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A Note on Extreme Bernoulli and Dependent Families of Bivariate Distributions

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On Extreme Bernoulli and Dependent Families of Bivariate Distributions

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Abstract

The objective and purpose of this note is to generate bivariate distributions via extreme Bernoulli distributions and obtain results on positively and negatively dependent families of bivariate binomial distributions generated by extreme Bernoulli distributions. Some distributional properties and results are presented. The factorial moment generating functions, correlation functions, conditional distributions and the regression functions are given.

Mathematics Subject Classifications: 60E05; 60F99

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1 Introduction

Bernoulli distribution is an important distribution that arises in many applications in probability and statistics. There are several distributions and models that take into account the concept of dependence in the parameters for certain classes of distributions. Lancaster (1958), Hamdan and Jensen (1976), Takeuchi and Takemura (1987) obtained expressions for association (correlation) functions in the form of series bilinear in suitable orthogonal polynomials.

Oluyede (1994) studied a family of bivariate binomial distributions generated by extreme Bernoulli distributions, see also references therein, including results by Marshall and Olkin (1985) on bivariate distributions generated by the Bernoulli distribution. Hamdan (1972) provided applications of canonical expansion of the bivariate binomial distribution.

In this paper, we present new probability distributions generated by extreme Bernoulli distributions and discuss the properties of these distributions including the factorial moment generating functions, correlation functions, conditional distributions and the regression functions. In section 2, some basic results leading to the generated distributions are presented. In section 3, distributional results including their properties and some comparisons are presented. Section 4 offers useful results on families of positive and negative dependent distributions including some comparisons. In Section 5, a brief discussion and concluding remarks are given.

2 Extreme Bernoulli Distributions and Some Basic Results

Let $P = (p_{ij})$ and $Q = (q_{ij})$ be $r \times c$ probability matrices with

$$\sum_{i=1}^{r} p_{ij} = \sum_{i=1}^{r} q_{ij} \quad \text{for} \quad j = 1, 2, \ldots, c, \sum_{j=1}^{c} p_{ij} = \sum_{j=1}^{c} q_{ij} \quad \text{for} \quad i = 1, 2, ..., r.$$

Then P majorizes Q, denoted by $P >^m Q$ if $\sum_{h=1}^{k} p_{(h)} \geq \sum_{h=1}^{k} q_{(h)}$, $k = 1, 2, \ldots, rc - 1$, where $p_{(h)}$ and $q_{(h)}$, $h = 1, 2, \ldots, k$, are elements of P and Q respectively, arranged in a nondecreasing order.

Now let p_{11}, p_{10}, p_{01} and p_{00} be the probabilities of classification of individuals into one of four mutually exclusive classes $\overline{AB}$, $\overline{A}B$, $A\overline{B}$, AB ($\overline{A}$ and $\overline{B}$ denotes negation). An example will be to classify patients as diseased or healthy using standard diagnostic as opposed to surgical procedures. The number of occurrences of A and B, denoted by X and Y are random variables having the binomial distributions with parameters (n, p_{1+}) and (n, p_{+1}) respectively, where $p_{1+} = p_{11} + p_{10}$ and $p_{+1} = p_{11} + p_{01}$. The joint probability function of X and Y is given by Aitken and Gonin (1935). The distribution is given by

$$P(X = x, Y = y) = \sum_{s} \frac{n!}{s!(x-s)!(y-s)!(n-x-y+s!} p_{11}^{s} p_{10}^{x-s} p_{01}^{y-s} p_{00}^{n-x-y+s}, \tag{1}$$

for $x, y = 0, 1, 2, ..., n$. When sampling without replacement from a finite population, a bivariate hypergeometric distribution is obtained leading to a bivariate binomial distribution in the limit.

