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Upper bounds for inverse domination in graphs

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1. Introduction

In this paper all graphs are simple. A *dominating set* for a graph G is a set of vertices D such that every vertex of G either lies in D or has a neighbor in D. The *domination number* of G, written $\gamma(G)$, is the size of a smallest dominating set in G. Note that a maximum independent set is a dominating set, so $\gamma(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.

If a graph G has no isolates and D is a minimum dominating set in G, then $V(G) - D$ is also a dominating set in G (owing to the minimality of D); this was first observed by Ore [10]. In general we say that a dominating set D' is an *inverse dominating set* for a graph G if there is some minimum dominating set D such that $D \cap D' = \emptyset$. A graph with isolates cannot have an inverse dominating set, but otherwise, given Ore's observation, we can define the *inverse domination number* of a graph G, written $\gamma^{-1}(G)$, as the smallest size of an inverse dominating set in G. The *Inverse Domination Conjecture* asserts that $\gamma^{-1}(G) \leq \alpha(G)$ for every isolate-free G.

The Inverse Domination Conjecture originated with Kulli and Sigarkanti [9], who in fact provided an erroneous proof. Discussion of this error and further consideration of the conjecture first appeared in a paper of Domke, Dunbar, and Markus [3]. It has since been shown by Driscoll and Krop [4] that the weaker bound of $\gamma^{-1}(G) \leq 2\alpha(G)$ holds in general, and Johnson, Prier and Walsh [7] showed that the conjecture itself holds whenever $\gamma(G) \leq 4$. Johnson and Walsh [8] have also proved two fractional analogs of the conjecture, and Frendrup, Henning, Randerath and Vestergaard [5] have shown that the conjecture holds for a number of special families, including bipartite graphs and claw-free graphs.

In this paper we prove two main results in support of the Inverse Domination Conjecture. The first is an improvement on the $2\alpha(G)$ approximation to the conjecture.

Theorem 1.1. *If G is a graph with no isolated vertices and G is not a clique, then $\gamma^{-1}(G) \leq \frac{3}{2}\alpha(G) - 1$.*

Note that if G is a clique and $G \neq K_1$, then trivially $\gamma^{-1}(G) = \alpha(G) = 1$, which is why we must exclude cliques in Theorem 1.1.

Our second main result improves the range of $\gamma(G)$ for which the conjecture is known.

Theorem 1.2. *If G is a graph with no isolated vertices and $\gamma(G) \leq 5$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

As a corollary of Theorem 1.2 we are also able to obtain the following.

Corollary 1.3. *If G is a graph with no isolated vertices and $|V(G)| \leq 16$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

It is worth noting that Asplund, Chaffee, and Hammer [2] have formulated a stronger form of the Inverse Domination Conjecture. In the strengthened version one requires, for *every* minimum dominating set D, the existence of a dominating set D' with $D \cap D' = \emptyset$ and $|D'| \leq \alpha(G)$. It is not hard to see that our proof for Theorem 1.1 also works for this stronger conjecture. However, the same is not true for Theorem 1.2, where we pick our minimum dominating set D very carefully.

The rest of the paper is organized as follows. In Section 2 we introduce the notion of an *independent set of representatives*, or ISR, and explore the connections between ISRs and inverse domination. (In this section, we also obtain, as a corollary, the inequality $\gamma^{-1}(G) \leq b(G)$ for graphs without isolated vertices, where $b(G)$ is the largest number of vertices in an induced bipartite subgraph of G.) In Section 3 we prove Theorem 1.1. In Section 4 we leverage the machinery of Section 2 to prove Theorem 1.2 and Corollary 1.3.

