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A Complete Characterisation of Vertex-multiplications of Trees with Diameter 5

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Abstract

For a connected graph G , let $\mathcal{D}(G)$ be the family of strong orientations of G ; and for any $D \in \mathcal{D}(G)$, we denote by $d(D)$ the diameter of D . The *orientation number* of G is defined as $\bar{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}$. In 2000, Koh and Tay introduced a new family of graphs, G vertex-multiplications, and extended the results on the orientation number of complete n -partite graphs. Suppose G has the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For any sequence of n positive integers (s_i) , a G *vertex-multiplication*, denoted by $G(s_1, s_2, \dots, s_n)$, is the graph with vertex set $V^* = \bigcup_{i=1}^n V_i$ and edge set E^* , where V_i 's are pairwise disjoint sets with $|V_i| = s_i$, for $i = 1, 2, \dots, n$; and for any $u, v \in V^*$, $uv \in E^*$ if and only if $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. They proved a fundamental classification of G vertex-multiplications, with $s_i \geq 2$ for all $i = 1, 2, \dots, n$, into three classes $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$, and any vertex-multiplication of a tree with diameter at least 3 does not belong to the class $\mathcal{C}_2$. Furthermore, some necessary and sufficient conditions for $\mathcal{C}_0$ were established for vertex-multiplications of trees with diameter 5. In this paper, we give a complete characterisation of vertex-multiplications of trees with diameter 5 in $\mathcal{C}_0$ and $\mathcal{C}_1$.

Keywords: optimal orientation, orientation number, vertex-multiplication, tree.
MSC 2020: 05C12, 05C20.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only graphs with no loops or parallel edges. For any vertices $v, x \in V(G)$, the *distance* from v to x , $d_G(v, x)$, is defined as the length of a shortest path from v to x . For $v \in V(G)$, its *eccentricity* $e_G(v)$ is defined as $e_G(v) = \max\{d_G(v, x) \mid x \in V(G)\}$. The *diameter* of G , denoted by $d(G)$, is defined as $d(G) = \max\{e_G(v) \mid v \in V(G)\}$ while the *radius* of G , denoted by $r(G)$, is defined as $r(G) = \min\{e_G(v) \mid v \in V(G)\}$. The above notions are defined similarly for a digraph D ; and we refer the reader to [1] for any undefined terminology. For a digraph D , a vertex x is said to be *reachable* from another vertex v if $d_D(v, x) < \infty$. The *outset* and *inset* of a vertex $v \in V(D)$ are defined to be $O_D(v) = \{x \in V(D) \mid v \rightarrow x\}$ and $I_D(v) = \{y \in V(D) \mid y \rightarrow v\}$ respectively. If there is no ambiguity, we shall omit the subscript for the above notation.

An *orientation* D of a graph G is a digraph obtained from G by assigning a direction to every edge $e \in E(G)$. An orientation D of G is said to be *strong* if every two vertices in $V(D)$ are mutually reachable. An edge $e \in E(G)$ is a *bridge* if $G - e$ is disconnected. Robbins' well-known One-way Street Theorem [12] states the following.

Theorem 1.1. (Robbins [12]) *Let G be a connected graph. Then, G has a strong orientation if and only if G is bridgeless.*

Given a connected and bridgeless graph G , let $\mathcal{D}(G)$ be the family of strong orientations of G . The *orientation number* of G is defined as

$$\bar{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}.$$

Any orientation D in $\mathcal{D}(G)$ with $d(D) = \bar{d}(G)$ is called an *optimal orientation* of G . The general problem of finding the orientation number of a connected and bridgeless graph is very difficult. Moreover, Chvátal and Thomassen [3] proved that it is NP-hard to

determine if a graph admits an orientation of diameter 2. Hence, it is natural to focus on special classes of graphs. The orientation number was evaluated for various classes of graphs, such as the complete graphs [2, 9, 11] and complete bipartite graphs [4, 13].

In 2000, Koh and Tay [7] introduced a new family of graphs, G vertex-multiplications, and extended the results on complete n -partite graphs. Let G be a given connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For any sequence of n positive integers (s_i) , a G vertex-multiplication, denoted by $G(s_1, s_2, \dots, s_n)$, is the graph with vertex set $V^* = \bigcup_{i=1}^n V_i$ and edge set E^* , where V_i 's are pairwise disjoint sets with $|V_i| = s_i$, for $i = 1, 2, \dots, n$; and for any $u, v \in V^*$, $uv \in E^*$ if and only if $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. For instance, if $G \cong K_n$, then the graph $G(s_1, s_2, \dots, s_n)$ is a complete n -partite graph with partite sizes $s_1, s_2, \dots, s_n$. Also, we say G is a *parent graph* of $G(s_1, s_2, \dots, s_n)$.