The extreme Bernoulli distributions can be displayed as follows

$$
\begin{aligned}
E_1 &= \begin{bmatrix} p_{1+} & 0 \\ p_{+1}-p_{1+} & 1-p_{+1} \end{bmatrix} \text{ if } \; p_{+1} \geq p_{1+}; \\
E_2 &= \begin{bmatrix} 0 & p_{1+} \\ p_{+1} & 1-p_{1+}-p_{+1} \end{bmatrix} \text{ if } \; p_{1+} \leq p_{+1} \leq \frac{1}{2}; \\
E_3 &= \begin{bmatrix} p_{+1} & p_{1+}-p_{+1} \\ 0 & 1-p_{1+} \end{bmatrix} \text{ if } \; p_{1+} \geq p_{+1};
\end{aligned}
$$

and

$$E_4 = \begin{bmatrix} p_{1+}+p_{+1}-1 & 1-p_{+1} \\ 1-p_{1+} & 0 \end{bmatrix} \text{ if } \frac{1}{2} \leq p_{+1} \leq p_{1+}. \tag{2}$$

The extreme Bernoulli distributions E_k, $k = 1, 2, 3, 4$ can be obtained as follows. Let (p_{ij}), $i = 0, 1$ $j = 0, 1$, be a probability matrix with $p_{1+} = p_{11}+p_{10}$, $p_{+1} = p_{11} + p_{01}$. If $p_{+1} \geq p_{1+}$, then $(p_{1+}, 0) >^m (p_{11}, p_{10})$ and $(p_{+1} - p_{1+}, 1 - p_{+1}) >^m (p_{01}, p_{00})$. Similarly, if $p_{1+} \geq p_{+1}$, then $(p_{+1}, p_{1+} - p_{+1}) >^m (p_{11}, p_{10})$ and $(1 - p_{1+}, 0) >^m (p_{01}, p_{00})$. The results now follows from the fact that $\max(0, p_{1+} + p_{1+} - 1) \leq p_{11} \leq \min(p_{1+}, p_{+1})$.

The bivariate distributions generated to E_k, $k = 1, 2, 3, 4$ are as follows. Suppose n independent drawings are made from these populations and let X_{ij}, $i, j = 0, 1$ be the number of occurrences of event whose cell probabilities are given by E_k, $k = 1, 2, 3, 4$.

For E_1,

$$P_{E_1}(X_{11} = n_1, X_{10} = n_2, X_{01} = n_3) = \frac{n! p_{1+}^{n_1} (p_{+1} - p_{1+})^{n_3} (1 - p_{+1})^{n-(n_1+n_3)}}{n_1! n_3! (n - (n_1 + n_3))!}, \tag{3}$$

where $n_i \geq 0$, $i = 1, 2, 3$, and $n_1 + n_2 + n_3 \leq n$.

With $X = X_{11} + X_{10}$ and $Y = X_{11} + X_{01}$, the joint distribution of X and Y is given by

$$P_{E_1}(X = x, Y = y) = \begin{cases} \frac{n! p_{1+}^x (p_{+1}-p_{1+})^{y-x} (1-p_{+1})^{n-y}}{x!(y-x)!(n-y)!}, & n \geq y \geq x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

For E_2,

$$P_{E_2}(X = x, Y = y) = \begin{cases} \frac{n! p_{1+}^x p_{+1}^y (1-p_{1+}-p_{+1})^{n-x-y}}{x! y! (n-x-y)!}, & \begin{array}{l} 0 \leq x, y \leq n \\ x + y \leq n \end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For E_3,

$$P_{E_3}(X = x, Y = y) = \begin{cases} \frac{n! p_{+1}^y (p_{1+}-p_{+1})^{x-y} (1-p_{1+})^{n-x}}{x!(x-y)!(n-x)!}, & n \geq x \geq y \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For E_4,

$$P_{E_4}(X=x, Y=y) = \begin{cases} \frac{n!(p_{1+}+p_{+1}-1)^x(1-p_{1+})^{n-y}(1-p_{+1})^{n-x}}{x!(n-y)!(n-x)!}, & 0 \le x, y \le n \\ & x+y \le n \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

We now examine obtain the regression functions $E(Y|X=x)$ and $E(X|Y=y)$ for the distributions given in (1) and those generated by E_1 and E_2 respectively. The regression functions for the distributions generated by E_3 and E_4 can be obtained similarly obtained.

