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Contemporary Mathematics

Disturbing the q-Dyson Conjecture

Andrew V. Sills

Abstract. I discuss the computational methods behind the formulation of some conjectures related to variants on Andrews' q-Dyson conjecture.

1. Introduction

In 1962 [**D2**], Freeman Dyson made the following conjecture:

Dyson's Conjecture. *For positive integers n and $a_1, a_2, \ldots, a_n$, the constant term in the expansion of the Laurent polynomial*

$$\prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i} \tag{1.1}$$

is the multinomial coëfficient

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}.$$

Dyson's conjecture was settled independently by Gunson [**Gu**] and Wilson [**W**]. In 1970, Good [**Go**] supplied a particularly compact and elegant proof.

In 1975, George Andrews conjectured a q-analog of Dyson's conjecture [**A**]:

Andrews' q-Dyson Conjecture. *For nonnegative integers n and $a_1, a_2, \ldots, a_n$, the constant term in the expansion of the Laurent polynomial*

$$\prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j} q\right) \left(1 - \frac{x_i}{x_j} q^2\right) \cdots \left(1 - \frac{x_i}{x_j} q^{a_j}\right) \left(1 - \frac{x_j}{x_i}\right) \left(1 - \frac{x_j}{x_i} q\right) \cdots \left(1 - \frac{x_j}{x_i} q^{a_i - 1}\right)$$

is the q-multinomial coëfficient

$$\frac{[a_1 + a_2 + \cdots + a_n]_q!}{[a_1]_q! [a_2]_q! \cdots [a_n]_q!},$$

where

$$[a]_q := \frac{1 - q^a}{1 - q} = 1 + q + q^2 + \cdots + q^{a-1}$$

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is the usual q-analog of the nonnegative integer a and

$$[a]_q! = \prod_{j=1}^{a} [j]_q$$

is the q-factorial. Clearly, the $q = 1$ case of the q-Dyson conjecture is the original Dyson conjecture. The q-Dyson conjecture remained unsettled for a decade until it was proved by Zeilberger and Bressoud [**ZB**]. Two additional decades passed before a shorter proof was found by Gessel and Xin [**GX**].

In [**SZ**], Zeilberger and I set out to "disturb" the Dyson conjecture[1] by programming the computer to conjecture, and then provide proofs modeled after Good's proof [**Go**], for closed form expressions of coefficients of terms in the expansion of (1.1) *other* than the constant term. Using our Maple package `GoodDyson`, available for free download from our home pages [**SZ2**], the computer can (up to the limits imposed by time and memory) conjecture and prove a closed form expression for the coefficient of $x_1^{b_2} x_2^{b_2} \cdots x_n^{b_n}$ in the expansion of (1.1) for any *fixed* n and any *fixed* $b_1, b_2, \ldots, b_n$.

At this point, we should introduce some more notation. For n a positive integer, we define the following symbols:

(n-vector of symbolic nonnegative integers)

$$\mathbf{a} := \langle a_1, a_2, \ldots, a_n \rangle,$$

(n-vector of indeterminants) $\quad \mathbf{x} := \langle x_1, x_2, \ldots, x_n \rangle,$

(first elementary symmetric polynomial in n indeterminants)

$$\sigma_n(\mathbf{a}) := a_1 + a_2 + \cdots + a_n,$$

(rising q-factorial) $\quad (A; q)_n := \prod_{i=0}^{n-1} (1 - Aq^i),$

(Dyson product) $\quad F_n(\mathbf{x}; \mathbf{a}) := \prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i},$

(q-Dyson product) $\quad \mathcal{F}_n(\mathbf{x}; \mathbf{a}; q) := \prod_{1 \leqq i < j \leqq n} \left(\frac{x_i q}{x_j}; q\right)_{a_j} \left(\frac{x_j}{x_i}; q\right)_{a_i},$

and let $[Y]Z$ denote the coefficient of Y in the expression Z, thus the Dyson conjecture is

$$[x_1^0 x_2^0 \cdots x_n^0] F_n(\mathbf{x}; \mathbf{a}) = \frac{\sigma_n(\mathbf{a})!}{a_1! a_2! \cdots a_n!},$$

while the q-Dyson conjecture is

$$[x_1^0 x_2^0 \cdots x_n^0] \mathcal{F}_n(\mathbf{x}; \mathbf{a}; q) = \frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q! [a_2]_q! \cdots [a_n]_q!}.$$