2. ISRs and Inverse Domination

If $(X_1, \dots, X_k)$ is a collection of sets, a *set of representatives* for $(X_1, \dots, X_k)$ is a set $\{x_1, \dots, x_k\}$ such that $x_i \in X_i$ for each i. If G is a graph and $V_1, \dots, V_k$ are subsets of $V(G)$, an *independent set of representatives*, or ISR, for $(V_1, \dots, V_k)$ is a set of representatives for the sets $V_1, \dots, V_k$ that is also an independent set in G. A *partial ISR* for $V_1, \dots, V_k$ is an ISR for any subfamily of $V_1, \dots, V_k$.

Several authors have proved various sufficient conditions guaranteeing the existence of ISRs; many of the proofs are topological in nature. See [1] for a collection of such results. A fundamental result on ISRs is the following sufficient condition due to Haxell [6]. In what follows, given a graph G and a set $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A. Given a collection of sets $(V_1, \dots, V_k)$ and $J \subseteq [k]$, we write V_J for the union $\bigcup_{j \in J} V_j$.

Theorem 2.1 (Haxell [6]). *Let G be a graph and let $V_1, \dots, V_n$ be a partition of $V(G)$. If, for all $S \subseteq [n]$,*

$$\gamma(G[V_S]) \geq 2|S| - 1,$$

then G has an independent set $v_1, \dots, v_n$ such that $v_i \in V_i$ for each i (that is, $(V_1, \dots, V_n)$ has an ISR).

Our basic idea for using Theorem 2.1 to obtain results on inverse domination is to apply it to a specific partition of vertices outside D (where D is a minimum dominating set), namely to what we'll call a *standard partition*.

Let G be a graph and suppose that X, Y are disjoint sets of vertices where X dominates Y. The standard partition of Y, subject to a given ordering $(v_1, \dots, v_n)$ of X, is the partition $(V_1, \dots, V_n)$ with

$$V_i = N_Y(v_i) \setminus \bigcup_{j<i} V_j,$$

where $N_Y(v_i)$ indicates those neighbors of v_i that are in Y. Consider a minimum dominating set D, and the standard partition of $V(G) - D$ with respect to any ordering of D. If this partition has an ISR, then the ISR is an independent set disjoint from D that dominates D. Expanding this independent set to a maximal independent set in $G - D$ would give an independent dominating set disjoint from D, implying that $\gamma^{-1}(G) \leq \alpha(G)$. However, we cannot always find an ISR for a standard partition of $G - D$. Instead, we obtain more technical results.

In the following, given disjoint sets $X_1, \dots, X_k$ and $S \subset X_1 \cup \dots \cup X_k$, we write $i(S)$ for the set $\{j\colon\ S \cap X_j \neq \emptyset\}$. When $S = \{v\}$, we'll denote the unique element of $i(S)$ by $i(v)$.

Theorem 2.2. *Let G be a graph, let D be a minimum dominating set in G, and let F be a maximal independent set in D. Let $(d_1, \dots, d_n)$ be any ordering of $D - F$, and let $(V_1, \dots, V_n)$ be the standard partition of $G - D - N(F)$ subject to this ordering. Then there exist two partial ISRs R_1, R_2 of $(V_1, \dots, V_n)$ such that $i(R_1) \cap i(R_2) = \emptyset$ and $i(R_1) \cup i(R_2) = [n]$.*

Proof. Let H be a graph consisting of two disjoint copies of $G - D - N(F)$, and let $W_1, \dots, W_n$ be a partition of $V(H)$ obtained by letting each W_i consist of both copies of each vertex in V_i.

We will use Theorem 2.1 to obtain an ISR of $(W_1, \ldots, W_n)$. Let S be any subset of $[n]$, and let $H' = H[W_S]$. We will show that $\gamma(H') \geq 2\,|S|$.

