1}) - 4 \geqq 2 + 1 + 1 - 4 = 0.$$

$\square$

Claim 3. *All of the π_k are at least $2j$.*

Proof of Claim 3. By Claim 2, it is sufficient to show that $\pi_j \geqq 2j$.

If $\gamma_j = 1$, then

$$\begin{aligned}\pi_j &= \gamma_j + 4j - 2j - 2 + P(\gamma_j) + 2\sum_{i=j+1}^{j} P(\gamma_i) \\ &= \gamma_j + 2j - 2 + 1 \\ &\geqq 1 + 2j - 2 + 1 \\ &= 2j.\end{aligned}$$

Otherwise $\gamma_j \geqq 2$, and so

$$\begin{aligned}\pi_j &= \gamma_j + 4j - 2j - 2 + P(\gamma_j) + 2\sum_{i=j+1}^{j} P(\gamma_i) \\ &= \gamma_j + 2j - 2 + 1 \\ &\geqq 2 + 2j - 2 + 0 \\ &= 2j.\end{aligned}$$

□

Claim 4. *All parts of π are even.*

Proof of Claim 4.

$$\begin{aligned}\pi_k - \pi_{k+1} &= \gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2\sum_{i=k+1}^{j} P(\gamma_i) \\ &\equiv \gamma_k + P(\gamma_k) \\ &\equiv 0 \pmod 2.\end{aligned}$$

□

Claim 5. *All parts of ν are distinct, odd, and at most $2j - 1$.*

Proof of Claim 5. Claim 5 is clear from the definition of g together with the observation that $P(\gamma_i) \in \{0, 1\}$ for any i. □

Claim 6. *The map g is invertible.*

Proof of Claim 6. Let
$$h : \mathcal{S}_{n,j} \to \mathcal{G}_{n,j}$$
be given by
$$\Big((\pi_1, \pi_2, \ldots, \pi_j), \langle 1^{f_1} 3^{f_3} \cdots (2j-1)^{f_{2j-1}} \rangle\Big) \overset{h}{\mapsto} (\gamma_1, \gamma_2, \ldots, \gamma_j)$$
where
$$\gamma_k = \pi_k - 4k + 2j + 2 - f_{2k-1} - 2 \sum_{i=k+1}^{j} f_{2i-1}$$
for $1 \leqq k \leqq j$. Direct computation shows that $h(g(\gamma)) = \gamma$ for all $\gamma \in \mathcal{G}_{n,j}$, and $g(h(\sigma)) = \sigma$ for all $\sigma \in \mathcal{S}_{n,j}$. Thus h is the inverse of g. □

Hence, by the above claims g is a bijection. □

References

[1] G.E. Andrews, Euler's *De Partitio Numerorum*, Bull. Amer. Math. Soc. 44 (2007), 561–573.

[2] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, part 2*, Springer, to appear.

[3] H. Göllnitz, Einfache Partitionen, (unpublished), Diplomabeit W. S., 1960, Göttingen.

[4] B. Gordon, Some continued fractions of the Rogers-Ramanujan type, Duke Math. J. 32 (1965) 741748

[5] L.J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. 54 (1952) 147–167.