Using the output from many applications of the `GoodDyson` program for various values of n and $b_1, \ldots, b_n$, I was able to conjecture and prove the following "disturbed" versions of the Dyson conjecture [**Si1**]:

[1] In the interest of full disclosure, it is my esteemed coauthor for [**SZ**] who deserves full credit for the double pun in our title based on [**D2**] and [**Go**].

THEOREM 1.1. *Let r and s be fixed integers with $1 \leqq r \neq s \leqq n$ and $n \geqq 2$. Then*

$$\left[\frac{x_r}{x_s}\right] F_n(\mathbf{x};\mathbf{a}) = -\left(\frac{a_s}{1+\sigma_n(\mathbf{a})-a_s}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

THEOREM 1.2. *Let r, s, and t be distinct fixed integers with $1 \leqq r,s,t \leqq n$ and $n \geqq 3$. Then*

$$\left[\frac{x_r^2}{x_s x_t}\right] F_n(\mathbf{x};\mathbf{a}) = \left(\frac{a_s a_t\Big((1+\sigma_n(\mathbf{a}))+(1+\sigma_n(\mathbf{a})-a_s-a_t)\Big)}{(1+\sigma_n(\mathbf{a})-a_s-a_t)(1+\sigma_n(\mathbf{a})-a_s)(1+\sigma_n(\mathbf{a})-a_t)}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

THEOREM 1.3. *Let r, s, t, and u be distinct fixed integers with $1 \leqq r,s,t,u \leqq n$ and $n \geqq 4$. Then*

$$\left[\frac{x_r x_s}{x_t x_u}\right] F_n(\mathbf{x};\mathbf{a}) = \left(\frac{a_t a_u\Big((1+\sigma_n(\mathbf{a}))+(1+\sigma_n(\mathbf{a})-a_t-a_u)\Big)}{(1+\sigma_n(\mathbf{a})-a_t-a_u)(1+\sigma_n(\mathbf{a})-a_t)(1+\sigma_n(\mathbf{a})-a_u)}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

2. q-analogs of Theorems 1.1–1.3

Given that the Dyson conjecture has such a natural q-analog, it seemed reasonable to look for comparable q-analogs of Theorems 1.1–1.3.

2.1. Statements of the conjectures.

CONJECTURE 2.1 (q-analog of Theorem 1.1). *Let r and s be fixed integers with $1 \leqq r \neq s \leqq n$ and $n \geqq 2$. Then*

$$[x_r/x_s]\mathcal{F}_n(\mathbf{x};\mathbf{a};q) = -q^{L(r,s)}\left(\frac{[a_s]_q}{[1+\sigma_n(\mathbf{a})-a_s]_q}\right)\frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q![a_2]_q!\cdots[a_n]_q!},$$

where

$$L(r,s) = \begin{cases} 1+\sigma_n(\mathbf{a})-\sum_{k=r}^{s} a_k, & \text{if } r<s \\ \sum_{k=s+1}^{r-1} a_k, & \text{if } r>s. \end{cases}$$

CONJECTURE 2.2 (q-analog of Theorem 1.2). *Let r, s, and t be distinct fixed integers with $1 \leqq r,s,t \leqq n$ and $n \geqq 3$. Without loss of generality we may assume that $s<t$. Then*

$$\left[\frac{x_r^2}{x_s x_t}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q) = q^{L(r,s,t)}\left(\frac{[a_s]_q[a_t]_q\Big([1+\sigma_n(\mathbf{a})]_q+q^{M(r,s,t)}[1+\sigma_n(\mathbf{a})-a_s-a_t]_q\Big)}{[1+\sigma_n(\mathbf{a})-a_s-a_t]_q[1+\sigma_n(\mathbf{a})-a_s]_q[1+\sigma_n(\mathbf{a})-a_t]_q}\right) \times \frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q![a_2]_q!\cdots[a_n]_q!},$$