For $i = 1, 2, \dots, n$, we denote the x -th vertex in V_i by (x, v_i) , i.e., $V_i = \{(x, v_i) \mid x = 1, 2, \dots, s_i\}$. Hence, two vertices (x, v_i) and (y, v_j) in V^* are adjacent in $G(s_1, s_2, \dots, s_n)$ if and only if $i \neq j$ and $v_i v_j \in E(G)$. For convenience, we write $G^{(s)}$ in place of $G(s, s, \dots, s)$ for any positive integer s , and it is understood that the number of s 's is equal to the order of G , n . Thus, $G^{(1)}$ is simply the graph G itself.

The G vertex-multiplications is a natural generalisation of complete multipartite graphs. Optimal orientations minimising the diameter can also be used to solve a variant of the Gossip Problem on a graph G . The Gossip Problem attributed to Boyd by Hajnal et al. [6] is stated as follows:

“There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?”

The Gossip Problem has been the source of many papers that have studied the spread of information by telephone calls, conference calls, letters and computer networks. One can imagine a network of people modelled by a G vertex-multiplication where the underlying graph is G and persons within a partite set are not allowed to communicate directly with each other, for perhaps secrecy or disease containment reasons.

The following theorem by Koh and Tay [7] provides a fundamental classification on G vertex-multiplications.

Theorem 1.2. (Koh and Tay [7]) *Let G be a connected graph of order $n \geq 3$. If $s_i \geq 2$ for $i = 1, 2, \dots, n$, then $d(G) \leq \bar{d}(G(s_1, s_2, \dots, s_n)) \leq d(G) + 2$.*

In view of Theorem 1.2, all graphs of the form $G(s_1, s_2, \dots, s_n)$, with $s_i \geq 2$ for all $i = 1, 2, \dots, n$, can be classified into three classes $\mathcal{C}_j$, where

$$\mathcal{C}_j = \{G(s_1, s_2, \dots, s_n) \mid \bar{d}(G(s_1, s_2, \dots, s_n)) = d(G) + j\},$$

for $j = 0, 1, 2$. Henceforth, we assume $s_i \geq 2$ for $i = 1, 2, \dots, n$, unless otherwise stated. The following lemma was found useful in proving Theorem 1.2.

Lemma 1.3. (Koh and Tay [7]) *Let s_i, t_i be integers such that $s_i \leq t_i$ for $i = 1, 2, \dots, n$. If the graph $G(s_1, s_2, \dots, s_n)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m , then $\bar{d}(G(t_1, t_2, \dots, t_n)) \leq \max\{m, d(F)\}$.*

Theorem 1.2 was generalised to digraphs by Gutin et al. [5]. Koh and Tay [8] further investigated vertex-multiplications of trees. Since trees with diameter at most 2 are parent graphs of complete bipartite graphs, which are completely solved, they considered trees of diameter at least 3. They proved that vertex-multiplications of trees with diameter at least 3 do not belong to the class $\mathcal{C}_2$.

Theorem 1.4. (Koh and Tay [8]) *If T is a tree of order n and $3 \leq d(T) \leq 5$ ($d(T) \geq 6$ resp.), then $T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0 \cup \mathcal{C}_1$ ($\mathcal{C}_0$ resp.).*

Wong and Tay [14] almost completely characterised vertex-multiplications of trees with diameter 4. On the other hand, Ng and Koh [10] examined vertex-multiplications of cycles. To continue the discussion, we need some notation.

From here onwards, we consider a tree T of diameter 5 with vertex set $V(T) = \{v_1, v_2, \dots, v_n\}$. We let v_1 and v_2 be the two central vertices of T , i.e., $e_T(v_k) = r(T) = 3$ for $k = 1, 2$. For $k = 1, 2$, denote the neighbours of v_k , excluding v_{3-k} , by $[i]_k$, i.e., $N_T(v_k) - \{v_{3-k}\} = \{[i]_k \mid i = 1, 2, \dots, \deg_T(v_k) - 1\}$. For each $i = 1, 2, \dots, \deg_T(v_k) - 1$, we denote the neighbours of $[i]_k$, excluding v_k itself, by $[\alpha, i]_k$, i.e., $N_T([i]_k) - \{v_k\} = \{[\alpha, i]_k \mid \alpha = 1, 2, \dots, \deg_T([i]_k) - 1\}$. Also, we denote the set of neighbours $[i]_k$ of v_k which are not end-vertices to be $E'_k = \{[i]_k \in V(T) \mid 1 \leq i \leq \deg_T(v_k) - 1 \text{ and } [i]_k \text{ is not an end-vertex}\}$ for $k = 1, 2$. Of course, $|E'_k| \geq 1$, where $k = 1, 2$, for any tree T with $d(T) = 5$.