For the distribution given by (1), after some simplification, the conditional distribution of X given $Y=y$ is

$$P(X|Y=y)$$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \binom{n-y}{x-j}\binom{y}{j}\left(\frac{p_{11}}{p_{+1}}\right)^j \left(\frac{p_{01}}{p_{+1}}\right)^{y-j} \left(\frac{p_{10}}{p_{+0}}\right)^{x-j} \left(\frac{p_{00}}{p_{+0}}\right)^{n-x-y+j}. \tag{8}$$

This is the distribution of the sum of two random variables. One distributed as binomial with parameters y and $\frac{p_{11}}{p_{+1}}$, and the other as binomial with parameters $n-y$ and $\frac{p_{10}}{p_{+0}}$. The regression function is given by

$$E(X|Y=y) = n\left(\frac{p_{10}}{p_{+0}}\right) + y\left(\frac{p_{11}}{p_{+1}} - \frac{p_{10}}{p_{+0}}\right). \tag{9}$$

For the bivariate distribution generated by E_1, the regression function is $E(X|Y=y) = y\left(\frac{p_{1+}}{p_{+1}}\right)$ and for E_2, the corresponding regression function is given by

$$E(Y|X=x) = (n-x)\left(\frac{p_{+1}}{1-p_{1+}}\right). \tag{10}$$

3 Some Distributions and Properties

In this section, we examine some distributions and their properties including those generated by $\alpha I + (1-\alpha)E_i$, $i = 1, 2, 3, 4$, $0 \le \alpha \le 1$, where I is the fourfold independence table with entries $p_{hk} = p_{h+}p_{+k}$, $h, k = 0, 1$. First, consider the convex combination of $P_{E_1}(x, y)$ and $P_{E_2}(x, y)$. The corresponding distribution $P_E(x, y)$ has the covariance function $Cov(X, Y) = np_{1+}(\alpha - p_{+1})$. The regression function is given by

$$E(Y|X=x) = \frac{n(p_{+1} - \alpha p_{+1})}{1-p_{1+}} + \frac{x(\alpha p_{1+} - p_{+1})}{1-p_{1+}}. \tag{11}$$

Now consider the family generated by a linear combination of the independence model I and E_i, $i = 1, 2, 3, 4$. Here, we consider the family generated by $A_i = (1-\alpha)I + \alpha)E_i, 0 \le \alpha \le 1,\ i = 1,\ 2,\ 3,\ 4$. Suppose n independent drawings are made from the population $A_1 = (1-\alpha)I + \alpha E_1$, and let X_{hk}, $h, k = 0, 1$, be the number of occurrences of the event whose cell probabilities are given by A_1. Then, the joint distribution of X and Y is given by

$P_{A_1}(X = x, Y = Y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \frac{n!\,[(1-\alpha)p_{1+}p_{+1} + \alpha p_{1+}]^j\,[\alpha p_{1+}p_{+0}]^{x-j}}{j!(x-j)!}$$

$$\cdot \frac{[(1-\alpha)p_{0+}p_{+1} + \alpha(p_{+1} - p_{1+})]^{y-j}\,[(1-\alpha)p_{+0}p_{0+} + \alpha(1-p_{+1})]^{n-x-y+j}}{(y-j)!(n-x-y+j)!}$$

$$x, y = 0, 1, 2, ..., n, \quad 0 \le \alpha \le 1, \quad p_{+1} \ge p_{1+}. \tag{12}$$

The conditional distribution of X given $Y = y$ is given by

$P_{A_1}(X = x|Y = y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \binom{y}{j} \binom{n-y}{x-j} \left(\frac{(1-\alpha)p_{1+}p_{+1} + \alpha p_{1+}}{p_{+1}}\right)^j$$

$$\cdot \left(\frac{(1-\alpha)p_{0+}p_{+1} + \alpha(p_{+1} - p_{1+})}{p_{+1}}\right)^{y-j} \left(\frac{\alpha p_{1+}p_{+0}}{p_{+0}}\right)^{x-j}$$

$$\cdot \left(\frac{(1-\alpha)p_{0+}p_{+0} + \alpha(1-p_{1+})}{p_{+0}}\right)^{n-x-y+j}.$$

This is the distribution of the sum of two random variables. One distributed as binomial $\left(y, \frac{(1-\alpha)p_{1+}p_{+1}+\alpha p_{1+}}{p_{+1}}\right)$ and the other as binomial $(n-y, \alpha p_{1+})$. The regression function is

$$E(X|Y = y) = n\alpha p_{1+} + y\frac{(1-\alpha)p_{1+}p_{+1} + \alpha p_{1+} - \alpha p_{1+}p_{+1}}{p_{+1}}. \tag{13}$$

Also, the variance of X given $Y = y$ is given by

$$Var(X|Y = y) = (n-y)\frac{\alpha p_{1+}p_{+0}}{p_{+0}} + y\frac{(1-\alpha)p_{1+}p_{+1} + \alpha p_{1+} - \alpha p_{1+}p_{+1}}{p_{+1}}. \tag{14}$$