Observe that H' consists of two disjoint copies of the subgraph $G' := G[V_S]$, so that any dominating set in H' must dominate each of those copies. If $\gamma(H') < 2\,|S|$, then let C be a minimum dominating set of H'. We can partition C into $C = C_1 \cup C_2$, where C_1 dominates one copy of G' and C_2 dominates the other copy. Without loss of generality $|C_1| \leq |C_2|$, and since $|C| < 2\,|S|$, this implies $|C_1| < |S|$. Let C' be the set of vertices in G' corresponding to the vertices of C_1, and let $D^* = (D \setminus \{d_i \colon i \in S\}) \cup C'$. We know that D^* dominates $V(G) - D$, and moreover since $F \subseteq D^*$ and F dominates $D - F$, we see that D^* is a dominating set of G. Since $|D^*| < |D|$, this contradicts the minimality of D.

Thus $(W_1, \ldots, W_n)$ has some ISR R. We can partition $R = R_1 \cup R_2$ where R_1 consists of the R-vertices in one copy of G' and R_2 consists of the R-vertices in the other copy of G'. Now R_1 and R_2 are each independent subsets of G', and since R is an ISR we see that $i(R_1) \cap i(R_2) = \emptyset$ and $i(R_1) \cup i(R_2) = [n]$. □

As an immediate and useful corollary to Theorem 2.2, we get the following.

Corollary 2.3. *Let G be a graph, let D be a minimum dominating set in G, and let F be a maximal independent set in D. Let $(d_1, \ldots, d_n)$ be any ordering of $D - F$, and let $(V_1, \ldots, V_n)$ be the standard partition of $G - D - N(F)$ subject to this ordering. Then $(V_1, \ldots, V_n)$ has a partial ISR of size at least $n/2$.*

Observe that if D is a minimum dominating set in a graph G without isolates, then each vertex in D has a neighbor in $G - D$. These neighbors can be used to help build inverse dominating sets, and our first use of this will be in the following corollary.

Corollary 2.4. *Let G be a graph without isolated vertices and let D be a minimum dominating set in G. If $b(G)$ is the largest number of vertices in an induced bipartite subgraph of G, then $\gamma^{-1}(D) \leq b(G)$.*

Proof. Let F be a maximal independent set in D, and let R_1, R_2 be partial ISRs as in Theorem 2.2. As R_1 and R_2 are each independent and $R_1 \cap R_2 = \emptyset$, $R_1 \cup R_2$ induces a bipartite subgraph of G. Since $i(R_1) \cup i(R_2) = [n]$, the set $R_1 \cup R_2$ dominates $D - F$. Expand $R_1 \cup R_2$ to a maximal set $B \subseteq G - D$ inducing a bipartite subgraph.

The maximality of B implies that B dominates $G - F$. Let $F_0 = F - N(B)$, so that B dominates $G - F_0$. Observe that $B \cup F_0$ still induces a bipartite graph, so that $b(G) \geq |B| + |F_0|$. On the other hand, each vertex $v \in F_0$ has some neighbor $v' \in V(G) - D$. Augmenting B by adding in such a vertex v' for each $v \in F_0$ yields a inverse dominating set of size at most $|B| + |F_0|$, which is at most $b(G)$. □

3. Proof of Theorem 1.1

Theorem 3.1. *Let G be a graph, and let D be a minimum dominating set in G. There is a set $T \subset V(G) - D$ such that T is a dominating set in G and $|T| \leq \alpha(G) + \left\lfloor \frac{\gamma(G)-1}{2} \right\rfloor$.*

Proof. Let F be a maximal independent set in D, and write $D - F$ as $\{d_1, \ldots, d_n\}$. Let $(V_1, \ldots, V_n)$ be the standard partition of $N(D - F)$.

Let R be a largest possible partial ISR for $(V_1, \ldots, V_n)$. By Corollary 2.3, we have $|R| \geq n/2$. Expand R to a maximal independent set S in $G - D$. The set S dominates every vertex of $V(G) - D$ and at least $n/2$ vertices of $D - F$. We now expand S to dominate the rest of D.