Example 1.5. Let T be the tree of diameter 5 shown in Figure 1. We label the vertices as described above. Note also $E'_1 = \{[1]_1, [2]_1\}$ and $E'_2 = \{[2]_2\}$.

Figure 1: Labelling vertices in T .

In the vertex-multiplication graph $G = T(s_1, s_2, \dots, s_n)$ of T , the integer s_i corresponds to the vertex v_i , $i = 1, 2, \dots, n$. We will loosely use the two denotations of a vertex, for example, if $v_i = [j]_k$, then $s_i = s_{[j]_k}$.

In the following theorem, Koh and Tay [8] proved some necessary and sufficient conditions for $T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0$, where T is a tree of diameter 5.

Theorem 1.6. (Koh and Tay [8]) *Let T be a tree with diameter 5 and its central vertices be v_1 and v_2 . Suppose $A = \{x \in V(T) \mid d_T(x, u) = 5 = d_T(x, v) \text{ for some } u, v \in V(T), u \neq v\}$. Then,*

- (a) $T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0 \cup \mathcal{C}_1$.
- (b) If $|A| \leq 1$, then $T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0$.
- (c) If $\deg_T(v) \leq 2$ for all $v \notin \{v_1, v_2\}$ and $|A| \geq 2$, then $T^{(2)} \in \mathcal{C}_1$.

In this paper, we give a complete characterisation of vertex-multiplications of trees with diameter 5 in $\mathcal{C}_0$ and $\mathcal{C}_1$.

Theorem 1.7. *Let $G = T(s_1, s_2, \dots, s_n)$ with $d(T) = 5$, and $m_k = \min\{s_{[i]_k} \mid [i]_k \in E'_k\}$ for $k = 1, 2$.*

Table 1: Summary for $T(s_1, s_2, \dots, s_n)$ with $d(T) = 5$.

Underlying conditions	$G \in \mathcal{C}_0$ or $\mathcal{C}_1$?	Reference
$s_i \geq 3$ for some $i = 1, 2$	$G \in \mathcal{C}_0$.	Proposition 2.1
$m_1 \geq 4$ and $m_2 \geq 4$	$G \in \mathcal{C}_0$.	Proposition 2.5
$s_1 = s_2 = 2$, and $2 \leq m_i \leq 3$ for some $i = 1, 2$	$G \in \mathcal{C}_0 \iff E'_k = 1$ for some $k = 1, 2$.	Proposition 2.4

2. Proof of Theorem 1.7

For convenience, we shall introduce some notation. Let D be an orientation of $T(s_1, s_2, \dots, s_n)$ with $s_i \geq 2$ for $1 \leq i \leq n$. If v_p and v_q , $1 \leq p, q \leq n$ and $p \neq q$, are adjacent vertices in T , then for each i , $1 \leq i \leq s_p$, we denote by $O_D^{v_q}((i, v_p)) = \{(j, v_q) \mid (i, v_p) \rightarrow (j, v_q), 1 \leq j \leq s_q\}$ and $I_D^{v_q}((i, v_p)) = \{(j, v_q) \mid (j, v_q) \rightarrow (i, v_p), 1 \leq j \leq s_q\}$. If there is no ambiguity, we shall omit the subscript D for the above notation.

Proposition 2.1. *Let T be a tree of diameter 5. If $s_1 \geq 3$ or $s_2 \geq 3$, then $G = T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0$.*

Proof: Without loss of generality, assume $s_1 \geq 3$. Let H be a subgraph of G , where $s_1 = 3$ and $s_i = 2$ for all $i \neq 1$ in H . Define an orientation D of H as follows (see Figure 2).

$$\begin{aligned} (2, [i]_1) &\rightarrow \{(1, [\alpha, i]_1), (2, [\alpha, i]_1)\} \rightarrow (1, [i]_1), \\ (2, v_1) &\rightarrow (1, [i]_1) \rightarrow \{(1, v_1), (3, v_1)\}, \text{ and} \\ \{(1, v_1), (2, v_1)\} &\rightarrow (2, [i]_1) \rightarrow (3, v_1), \end{aligned}$$

for all $i = 1, 2, \dots, \deg_T(v_1) - 1$ and all $\alpha = 1, 2, \dots, \deg_T([i]_1) - 1$.