For the extreme Bernoulli distribution generated by $A_2 = (1-\alpha)I + \alpha E_2$, suppose n independent drawings are made from this population and let X_{hk}, $h, k = 0, 1$, be the number of occurrences of the event whose cell probabilities are given by A_2. Then the joint distribution of X and Y is given by

$$P_{A_2}(X=x, Y=Y)$$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \frac{n!\,[(1-\alpha)p_{1+}p_{+1}]^j\,[(1-\alpha)p_{1+}p_{+0}+\alpha p_{1+}]^{x-j}}{j!(x-j)!(y-j)!(n-x-y+j)!}$$

$$.\,[(1-\alpha)p_{0+}p_{+1}+\alpha p_{+1}]^{y-j}\,[(1-\alpha)p_{+0}p_{0+}+\alpha(1-p_{+1}-p_{+1})]^{n-x-y+j}$$

$$x, y = 0, 1, 2, ..., n, \quad 0 \leq \alpha \leq 1. \tag{15}$$

The corresponding conditional distribution of X given $Y=y$ is given by $P_{A_2}(X=x|Y=y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \begin{pmatrix} y \\ j \end{pmatrix} \begin{pmatrix} n-y \\ x-j \end{pmatrix} \left(\frac{(1-\alpha)p_{1+}p_{+1}}{p_{+1}}\right)^j$$

$$.\left(\frac{(1-\alpha)p_{0+}p_{+1}+\alpha p_{+1}}{p_{+1}}\right)^{y-j} \left(\frac{\alpha p_{1+}+(1-\alpha)p_{1+}}{p_{+0}}\right)^{x-j}$$

$$.\left(\frac{(1-\alpha)p_{0+}p_{+0}+\alpha(1-p_{1+}-p_{+1})}{p_{+0}}\right)^{n-x-y+j}.$$

The regression function reduces to

$$E(X|Y=y) = n\left[(1-\alpha)p_{1+}+\frac{\alpha p_{1+}}{p_{+0}}\right] - y\frac{\alpha p_{1+}}{p_{+0}}. \tag{16}$$

The factorial moment generating function is given by

$$M(t_1,t_2) = [(1-\alpha)p_{0+}p_{+0}+\alpha(1-p_{1+}-p_{+1})+((1-\alpha)p_{1+}p_{+0}+\alpha p_{1+})t_1$$

$$+((1-\alpha)p_{0+}p_{+0}+\alpha p_{+1})t_2+(1-\alpha)p_{1+}p_{+1}t_1t_2]^n,$$

from which the correlation

$$Corr(X,Y) = -\alpha\left(\frac{p_{1+}p_{+1}}{p_{0+}p_{+0}}\right)^{1/2}.$$

Proposition. For the bivariate binomial distribution generated by $A_2 = (1-\alpha)I + E_2$,

$$-\alpha \leq Corr(X,Y) \leq 0, \quad if \quad p_{1+}+p_{+1} \leq 1. \tag{17}$$

4 Positive and Negative Dependent Families of Distributions

In this section, we present positive and negative dependent families of bivariate binomial distributions generated by the extreme Bernoulli distributions discussed in section 2. For the distributions generated by $\{\Gamma : p_{11} \geq p_{1+}p_{+1}\}$, we have,

$P_2(X = x, Y = Y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \frac{n!\,[(1-\alpha)p_{1+} + \alpha p_{1+}p_{+1}]^j\,[\alpha p_{1+}p_{+0}]^{x-j}}{j!(x-j)!(y-j)!(n-x-y+j)!}$$

$$.\,[(1-\alpha)(p_{+1} - p_{1+}) + \alpha p_{+0}p_{+1}]^{y-j}\,[\alpha p_{+0}p_{0+} + (1-\alpha)(1-p_{+1})]^{n-x-y+j}$$

$$x, y = 0, 1, 2, ..., n, \quad p_{+1} \geq p_{1+}, \quad 0 \leq \alpha \leq 1. \tag{18}$$