Let $F' = F - N(S)$. Observe that $S \cup F'$ is an independent set, so $|S| + |F'| \leq \alpha(G)$. Expand S to a set S_1 by adding an arbitrary $(G - D)$-neighbor of v' for each $v' \in F'$; we have $|S_1| \leq |\alpha(G)|$. Next, expand S_1 to a set T by adding an arbitrary $(G - D)$-neighbor of w for each $w \in D - F - N(S_1)$; note that $|D - F - N(S_1)| \leq n/2$, so $|T| \leq \alpha(G) + n/2$. As $n \leq \gamma(G) - 1$ and $|T|$ is an integer, this implies that

$$|T| \leq \alpha(G) + \left\lfloor \frac{\gamma(G) - 1}{2} \right\rfloor.$$

Since T is a dominating set in G, the theorem is proved. □

The following lemma is more general than is necessary for proving Theorem 1.1, but stating it in this generality will be useful for later results.

Lemma 3.2. *If a graph G has a minimum dominating set D and an independent set S such that $S - D$ dominates $D - S$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Proof. Let $S_1 = S - D$ and let $S_2 = S \cap D$. Expand S_1 to a maximal independent set S_1' of $G - D$. Now S_1' dominates $G - D$. Let S_2' be the set of vertices in D not dominated by S_1'. Observe that $S_2' \subset S_2$, since by hypothesis S_1 dominates $D - S_2$. Hence $S_1' \cup S_2'$ is an independent set, so that $\alpha(G) \geq |S_1'| + |S_2'|$.

Since D is a minimum dominating set of G and G has no isolated vertices, each vertex of D has a neighbor outside of D. Let T be the vertex set obtained from S_1' by adding in, for each $v \in S_2'$, a neighbor of v outside of D. Now T is a dominating set in G and $|T| \leq |S_1'| + |S_2'| \leq \alpha(G)$. Hence $\gamma^{-1}(G) \leq \alpha(G)$. □

The proof of Theorem 1.1 now follows easily. If G has a minimum dominating set D that is independent, then we can choose $S = D$ to vacuously meet the hypothesis of Lemma 3.2, and hence $\gamma^{-1}(G) \leq \alpha(G) \leq (3/2)\alpha(G) - 1$. Otherwise, $\gamma(G) \leq \alpha(G) - 1$, so by Theorem 3.1, we have

$$\gamma^{-1}(G) \leq \alpha(G) + \left\lfloor \frac{\gamma(G) - 1}{2} \right\rfloor \leq \alpha(G) + \left\lfloor \frac{\alpha(G) - 2}{2} \right\rfloor \leq \frac{3}{2}\alpha(G) - 1.$$

4. Proof of Theorem 1.2

Our proof of Theorem 1.2 relies on a careful choice of minimum dominating set. For shorthand, it will be convenient to speak of the *independence number* of a dominating set D to refer to the independence number of the induced subgraph $G[D]$, and likewise to write $\alpha(D)$ for $\alpha(G[D])$. We will consider a dominating set D in a graph G to be *optimal* if it is of minimum size and, among minimum-size dominating sets, has greatest independence number and, subject to that, has the fewest edges in the induced subgraph $G[D]$. In order to build inverse dominating sets in a graph G, we previously used the fact that any vertex v in a minimum dominating set D has a neighbor in $G - D$ (provided G is isolate-free). In some arguments, it is helpful if such a neighbor is *private* with respect to D; that is, if we are able to choose $w \in V(G) - D$ with $N(w) \cap D = \{v\}$. In fact, the choice of a private neighbor for v is always possible when D is a minimum dominating set, unless v is isolated in $G[D]$. The following lemma tells us that if D is optimal, we can improve on this.