$$\begin{aligned} (2, [j]_2) &\rightarrow \{(1, [\beta, j]_2), (2, [\beta, j]_2)\} \rightarrow (1, [j]_2), \text{ and} \\ (2, [j]_2) &\rightarrow (1, v_2) \rightarrow (1, [j]_2) \rightarrow (2, v_2) \rightarrow (2, [j]_2), \end{aligned}$$

for all $j = 1, 2, \dots, \deg_T(v_2) - 1$ and all $\beta = 1, 2, \dots, \deg_T([j]_2) - 1$. Also,

$$\begin{aligned} \{(1, v_1), (3, v_1)\} &\rightarrow (1, v_2) \rightarrow (2, v_1), \text{ and} \\ (3, v_1) &\rightarrow (2, v_2) \rightarrow \{(1, v_1), (2, v_1)\}. \end{aligned}$$

We emphasize that the above orientation applies to the vertices $[i]_1$ and $[j]_2$ even if they are end-vertices. It suffices to verify $d(D) = 5$ and every vertex in $V(D)$ lies in a directed C_4 , where D is as shown in Figure 2. Then, by Lemma 1.3, $\bar{d}(G) \leq \max\{d(D), 4\} = 5$. $\square$

Figure 2: Orientation D for H .

The next two lemmas will be found useful in proving our propositions to come.

Lemma 2.2. *Let D be an orientation of $T(s_1, s_2, \dots, s_n)$ with $s_1 = s_2 = 2$. If $d(D) = 5$, then*

- (i) $|O^{v_1}((p, [i]_1))| \geq 1$ and $|I^{v_1}((p, [i]_1))| \geq 1$ for all $i = 1, 2, \dots, \deg_T(v_1) - 1$ and $p = 1, 2, \dots, s_{[i]_1}$, and
- (ii) $|O^{v_2}((q, [j]_2))| \geq 1$ and $|I^{v_2}((q, [j]_2))| \geq 1$ for all $j = 1, 2, \dots, \deg_T(v_2) - 1$ and $q = 1, 2, \dots, s_{[j]_2}$.

Proof: (i) follows from the fact that $d_D((p, [i]_1), (1, [1, j]_2)) = d_D((1, [1, j]_2), (p, [1]_1)) = 4$, where $[j]_2$ is not an end-vertex in T . By symmetry, (ii) follows from (i). $\square$

Lemma 2.3. (Duality) *Let D be an orientation of a graph G . Let $\tilde{D}$ be the orientation of G such that $uv \in E(\tilde{D})$ if and only if $vu \in E(D)$. Then, $d(\tilde{D}) = d(D)$.*

Proof: Suppose not. Then, there exist some vertices $u, v \in V(\tilde{D})$ such that $d_{\tilde{D}}(u, v) > d(D)$. By definition of $\tilde{D}$, $d_D(v, u) = d_{\tilde{D}}(u, v)$. It follows that $d_D(v, u) > d(D)$, a contradiction. $\square$

Proposition 2.4. *Let T be a tree of diameter 5 and $m_k = \min\{s_{[i]_k} \mid [i]_k \in E'_k\}$ for $k = 1, 2$. Suppose $G = T(s_1, s_2, \dots, s_n)$ satisfies $2 \leq m_1 \leq 3$ or $2 \leq m_2 \leq 3$, and $s_1 = s_2 = 2$. Then,*

$$G \in \mathcal{C}_0 \iff |E'_k| = 1 \text{ for some } k = 1, 2.$$

Proof: ($\Rightarrow$) Suppose $|E'_k| \geq 2$ for all $k = 1, 2$. Without loss of generality, assume $2 \leq m_1 \leq 3$, and $m_k = s_{[1]_k}$ where $[1]_k \in E'_k$ for $k = 1, 2$. Since $G \in \mathcal{C}_0$, there exists some orientation D of G with $d(D) = 5$.

Case 1. $|I^{v_2}((p, v_1))| = 2$ for some $p = 1, 2$.