The corresponding conditional distribution of X given $Y = y$ is given by $P_2(X = x|Y = y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \begin{pmatrix} y \\ j \end{pmatrix} \begin{pmatrix} n-y \\ x-j \end{pmatrix} \left(\frac{(1-\alpha)p_{1+} + \alpha p_{1+}p_{+1}}{p_{+1}}\right)^j$$

$$.\left(\frac{(1-\alpha)(p_{+1} - p_{+1}) + \alpha p_{0+}p_{+1}}{p_{+1}}\right)^{y-j} \left(\frac{\alpha p_{1+}p_{+0}}{p_{+0}}\right)^{x-j}$$

$$.\left(\frac{(1-\alpha)p_{0+}p_{+0} + \alpha(1-p_{+1})}{p_{+0}}\right)^{n-x-y+j}.$$

The regression function is given by

$$E(X|Y = y) = n\alpha p_{1+} + y\frac{\alpha p_{1+}p_{+1} + (1-\alpha)p_{1+} - \alpha p_{1+}p_{+1}}{p_{+1}}. \tag{19}$$

The factorial moment generating function is

$$M(t_1, t_2) = [\alpha p_{0+}p_{+0} + (1-\alpha)(1-p_{+1}) + \alpha p_{1+}p_{+0}t_1$$

$$+(\alpha p_{0+}p_{+1} + (1-\alpha)(p_{+1} - p_{1+}))t_2 + (\alpha p_{1+}p_{+1} + (1-\alpha)p_{+1})t_1t_2]^n.$$

The correlation $Corr(X, Y)$ is given by

$$Corr(X, Y) = (1-\alpha)\left(\frac{p_{1+}p_{+0}}{p_{+1}p_{0+}}\right)^{1/2} \geq 0.$$

Proposition. For the bivariate binomial distribution generated by $A_2 = (1-\alpha)I + E_2$,

$$0 \leq Corr(X, Y) \leq (1-\alpha), \quad if \quad p_{1+} + p_{+1} \leq 1. \tag{20}$$

The bivariate binomial distributions generated by $\{\Delta : p_{11} \leq p_{1+}p_{+1}\}$, is $P_3(X = x, Y = Y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \frac{n!\,[\alpha p_{1+}p_{+1}]^j\,[\alpha p_{1+}p_{+0} + (1-\alpha)p_{1+}]^{x-j}}{j!(x-j)!(y-j)!(n-x-y+j)!}$$

$$.\,[(1-\alpha)p_{+1} + \alpha p_{0+}p_{+1}]^{y-j}\,[\alpha p_{+0}p_{0+} + (1-\alpha)(1-p_{+1}-p_{1+})]^{n-x-y+j}$$

$$x, y = 0, 1, 2, ..., n, \quad p_{+1} \geq p_{1+}, \quad 0 \leq \alpha \leq 1. \tag{21}$$

The corresponding conditional distribution of X given $Y = y$ is given by $P_3(X = x|Y = y)$

$$= \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \binom{y}{j}\binom{n-y}{x-j}\left(\frac{\alpha p_{1+}p_{+1}}{p_{+1}}\right)^j$$

$$\cdot\left(\frac{(1-\alpha)p_{+1} + \alpha p_{0+}p_{+1}}{p_{+1}}\right)^{y-j}\left(\frac{\alpha p_{1+}p_{+0} + (1-\alpha)p_{1+}}{p_{+0}}\right)^{x-j}$$

$$\cdot\left(\frac{\alpha p_{0+}p_{+0} + (1-\alpha)(1-p_{1+}-p_{+1})}{p_{+0}}\right)^{n-x-y+j}.$$

The regression function is given by

$$E(X|Y = y) = n\frac{\alpha p_{0+}p_{+0} + (1-\alpha)p_{1+}}{p_{+0}} + y\frac{\alpha p_{1+}p_{+0} + (1-\alpha)p_{1+} + \alpha p_{0+}p_{+0}}{p_{+0}}. \tag{22}$$

Proposition. For the bivariate binomial distribution generated by $A_2 = (1-\alpha)I + E_2$,

$$Corr(X, Y) = -(1-\alpha)\left(\frac{p_{1+}p_{+1}}{p_{0+}p_{+0}}\right)^{1/2} \leq 0. \tag{23}$$

Let $P_C(X = x, Y = y)$ be the bivariate binomial distribution generated by $\alpha E_1 + (1-\alpha)E_2$, $0 \leq \alpha \leq 1$. If $y < x$ and $x + y \leq n$, then the joint probability function is given by