Lemma 4.1. *Let G be an isolate-free graph and let D be an optimal dominating set in G. If $v \in D$ is not an isolated vertex in $G[D]$, then v has at least 2 private neighbors with respect to D.*

Proof. Let G_v be the subgraph of G induced by the private neighbors of v. We in fact show $\gamma(G_v) > 1$. Suppose to the contrary that G_v has a dominating vertex w. Let $D' = (D - v) \cup \{w\}$. Every vertex of $G - D'$ is either v itself, hence dominated by w, or a private neighbor of v, hence dominated by w, or a vertex of $G - D$ that is not a private neighbor of D, hence dominated by $D - v$. Thus, D' is a dominating set. Furthermore, as w was a private neighbor of v, the vertex w is an isolated vertex in D'. In particular, for any maximum independent set S in D, we see that $(S - v) \cup \{w\}$ is also a maximum independent set in D', so D' has at least as large an independence number as D did. As w is isolated in D' but v was not isolated in D, we see that D' has fewer edges than D, contradicting the optimality of D. □

Lemma 4.2. *Let G be an isolate-free graph and let D be an optimal dominating set in G. Suppose that the number of isolates in $G[D]$ is a. Then either G has an independent set S such that $S - D$ dominates $D - S$, or all of the following are true:*

(1) $a + 1 \leq \alpha(D) \leq |D| - 3$,
(2) $|V(G)| + a \geq 3\,|D|$, *and*
(3) $|D| \geq a + 5$.

Proof. Assuming that G has no such independent set S, we prove each part of the conclusion separately.

(1) If D is an independent set, then taking $S = D$ gives the desired independent set. Hence $a + 1 \leq \alpha(D) \leq |D| - 1$, and we may choose a vertex $d^* \in D$ that is not isolated in $G[D]$. If $\alpha(D) = |D| - 1$, then letting v^* be a private neighbor of d^* and taking $S = (D - d^*) \cup \{v^*\}$ gives the desired independent set.

Hence we may assume that $\alpha(D) = |D| - 2$. Let $\{d_1, \ldots, d_n\}$ be an ordering of D with $\{d_1, \ldots, d_{n-2}\}$ independent, and let $(V_1, \ldots, V_n)$ be the standard partition of $N(D)$ with respect to this ordering.

If there is a pair of nonadjacent vertices $v_{n-1} \in V_{n-1}$, $v_n \in V_n$, then taking $S = \{d_1, \ldots, d_{n-2}, v_{n-1}, v_n\}$ yields an independent set S such that $S - D$ dominates $D - S$. Otherwise, there is a complete bipartite graph between V_{n-1} and V_n. Taking v^*_{n-1} and v^*_n to be private neighbors of d_{n-1} and d_n respectively, we see that $\{d_1, \ldots, d_{n-2}, v^*_{n-1}, v^*_n\}$ is a dominating set in G having independence number $n - 1$, contradicting the optimality of D.

(2) Let A be the set of a isolated vertices in D. Notice that if $|N(A)| < |A|$, then $(D - A) \cup N(A)$ is a dominating set of size less than D, which is impossible. Hence, $|N(A)| \geq |A|$. We count $|A|$ as well as $|N(A)|$ and then apply Lemma 4.1, which implies that $|V(G)| \geq |D| + a + 2(|D| - a) = 3\,|D| - a$.

(3) Suppose $|D| \leq a + 4$. By (1) we get $|D| - 3 \geq a + 1$, so in fact $|D| = a + 4$. Moreover, by (1), this means that $\alpha(D) = a + 1$, so $G[D] \cong aK_1 + K_4$.

Write $D = \{d_1, \ldots, d_n\}$ with $d_1, \ldots, d_{n-4}$ isolated in $G[D]$, and let $(V_1, \ldots, V_n)$ be the standard partition of $V(G) - D$ with respect to this ordering. Suppose first that there is an independent set S_0 in $G - D$ hitting at least three of the sets $\{V_{n-3}, \ldots, V_n\}$. Then define S to be $S_0 \cup \{d_1, \ldots, d_{n-4}\}$; note S is independent. Out of the four vertices in $D - S$, at most one is not dominated by $S - D$. However, if such a vertex exists, then we can add it to S as well, without violating independence. Thus, we may assume that no such set S_0 exists.

