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On Graphs with Proper Connection Number 2

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Abstract

An edge-colored graph is properly connected if for every pair of vertices u and v there exists a properly colored uv-path (i.e. a uv-path in which no two consecutive edges have the same color). The proper connection number of a connected graph G, denoted $pc(G)$, is the smallest number of colors needed to color the edges of G such that the resulting colored graph is properly connected. An edge-colored graph is flexibly connected if for every pair of vertices u and v there exist two properly colored paths between them, say P and Q, such that the first edges of P and Q have different colors and the last edges of P and Q have different colors. The flexible connection number of a connected graph G, denoted $fpc(G)$, is the smallest number of colors needed to color the edges of G such that the resulting colored graph is flexibly connected. In this paper, we demonstrate several methods for constructing graphs with $pc(G) = 2$ and $fpc(G) = 2$. We describe several families of graphs such that $pc(G) \geq 2$ and we settle a conjecture from [3]. We prove that if G is connected and bipartite, then $pc(G) = 2$ is equivalent to being 2-edge-connected and $fpc(G) = 2$ is equivalent to the existence of a path through all cut-edges. Finally, it is proved that every connected, k-regular, Class 1 graph has flexible connection number 2.

1 Introduction

An edge-colored graph is said to be *properly colored* if no two adjacent edges share a color. An edge-colored graph is said to be *properly connected* if between every pair of distinct vertices there exists a path that is properly colored. If only two colors are used to color the edges of a graph, we often refer to a properly colored path as an *alternating path.* The *proper connection number* of a connected graph G (introduced in [3], [1]), denoted by $pc(G)$, is the minimum number of colors needed to color the edges of G to make it properly connected. Walk and trail analogs of proper connection exist and can be found in [7, 13, 3].

Properly connected graphs have applications to communication networks, genetics and social sciences. (See for example: [6, 5, 4].) There are multiple surveys on the topic including one by Bang-Jensen and Gutin [2] that focuses on cycles and paths and includes a section devoted to 2-colored graphs and a dynamic survey on proper connection number maintained by Li and Magnant, [10]. Recently, a text by Li, Magnant, and Qin [11] has been published with paper and electronic versions.

Many previous results have hypotheses concerning the number of bridges; for example, Borozan et al. [3] proved that if the graph G is bridgeless and not a tree, then $pc(G) \leq 3$. Another example can be found in Theorem 4.4 [13]. Theorem 4.3 in this manuscript uses a different but equivalent hypothesis concerning bridges to obtain the same conclusion as Theorem 4.4 from [13].

If P is an alternating uv-path in the edge-colored graph G, we use $start(P)$ and $end(P)$ to denote the color of the edge incident to u in P and the color of the edge incident to v in P respectively. In several places in [3] and [8], the authors reference an additional "strong property." In this paper, we formally define this property as "flexible." (See Definition 1.1 below.) Note that we do not use the word "strong" since "strong proper path coloring" was defined in [12] to be an edge coloring in which there exists properly colored *shortest* paths between vertices.

Definition 1.1. *An edge-colored graph is* flexibly connected *if between every pair of vertices* u *and* v *there exist properly colored paths* P *and* Q *such that* $start(P) \neq start(Q)$ *and* $end(P) \neq end(Q)$.

The flexible connection number *of a connected graph* G*, denoted* $fpc(G)$*, is the smallest number of colors needed to color the edges of* G *such that the resulting colored graph is flexibly connected.*

Observe that by definition $fpc(G) \geq 2$ for all graphs G.

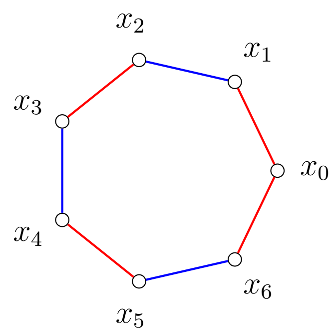

Figure 1: The 7-cycle above is 2-edge-colored such that it is properly connected. It is not flexibly connected since every path starting at x_0 must begin with red edge. If edge x_0x_1 was changed to a third color, the 7-cycle would be flexibly connected.