where

$$L(r,s,t) = \begin{cases} 2+2\sigma_n(\mathbf{a}) - 2\sum_{k=r}^{t} a_k + \sum_{k=s+1}^{t-1} a_k, & \textit{if } r<s<t,\\ 1+\sigma_n(\mathbf{a}) - \sum_{k=s}^{t} a_k + 2\sum_{k=s+1}^{r-1} a_k, & \textit{if } s<r<t,\\ 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=s+1}^{t-1} a_k, & \textit{if } s<t<r, \end{cases}$$

and

$$M(r,s,t) = \begin{cases} a_t, & \textit{if } r<s<t \textit{ or } s<t<r,\\ a_s, & \textit{if } s<r<t. \end{cases}$$

CONJECTURE 2.3 (q-analog of Theorem 1.3). *Let r, s, t and u be distinct fixed integers with $1 \leqq r,s,t,u \leqq n$ and $n \geqq 4$. Without loss of generality we may assume that $r<s$ and $t<u$. Then*

$$\begin{aligned} &\left[\frac{x_r x_s}{x_t x_u}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q)\\ &= q^{L(r,s,t,u)} \left(\frac{[a_t]_q [a_u]_q \Big([1+\sigma_n(\mathbf{a})]_q) + q^{M(r,s,t,u)} [1+\sigma_n(\mathbf{a}) - a_t - a_u]_q \Big)}{[1+\sigma_n(\mathbf{a}) - a_t - a_u]_q [1+\sigma_n(\mathbf{a}) - a_t]_q [1+\sigma_n(\mathbf{a}) - a_u]_q} \right)\\ &\qquad\qquad \times \frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q! [a_2]_q! \cdots [a_n]_q!}, \end{aligned}$$

where

$$\begin{aligned} &L(r,s,t,u)\\ &= \begin{cases} 2+2\sigma_n(\mathbf{a}) - 2\sum_{k=r}^{u} a_k + \sum_{k=r}^{s-1} a_k + \sum_{k=t+1}^{u-1} a_k, & \textit{if } r<s<t<u,\\ 1+\sigma_n(\mathbf{a}) - \sum_{k=r}^{u} a_k + \sum_{k=t+1}^{s-1} a_k, & \textit{if } r<t<s<u,\\ 1+\sigma_n(\mathbf{a}) - \sum_{k=r}^{s-1} a_k + 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=t+1}^{u-1} a_k & \\ \qquad +2\sum_{k=u+1}^{s-1} a_k, & \textit{if } r<t<u<s,\\ 1+\sigma_n(\mathbf{a}) - \sum_{k=t}^{u} a_k + \sum_{k=r}^{s-1} a_k + 2\sum_{k=t+1}^{r-1} a_k, & \textit{if } t<r<s<u,\\ \sum_{k=t+1}^{r-1} a_k + \sum_{k=u+1}^{s-1} a_k, & \textit{if } t<r<u<s,\\ \sum_{k=r}^{s-1} a_k + \sum_{k=t+1}^{u-1} a_k + 2\sum_{k=u+1}^{r-1} a_k, & \textit{if } t<u<r<s, \end{cases} \end{aligned}$$

and

$$M(r,s,t,u) = \begin{cases} a_u, & \textit{if } r<s<t<u \textit{ or } r<t<u<s \textit{ or } t<u<r<s,\\ 1+\sigma_n(\mathbf{a}) & \textit{if } r<t<s<u \textit{ or } t<r<u<s,\\ a_t, & \textit{if } t<r<s<u. \end{cases}$$

2.2. How the conjectures were formed. Conjecture 2.1 was found first since it is the simplest. It is straightforward to program a Maple procedure which extracts the coefficient of x_r/x_s of $\mathcal{F}_n(\mathbf{x};\mathbf{a};q)$ for specific values of n, r, s, $a_1, a_2, \ldots, a_n$, and to divide out the multinomial coefficient from the resulting q-expression. Furthermore, the `qfactor` procedure in Frank Garvan's `qseries.m` Maple package [**Ga**] was helpful for putting the result in a tractable form. For $n=3$ and 4 and various small values of $a_1, \ldots, a_n$, it became clear that to move from Theorem 1.1 to its q-analog, all that was necessary was to replace each factor z by $[z]_q$, and multiply the resulting expression by q^L, where L was an (as yet unknown) function of the a_i's that depended on r and s. Upon examining the data, I was led to the working hypothesis that L was *piecewise linear* in the a_i's with different pieces arising from some condition on r and s.