If the pair (V_{n-3}, V_{n-2}) is joined by a complete bipartite graph, then we may take v^*_{n-3} and v^*_{n-2} to be private neighbors of d_{n-3} and d_{n-2} respectively. Now

$$(D \setminus \{d_{n-3}, d_{n-2}\}) \cup \{v^*_{n-3}, v^*_{n-2}\}$$

is a dominating set of G of size $|D|$ containing the independent set

$$\{d_1, \ldots, d_{n-4}, v^*_{n-3}, d_{n-1}\}$$

of size $n-2$, contradicting the optimality of D.

Otherwise, there is a pair of nonadjacent vertices $v_{n-3} \in V_{n-3}$ and $v_{n-2} \in V_{n-2}$. Since, by assumption, this pair cannot be extended to an independent set that also hits one of the sets V_{n-1} or V_n, we see that $\{v_{n-3}, v_{n-2}\}$ dominates $V_{n-1} \cup V_n$. Thus

$$\{d_1, \ldots, d_{n-4}\} \cup \{d_{n-3}, d_{n-2}, v_{n-3}, v_{n-2}\}$$

is a dominating set in G containing the independent set

$$\{d_1, \ldots, d_{n-4}, v_{n-3}, v_{n-2}\},$$

contradicting the optimality of D. □

In the remainder of the section we will prove the inverse domination conjecture for graphs G with $\gamma(G) \leq 5$. In light of the following lemma, it will suffice to prove the conjecture for graphs with domination number exactly 5.

Lemma 4.3. *Let k be a positive integer. If $\gamma^{-1}(G) \leq \alpha(G)$ for every isolate-free graph G with $\gamma(G) = k$, then $\gamma^{-1}(G) \leq \alpha(G)$ for every isolate-free graph G with $\gamma(G) \leq k$.*

Proof. Let G be an isolate-free graph with $\gamma(G) \leq k$, and let $t = k - \gamma(G)$. Let G' be the disjoint union of G and t copies of K_2. Now $\gamma(G') = \gamma(G) + t = k$, so by hypothesis, $\gamma^{-1}(G') \leq \alpha(G') = \alpha(G) + t$. In particular, in G' we can choose a minimum dominating set D' and a second disjoint dominating set T' with $|T'| \leq \alpha(G')$. Observe that D' and T' must each contain one vertex from every added copy of K_2. Hence, letting $D = D' \cap V(G)$ and $T = T' \cap V(G)$, we see that $|D| = |D'| - t = \gamma(G)$ and $|T| \leq \alpha(G') - t = \alpha(G)$. Furthermore, D and T are dominating sets in G. Hence, $\gamma^{-1}(G) \leq \alpha(G)$. □

We wish to strengthen the conclusion of Theorem 2.2 by eliminating the maximal independent set F inside D, and instead finding a pair of ISRs that jointly dominate the entire minimum dominating set D. When $\gamma(G) = 5$ and $\alpha(D) \leq 2$, we are able to do this.

Lemma 4.4. *Let D be an optimal dominating set in an isolate-free graph G. Suppose that $|D| = 5$, that $\alpha(D) \leq 2$, and that $G[D]$ has no isolated vertices. Then there is an ordering $(d_1, \ldots, d_5)$ of D and a pair of independent sets R_1 and R_2 such that R_1 is an ISR for (V_1, V_2, V_3) and R_2 is an ISR for (V_4, V_5), where $(V_1, \ldots, V_5)$ is the standard partition of $G - D$ with respect to this ordering.*

Proof. Choose $d_1, d_2 \in D$ so that $\{d_1, d_2\}$ is an independent set, if possible. (Thus, $d_1 d_2 \in E(G)$ only if D is a clique.) Note that since $\alpha(D) \leq 2$, the set $\{d_1, d_2\}$ contains a maximal independent set in D, hence dominates $D - \{d_1, d_2\}$. This implies that there are at least 3 edges from $\{d_1, d_2\}$ to the rest of D.