In Section 2 we describe several ways of building properly connected graphs and flexibly connected graphs. In Section 3 we describe several families of graphs that motivate the conjecture below.

Conjecture 1.1. *If* G *is 2-connected and regular, then* $pc(G) \leq 2$.

In Section 4, we characterize bipartite graphs G with $pc(G) = 2$ and with $fpc(G) = 2$. In addition, we prove that every connected, k-regular, Class 1 graph has flexible connection number 2 provided $k \geq 2$.

2 Constructions

Here we describe methods of constructing graphs with proper connection number 2 and with flexible proper connection number 2. Construction 2.1 appears as Proposition 2 in [3]. The proof is included for completeness.

Construction 2.1. *Let* G_1 *be a 2-edge-colored graph that is properly connected. Let* G *be obtained from* G_1 *by adding vertex* v *and two edges from* v *to* G_1*. Then* $pc(G) = 2$.

Proof. Let G_1 be a 2-edge-colored graph that is properly connected and let v be a vertex added to G_1 such that $N(v) = \{x, y\}$. Since the 2-colored graph G_1 is properly connected, there exists an alternating xy-path, P, in G_1. Color edge xv with the color different from

$start(P)$ and color edge yv with color different from $end(P)$. Observe that with this coloring, every vertex on P has two alternating paths to v, one along P via x and the other along P via y and these two paths to v begin with different colors.

Since G_1 is properly connected, for every $z \in V(G_1)$, there exists an alternating zx-path in G_1, say Q. Observe that Q must share at least one vertex in common with P (perhaps only x). Thus there exists a first vertex shared by both paths Q and P. Now construct an alternating zv-path by following Q to the first vertex it shares with P. Then follow P to v via whichever direction gives a proper coloring. □

Construction 2.2. *Let G_1 and G_2 be disjoint 2-edge-colored graphs such that G_1 is flexibly connected and G_2 is properly connected. Let G be obtained by adding two independent edges between G_1 and G_2. Then $pc(G) = 2$.*

Proof. Let $u, v \in V(G_1)$ and $x, y \in V(G_2)$ such that $ux, vy \in E(G)$. Observe that since G_1 is flexibly connected, it is sufficient to demonstrate a coloring of edges ux and vy such that for every $z \in V(G_2)$, there exists an alternating zu-path or an alternating zv-path. Such a coloring can be found by adding vertex w to G_2 with edges wx and wy and then applying Construction 2.1 to determine the colors of edges wx and wy. Use these colors on ux and vy to demonstrate that $pc(G) = 2$. □

Construction 2.3. *Let G_1 and G_2 be 2-edge-colored graphs such that G_1 is flexibly connected and G_2 is properly connected. Let G be obtained from G_1 and G_2 by identifying a vertex in G_1 with a vertex in G_2. Then $pc(G) = 2$.*

Proof. Let x be the identified vertex. Since G_2 is properly connected, for every $z \in V(G_2)$, there exists an alternating zx-path, P, in G_2. Since G_1 is flexibly connected, for every $w \in V(G_1)$, there exists an alternating xw-path that begins on the color different from $end(P)$. □

Construction 2.4. *Let G_1 be a 2-edge-colored graph that is properly connected and let $u, v \in V(G_1)$. Let P_1 be an alternating uv-path in G_1. Let G be obtained from G_1 by adding a uv-path P such that all internal vertices of P are new and such that the cycle obtained from P and P_1 is even. Then $pc(G) = 2$.*

Proof. Alternately color the edges P such that the cycle $P \cup P_1$ is alternately colored. Every vertex z in G_1 has a shortest alternating path to some vertex of P_1. Proceed around the cycle $P_1 \cup P$ in the direction that makes the coloring proper to find an alternating path to any vertex in P. □