At this point, I began to create the "`qDysonConj`" package [**Si2**]. I programmed the "`Conj1m1`" procedure in Maple, which takes as input the ordered pair (r, s) and n, and finds the linear function

$$L(r,s) = \lambda_0 + \sum_{i=1}^{n} \lambda_i a_i$$

which fits the internally generated data.

(The name "Conj1m1" is meant to suggest that we wish to conjecture the missing exponent L for the coefficient of $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ in the q-Dyson product (for a specific n) where one of the b_i is 1, one of the b_i is -1 and the rest are zero.)

The idea behind the `Conj1m1` is quite simple. Based on the assumption

$$\left[\frac{x_r}{x_s}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q) = -q^{\lambda_0 + a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n} \left(\frac{[a_s]_q}{[1+\sigma_n(\mathbf{a}) - a_s]_q}\right) \frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q! \cdots [a_n]_q!},$$

the `Conj1m1` procedure, for a given r, s, and n effectively computes

$$\log_q \left(\frac{[x_r/x_s]\mathcal{F}_n(\mathbf{x};\mathbf{a};q)}{-\left(\frac{[a_s]_q}{[1+\sigma_n(\mathbf{a})-a_s]_q}\right)\frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q!\cdots[a_n]_q!}} \right)$$

for $n+1$ linearly independent values of the vector $\mathbf{a}$ and solves the resulting linear system for $\lambda_0, \lambda_1, \ldots, \lambda_n$.

Let us recreate a Maple session to guess $L(r, s)$ using the case $n = 6$.

```
>  read "qDysonConj";
              Generalized qDyson conjecture package
                          by A.V. Sills
               Enter 'ez()' for a list of procedures
> C:=combinat[permute](6,2);
C := [[1, 2], [1, 3], [1, 4], [1, 5], [1, 6], [2, 1], [2, 3], [2, 4],
[2, 5], [2, 6], [3, 1], [3, 2], [3, 4], [3, 5], [3, 6], [4, 1], [4, 2],
[4, 3], [4, 5], [4, 6], [5, 1], [5, 2], [5, 3], [5, 4], [5, 6], [6, 1],
[6, 2], [6, 3], [6, 4], [6, 5]]
```

In order to have Maple run the `Conj1m1` procedure on all ordered pairs (r, s) with $1 \leqq r \neq s \leqq 6$, we use the built-in `combinat[permute]` procedure, and have Maple loop through all 30 permutations of length 2 on the set $\{1, 2, 3, 4, 5, 6\}$.

```
> for k from 1 to nops(C) do Conj1m1( op(C[k]), 6) od;
                    [1, 2], 1 + a3 + a4 + a5 + a6
                      [1, 3], 1 + a4 + a5 + a6
                        [1, 4], 1 + a5 + a6
                           [1, 5], 1 + a6
                             [1, 6], 1
                             [2, 1], 0
                   [2, 3], 1 + a1 + a4 + a5 + a6
                      [2, 4], 1 + a1 + a5 + a6
                        [2, 5], 1 + a1 + a6
                           [2, 6], 1 + a1
                             [3, 1], a2
                             [3, 2], 0
```

```
[3, 4], 1 + a1 + a2 + a5 + a6
[3, 5], 1 + a1 + a2 + a6
[3, 6], 1 + a1 + a2
[4, 1], a2 + a3
[4, 2], a3
[4, 3], 0
[4, 5], 1 + a1 + a2 + a3 + a6
[4, 6], 1 + a1 + a2 + a3
[5, 1], a2 + a3 + a4
[5, 2], a3 + a4
[5, 3], a4
[5, 4], 0
[5, 6], 1 + a1 + a2 + a3 + a4
[6, 1], a2 + a3 + a4 + a5
[6, 2], a3 + a4 + a5
[6, 3], a4 + a5
[6, 4], a5
[6, 5], 0
```