First we argue that there is an independent set $\{r_1, r_2\}$ with $r_i \in V_i$. If not, then V_1 and V_2 are joined by a complete bipartite graph. Let v^*_1 and v^*_2 be private neighbors of d_1 and d_2 respectively. Observe that $\{v^*_1, v^*_2\} \cup (D \setminus \{d_1, d_2\})$ is a dominating set of D. Furthermore, there are no edges between $\{v_1^*, v_2^*\}$ and $D \setminus \{d_1, d_2\}$. This implies that $|E(D')| \leq |E(D)| - 2$, contradicting the optimality of D. (Note that $\alpha(D') \geq \alpha(D)$ since $\alpha(D) \leq 2$.)

Now, since D is a minimal dominating set of G, there is some vertex $r_3 \in V(G)$ not dominated by $\{d_1, d_2, r_1, r_2\}$. As $\{d_1, d_2\}$ dominates D, we have $r_3 \in V(G) - D$. Choose d_3 to be a neighbor of r_3 in D. Let $R_1 = \{r_1, r_2, r_3\}$, and let d_4 and d_5 be the remaining vertices of D, ordered arbitrarily. Observe that R_1 is an ISR for (V_1, V_2, V_3) in the standard partition of $V(G) - D$ with respect to this ordering. It remains to find the desired R_2.

We claim that there are nonadjacent vertices r_4, r_5 each with $r_i \in V_i$. If not, then V_4 and V_5 are joined by a complete bipartite graph. Let v_4^* and v_5^* be private neighbors of d_4 and d_5 respectively. Now $D' = \{d_1, d_2, d_3, v_4^*, v_5^*\}$ is a dominating set in D. Furthermore, since $\{d_1, d_2\}$ is a dominating set in D, there are at least two edges in the cut $[\{d_1, d_2, d_3\}, \{d_4, d_5\}]$, while by contrast there are no edges joining $\{v_4^*, v_5^*\}$ with $\{d_1, d_2, d_3\}$. Hence $|E(D')| \leq |E(D)| - 1$, contradicting the optimality of D. (Again $\alpha(D') \geq \alpha(D)$ since $\alpha(D) \leq 2$.) □

Theorem 4.5. *If G is an isolate-free graph with $\gamma(G) = 5$, then G has a minimum dominating set D such that $\gamma^{-1}(D) \leq \alpha(G)$.*

Proof. Let D be an optimal dominating set in G. By Lemma 3.2 and by parts (1) and (3) of Lemma 4.2, we may assume that $\alpha(D) \leq 2$ and that D has no isolated vertices. In particular, since D is not an independent set, we have $\alpha(G) \geq 6$, a fact we will use later.

By Lemma 4.4, we see that there is an ordering $(d_1, \ldots, d_5)$ of D and a pair of independent sets R_1, R_2 such that R_1 is an ISR for (V_1, V_2, V_3) and R_2 is an ISR for (V_4, V_5), where $(V_1, \ldots, V_5)$ is the standard partition of $G - D$ for the given ordering. Among all such pairs (R_1, R_2), choose R_1 and R_2 to minimize the number of edges from R_1 to R_2.

If $(V_1, \ldots, V_5)$ has a partial ISR of size 4, then we immediately get the desired conclusion: taking R to be such an ISR, we see that R dominates all of D except possibly for a single vertex $w \in D$, so we win by letting $S = R \cup \{w\}$ (or $S = R$) and applying Lemma 3.2.

Thus, $(V_1, \ldots, V_5)$ has no partial ISR of size 4, which implies that R_1 is a maximal partial ISR of this family, and so R_1 dominates $V_4 \cup V_5$.

Let T be the set of vertices in G that are not dominated by $R_1 \cup R_2$. If $T = \emptyset$ then we immediately have the desired conclusion, as $R_1 \cup R_2$ is an inverse dominating set of size γ. Thus we may assume that T is a nonempty subset of $V(G) - D$, and in particular, $T \subseteq V_1 \cup V_2 \cup V_3$.