Without loss of generality, assume $\{(1, v_2), (2, v_2)\} \rightarrow (2, v_1)$. Since for all $i = 1, 2, \dots, \deg_T(v_1) - 1$, and $q = 1, 2, \dots, s_{[i]_1}$, $d_D((q, [i]_1), (1, [1, 1]_2)) \leq 5$ and by Lemma 2.2, we have $(2, v_1) \rightarrow (q, [i]_1) \rightarrow (1, v_1)$. Then, $d_D((1, [1, 1]_1), (1, [1, i]_1)) > 5$ for any $[i]_1 \in E'_1$, $i \neq 1$, a contradiction.

Case 2. $|O^{v_2}((p, v_1))| = 2$ for some $p = 1, 2$.

By Duality Lemma and Case 1, this case follows.

Case 3. $|I^{v_1}((p, v_2))| = |I^{v_2}((p, v_1))| = 1$ for all $p = 1, 2$.

Without loss of generality, assume $(1, v_1) \rightarrow (1, v_2) \rightarrow (2, v_1) \rightarrow (2, v_2) \rightarrow (1, v_1)$.

Subcase 3.1. $|O((1, [1, 1]_1))| = 1$.

Assume without loss of generality that $O((1, [1, 1]_1)) = \{(1, [1]_1)\}$. By Lemma 2.2, $|O^{v_1}((1, [1]_1))| = 1$. By symmetry, we may assume $O^{v_1}((1, [1]_1)) = \{(1, v_1)\}$. Since for all $i = 1, 2, \dots, \deg_T(v_2) - 1$, and $q = 1, 2, \dots, s_{[i]_2}$, $d_D((1, [1, 1]_1), (q, [i]_2)) \leq 5$ and by Lemma 2.2, we have $(1, v_2) \rightarrow (q, [i]_2) \rightarrow (2, v_2)$. However, $d_D((1, [1, 1]_2), (1, [1, j]_2)) > 5$ for any $[j]_2 \in E'_2$, $j \neq 1$, a contradiction.

Subcase 3.2. $|I((1, [1, 1]_1))| = 1$.

By Duality Lemma and Subcase 3.1, this subcase also results in a contradiction.

($\Leftarrow$) Assume without loss of generality that $|E'_1| = 1$. Let H be a subgraph of G , where $s_i = 2$ for all i in H . Define an orientation D of H as follows (see Figure 3).

$$(1, [\alpha, 1]_1) \rightarrow (1, [1]_1) \rightarrow (2, [\alpha, 1]_1) \rightarrow (2, [1]_1) \rightarrow (1, [\alpha, 1]_1), \text{ and} \\ \{(1, [i]_1), (2, [i]_1)\} \rightarrow (1, v_1) \rightarrow \{(1, v_2), (2, v_2)\} \rightarrow (2, v_1) \rightarrow \{(1, [i]_1), (2, [i]_1)\},$$

for all $i = 1, 2, \dots, \deg_T(v_1) - 1$ and $\alpha = 1, 2, \dots, \deg_T([1]_1) - 1$.

$$(2, [j]_2) \rightarrow \{(1, [\beta, j]_2), (2, [\beta, j]_2)\} \rightarrow (1, [j]_2), \text{ and} \\ (1, [j]_2) \rightarrow (1, v_2) \rightarrow (2, [j]_2) \rightarrow (2, v_2) \rightarrow (1, [j]_2),$$

for all $j = 1, 2, \dots, \deg_T(v_2) - 1$ and $\beta = 1, 2, \dots, \deg_T([j]_2) - 1$.

We emphasize that the above orientation applies to the vertices $[i]_1$ and $[j]_2$ even if they are end-vertices. It suffices to verify $d(D) = 5$ and every vertex in $V(D)$ lies in a directed C_4 , where D is as shown in Figure 3. Then, by Lemma 1.3, $\bar{d}(G) \leq \max\{d(D), 4\} = 5$. $\square$

Figure 3: Orientation D for H .

Proposition 2.5. Let T be a tree of diameter 5 and $m_k = \min\{s_{[i]_k} \mid [i]_k \in E'_k\}$ for $k = 1, 2$. If $m_1 \geq 4$ and $m_2 \geq 4$, then $G = T(s_1, s_2, \dots, s_n) \in \mathcal{C}_0$.