The above output shows the conjectured form of $q^{L(r,s)}$ for each of the thirty possible values of (r,s) in the case $n=6$. Notice that when $r<s$,

$$L(r,s) = 1 + \sum_{i\in\{1,2,3,4,5,6\}\setminus\{r,r+1,\dots,s\}} a_i,$$

while if $r>s$,

$$L(r,s) = \sum_{i\in\{1,2,3,4,5,6\}\setminus\{s,s+1,\dots,r\}} a_i.$$

The data above, combined with the analogous data for many different values of n led me to conjecture $L(r,s)$ as given in Conjecture 2.1.

Once I had Conjecture 2.1, it seemed reasonable to guess that the q-analog of Theorem 1.2 would have an analogous form, noting that this time the expression broke down neatly into a sum of two terms. I guessed that each of the two terms included a factor of the form q^L, where again, L is a piecewise linear function of the a_i's that depended on r, s, and t; piecewise according to the ordering of r, s, and t from smallest to largest. This time I programmed the `Conj2m1m1` procedure which works similarly to the `Conj1m1` procedure except that now two piecewise linear functions must be found simultaneously. By extracting the coefficient of $x_r^2/x_s x_t$ from the expanded q-Dyson product for a given r, s, t, and n, and dividing through by

$$\left(\frac{[a_s]_q[a_t]_q}{(1-q)[1+\sigma_n(\mathbf{a})-a_s-a_t]_q[1+\sigma_n(\mathbf{a})-a_s]_q[1+\sigma_n(\mathbf{a})-a_t]_q}\right)\frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q!\cdots[a_n]_q!},$$

what remains is a four-term polynomial in q which we presume to be of the form

$$q^L(1-q^{1+\sigma_n(\mathbf{a})}) + q^M(1-q^{1+\sigma_n(\mathbf{a})-a_s-a_t}).$$

Evaluating the above expression at $n+1$ linearly independent values of $\mathbf{a}$ allows us to conjecture L and M. Furthermore, the data revealed that inevitably $L<M$, so q^L was factored out front of the expression, and we renamed q^{M-L} by q^M. Conjecture 2.3 was obtained similarly.

3. Status of the conjectures

As of this writing, the conjectures remain open. Some twenty years ago, J. Stembridge [**St**, p. 347, Corollary 7.4], in a different context, proved that in the special case where $\mathbf{a} = \langle a, a, a, \dots, a\rangle$, and $b_{\rho+1} = b_{\rho_2} = \cdots = b_{\rho_\tau} = -1$, for ρ and τ satisfying $0 \leqq \rho \leqq n$ and $1 \leqq \tau \leqq n - \rho$,

(3.1)
$$[x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}]\mathcal{F}_n(\mathbf{x}; \mathbf{a}; q) = (-1)^\tau q^{b_1+b_2+\cdots+b_\rho+am} \frac{(q;q)_{an}(q^a;q^a)_\tau (q;q^a)_{\rho+\tau}}{(q;q)_a^n (q;q^a)_n},$$

where $m = \rho\tau + \sum_{i=1}^{\rho}(i-1)b_i - \sum_{i=1}^{n-\rho-\tau} ib_{n-i+1}$. One can check that Conjectures 2.1–2.3 do in fact agree with (3.1) in the instances where they overlap.

It would of course be natural to investigate whether either or both proofs of the q-Dyson conjecture ([**ZB**] and [**GX**]) could be adapted to prove Conjectures 2.1–2.3.

4. Possibilities for additional results

It is likely that tractable formulas for additional coëfficients in the Dyson and q-Dyson products exist. It seems quite plausible that the methods of this paper would be sufficient for finding such formulas. In particular, the next simplest case would likely be the coëfficient of $x_r x_s / x_t^2$ in the q-Dyson product, where r, s, and t are distinct integers between 1 and n, with $n \geqq 3$.

Note added in proof: After the submission of this paper, Lv, Xin, and Zhou announced a proof of Conjectures 2.1–2.3 in "A family of q-Dyson style constant term identities," arXiv:0706.1009.

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