$P_{\alpha E_1 + (1-\alpha)E_2}(X = x, Y = y) = P_C(X = x, Y = y)$

$$= \binom{n}{x} p_{1+}^x (1-p_{1+})^{n-x} \sum_{j=\max(0,x+y-n)}^{\min(x,y)} \binom{n-x}{y-j}\binom{x}{j}\left(\frac{\alpha p_{1+}}{p_{1+}}\right)^j$$

$$\left(1 - \frac{\alpha p_{1+}}{p_{1+}}\right)^{x-j}\left(\frac{p_{+1} - \alpha p_{1+}}{p_{0+}}\right)^{y-j}\left(1 - \frac{p_{+1} - \alpha p_{1+}}{p_{0+}}\right)^{n-x-y+j},$$

$$x, y = 0, 1, 2, ..., n, \quad 0 \leq \alpha \leq 1. \tag{24}$$

The following result provides a lower bound for the joint cumulative distribution function $P_C(X = x, Y = y)$ in terms of the bivariate binomial distributions generated by E_1 and E_2 respectively.

Theorem. For $y \geq x$ and $p_{+1} \geq p_{1+}$,

$$P_C(X = x, Y = y) \geq P_{E_1}(X = x, Y = y) P_{E_2}(X = x, Y = y)$$

provided $\alpha \geq p_{+1}$.

Proof: For $y \geq x$, and $p_{+1} \geq p_{1+}$,

$$P_C(X = x, Y = y) = P_{E_1}(X = x, Y = y) P_{E_2}(X = x, Y = y) K^*(x, y)$$

$$\cdot \left\{ \sum_{k=0}^{n} \sum_{h=0}^{n} \frac{\tau_n(k,h)}{k!h!} \phi_k(x, p_{1+}) \phi_h(y, p_{+1}) \right\},$$

where

$$\tau_n(k,h) = \begin{cases} 0 & \text{if } k \neq h \\ \frac{(\alpha p_{1+} - p_{1+}p_{+1})^k}{\binom{n}{k}}, & \text{if } k = h, \end{cases}$$

$$K^*(x,y) = \frac{(y-x)!x!}{y!} \frac{(n-x-y)!y!}{(n-x)!} \left[\frac{(1-p_{1+})^{n-x}}{(p_{+1}-p_{1+})^{y-x} p_{1+}^x} \right] \left[\frac{1}{1-p_{1+}-p_{+1}} \right]^{n-x-y},$$

and $\phi_k(x, p_{1+})$ is the kth Krawchouk polynomial. Note that

$$\begin{aligned} \lim_{n \to \infty} K^*(x,y) &= \lim_{n \to \infty} \left\{ \frac{(y-x)!x!}{y!} \frac{(n-x-y)!y!}{(n-x)!} \left[\frac{1-p_{1+}}{1-p_{1+}-p_{+1}} \right]^n \right. \\ &\cdot \left. \left[\frac{(p_{+1}-p_{1+})(1-p_{1+}-p_{+1})}{p_{1+}(1-p_{1+})} \right]^x \left[\frac{1-p_{1+}-p_{+1}}{p_{+1}-p_{1+}} \right]^y \right\} = 1, \end{aligned}$$

so that

$$P_C(X = x, Y = y) \approx P_{E_1}(x,y) P_{E_2}(x,y)$$

$$\cdot \left\{ \sum_{k=0}^{n} \sum_{h=0}^{n} \frac{\tau_n(k,h)}{k!h!} \phi_k(x, p_{1+}) \phi_h(y, p_{+1}) \right\}.$$

Consequently, for $\alpha \geq p_{+1}$,

$$P_C(X = x, Y = y) \geq P_{E_1}(X = x, Y = y) P_{E_2}(X = x, Y = y).$$

Corollary. For $y \geq x$, $p_{+1} \geq p_{1+}$, and $\alpha \geq p_{+1}$,

$$P_C(X \leq x, Y \leq y) \geq P_{E_1}(X \leq x, Y \leq y) P_{E_2}(X \leq x, Y \leq y).$$

5 Concluding Remarks

Several bivariate binomial distributions generated by extreme Bernoulli distributions are derived. Important properties of these distributions including factorial moment generating functions and the regression functions are presented. Families of positive and negative dependent bivariate binomial distributions generated by the extreme Bernoulli distributions as well as their properties are presented. The representations of convex combinations of these extreme distributions are given in terms of Krawchouk polynomials. Some comparisons and dependence of the random variables X and Y are explored. Also, bounds on certain joint probabilities are given.

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