Proof: Let H be a subgraph of G , where $s_{[i]_k} = 4$ for all $[i]_k \in E'_k$, $k = 1, 2$, and $s_j = 2$ otherwise. Define an orientation D of H as follows (see Figure 4). For $k = 1, 2$,

$$\begin{aligned} \{(3, [i]_k), (4, [i]_k)\} &\rightarrow \{(1, [\alpha, i]_k), (2, [\alpha, i]_k)\} \rightarrow \{(1, [i]_k), (2, [i]_k)\}, \\ (2, v_k) &\rightarrow \{(1, [i]_k), (3, [i]_k)\} \rightarrow (1, v_k) \rightarrow \{(2, [i]_k), (4, [i]_k)\} \rightarrow (2, v_k), \text{ and} \\ (1, v_1) &\rightarrow (1, v_2) \rightarrow (2, v_1) \rightarrow (2, v_2) \rightarrow (1, v_1), \end{aligned}$$

for all $[i]_k \in E'_k$ and $\alpha = 1, 2, \dots, \deg_T([i]_k) - 1$; and

$$(2, [j]_k) \rightarrow (1, v_k) \rightarrow (1, [j]_k) \rightarrow (2, v_k) \rightarrow (2, [j]_k),$$

for all $[j]_k \notin E'_k$.

We emphasize that the above orientation applies to the vertices $[i]_1$ and $[j]_2$ even if they are end-vertices. Note that for any $p, q, r, k = 1, 2$; $1 \leq i, j \leq \deg_T(v_k)$, $i \neq j$, $\alpha = 1, 2, \dots, \deg_T([i]_k) - 1$, and $\beta = 1, 2, \dots, \deg_T([j]_k) - 1$, we have $d_D((p, [\alpha, i]_k), (q, v_k)) = 2$ and $d_D((q, v_k), (r, [\beta, j]_k)) = 2$ so that $d_D((p, [\alpha, i]_k), (r, [\beta, j]_k)) \leq d_D((p, [\alpha, i]_k), (q, v_k)) + d_D((q, v_k), (r, [\beta, j]_k)) = 4$. Similar considerations apply for cases involving the vertices $[j]_k \notin E'_k$. Furthermore, in view of the orientation's similarity, it suffices to verify $d(D) = 5$ and every vertex in $V(D)$ lies in a directed C_4 , where D is as shown in Figure 4. Hence, by Lemma 1.3, $\bar{d}(G) \leq \max\{d(D), 4\} = 5$. $\square$

Figure 4: Orientation D for H .

It remains open to characterise vertex-multiplications of trees with diameter 3 belonging to $\mathcal{C}_0$. We conclude the paper by proposing the following problem.

Problem 2.6. For trees T with $d(T) = 3$, characterise the tree vertex-multiplications $T(s_1, s_2, \dots, s_n)$ that belong to $\mathcal{C}_0$.

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References

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, algorithms and applications*, Springer Verlag, London, (2009).
- [2] F. Boesch and R. Tindell, Robbins's Theorem for mixed multigraphs, *Amer. Math. Monthly*, 87, (1980), 716–719.
- [3] V. Chvátal and C. Thomassen, Distances in orientations of graphs, *J. Combin. Theory Ser. B*, 24, (1978), 61–75.
- [4] G. Gutin, m -sources in complete multipartite digraphs, *Vestsi Acad. Navuk BSSR Ser. Fiz-Mat*, 5, (1989), 101–106 (in Russian).
- [5] G. Gutin, K.M. Koh, E.G. Tay and A. Yeo, Almost minimum diameter orientations of semicomplete multipartite and extended digraphs, *Graphs Combin.*, 18, (2002), 499–506.
- [6] A. Hajnal, E.C. Milner and E. Szemerédi, A cure for the telephone disease, *Canad. Math. Bull.*, 15, (1972), 447–450.

- [7] K.M. Koh and E.G. Tay, On optimal orientations of G vertex-multiplications, *Discrete Math.*, 219, (2000), 153–171.
- [8] K.M. Koh and E.G. Tay, On optimal orientations of tree vertex-multiplications, *Australas. J. Combin.*, 34, (2006), 69–87.
- [9] S.B. Maurer, The king chicken theorems, *Math. Mag.*, 53, (1980), 67–80.
- [10] K.L. Ng and K.M. Koh, On optimal orientation of cycle vertex multiplications, *Discrete Math.*, 297, (2005), 104–118.
- [11] J. Plesník, Diametrically critical tournaments, *Časopis pro pěstování matematiky*, 100, (1975), 361–370.
- [12] H.E. Robbins, A theorem on graphs, with an application to a problem of traffic control, *Amer. Math. Monthly*, 46, (1939), 281–283.
- [13] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, *Math. Slovaca*, 36, (1986), 289–296.
- [14] W.H.W. Wong and E.G. Tay, Application of some techniques in Sperner Theory: Optimal orientations of vertex-multiplications of trees with diameter 4, arXiv:2110.09003 [math.CO], (2021).