Write $R_1 = \{r_1, r_2, r_3\}$ with $r_i \in V_i$. We claim that if T intersects V_j for some $j \in \{1, 2, 3\}$, then the corresponding vertex r_j is not adjacent to any vertex of R_2. Otherwise, let $r_j' \in T \cap V_j$, and let $R_1' = (R_1 \setminus \{r_j\}) \cup \{r_j'\}$. Now R_1' is an ISR of (V_1, V_2, V_3) and, since r_j' is not dominated by $R_1 \cup R_2$, there are fewer edges between R_1' and R_2 than there were between R_1 and R_2. This contradicts the choice of $R_1 \cup R_2$, establishing the claim.

In particular, the above claim implies that $|i(T)| = 1$, since if $|i(T)| \geq 2$, then taking distinct $j, k \in i(T)$, we see that $R_2 \cup \{r_j, r_k\}$ is a partial ISR of $(V_1, \ldots, V_5)$ having size 4, contradicting our earlier claim that the largest such partial ISR has size 3.

Let k be the unique index in $i(T)$. Let $R^* = (R_1 \cup R_2) \setminus \{r_k\}$. We next claim that any vertex of $\bigcup_{j \neq k} V_j$ not dominated by R^* is adjacent to all of T. Otherwise, let v_j be such a vertex that is not adjacent to all of T, with $v_j \in V_j$.

Let v_k be a vertex of T not adjacent to v_j. If $j \in \{1, 2, 3\}$, then let $R_2' = R_2 \cup \{v_j, v_k\}$. Now R_2' is an independent set, since $R_2 \subset R^*$ and neither v_j nor v_k is dominated by R^* (by choice of v_j and because $v_k \in T$). As $i(R_2) = \{4, 5\}$ this implies that R_2' is a partial ISR of

$(V_1, \ldots, V_5)$ having size 4, contradicting the earlier claim that the largest such ISR has size 3. If instead $j \in \{4,5\}$, then taking $R_1' = (R_1 \setminus \{r_k\}) \cup \{v_j, v_k\}$ gives the same contradiction.

Hence, any vertex of $\bigcup_{j \neq k} V_j$ not dominated by R^* is adjacent to all of T. If there is any vertex of $\bigcup_{j \neq k} V_j$ not dominated by R^*, then let w be such a vertex; now $R_1 \cup R_2 \cup \{w\}$ is an inverse dominating set of size 6, where $\alpha(G) \geq 6$, and we are done. Hence, we may assume that R^* dominates $\bigcup_{j \neq k} V_j$.

In this case, let $D' = R^* \cup \{d_k\}$. Since R^* dominates $\bigcup_{j \neq k} V_j$, we see that D' is a dominating set of G. Since $k \leq 3$, the set $\{d_k, r_4, r_5\}$ is an independent set: if d_k were adjacent to r_4, this would imply $r_4 \in V_k$, contradicting $r_4 \in V_4$, and likewise for r_5. This contradicts the optimality of D. □

Corollary 4.6. *If* $|V(G)| \leq 16$ *then* $\gamma^{-1}(G) \leq \alpha(G)$.

Proof. Let G be some graph with $\gamma^{-1}(G) > \alpha(G)$, and let D be an optimal dominating set in G. Let a be the number of isolated vertices in $G[D]$. By Lemma 3.2, there cannot be any independent set S such that $S - D$ dominates $D - S$, so by Lemma 4.2, we have:

(2) $|V(G)| + a \geq 3\,|D|$, and
(3) $|D| \geq a + 5$.

By Theorem 4.5 and Lemma 4.3, we have $\gamma(G) \geq 6$, so that $|D| \geq 6$. If $a = 0$ then (2) yields $|V(G)| \geq 18$. Otherwise, $a \geq 1$, and then (2) combined with (3) yields $|V(G)| \geq 2a + 15 \geq 17$. □

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