Lemma 2.1. *If G is a 2-edge-colored graph such that between every pair of vertices u and v there exists a pair of alternating uv-paths, P and Q such that $start(P) \neq start(Q)$, then $fpc(G) = 2$.*

Proof. Suppose G is a 2-edge-colored graph such that between every pair of vertices u and v there exists a pair of alternating paths from v to u, say P_1 and P_2, such that $start(P_1) \neq start(P_2)$. If $end(P_1) \neq end(P_2)$, then paths satisfying the Lemma have been found. If not, by assumption, there exists an alternating path from u to v, say P_3, such that $start(P_3) \neq end(P_1) = end(P_2)$. Since $start(P_1) \neq start(P_2)$, the color of $end(P_3)$ must be different from one of $start(P_1)$ or $start(P_2)$ and a pair of paths satisfying the Lemma have been found. □

Construction 2.5. *Let G_1 be a 2-edge-colored graph that is flexibly connected and let $u, v \in V(G_1)$. Let P_1 be an alternating uv-path in G_1. Let G be obtained from G_1 by adding a uv-path P such that all internal vertices of P are new and such that the cycle obtained from P and P_1 is even. Then $fpc(G) = 2$.*

Proof. Color the edges of P such that the even cycle obtained from $P_1 \cup P$ is alternately colored. Lemma 2.1 implies that it is sufficient to show that for every $x, y \in G$ there exists a pair of paths from x to y starting on different colors.

Let $x \in G_1$. Since $fpc(G_1) = 2$, we need only consider $y \in P - \{u, v\}$. Construct the desired paths by selecting two alternating paths from x to P_1 that start on different colors. Each path can be extended to an alternating x, y-path by selecting the direction around the cycle $P_1 \cup P$.

Let x be an internal vertex on path P. If $y \in P_1 \cup P$, a pair of xy-paths starting with different colors can be found by going different directions around the cycle $P_1 \cup P$. If $y \in G_1 - P_1$, then find one xy-path by using xPu and the other using xPv. Since G_1 is flexibly colored, the path to u (respectively v) can be extended to an alternating path to y. □

3 Examples

Observation 3.1. *Let u and v be vertices in bipartite graph G. Assume the edges of G are 2-colored. Since the parity of the length of any uv-path in G is fixed, in any alternating uv-path P, the color of start(P) determines the color of end(P).*

The next example is an infinite family of graphs, each containing roughly half of all possible edges, yet with flexible proper connection number greater than 2.

Example 3.2. *Let G be constructed from a complete bipartite graph $K_{m,m}$ by adding one additional vertex u and two edges incident to u with endpoints, v and w, in different partite sets of $K_{m,m}$. We claim $fpc(G) > 2$.*

Observe that if $fpc(G) = 2$, then without loss of generality, uv is red and uw is blue. Let x be any other vertex of G. If G is flexibly colored, then there must exist a vx-path, P_v, that starts on a blue edge and a wx-path, P_w, that starts on a red edge. But the parity of P_v and P_w are different. Thus, end(P_v)=end(P_w). Thus, every alternating xu-path starts on the same color. Thus, $fpc(G) > 2$.

Example 3.3. *The graph in Figure 2, labeled M and described in [3], is the smallest 2-connected graph with proper connection number 3. It is the foundation of several interesting examples.*

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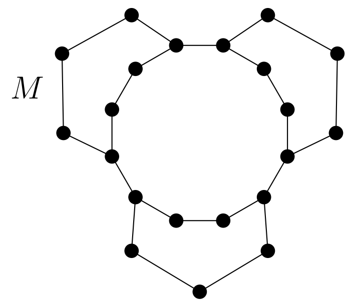

Figure 2: The smallest 2-connected graph with proper connection number 3

Example 3.4. *The graph G_5, in Figure 3, is constructed from the graph M by adding edges from each vertex of M to a graph such that the resulting graph is 5-regular and 2-edge-connected. Vertices of degree 3 in M are connected to a $K_6 - e$ (in red) and vertices of degree 2 in M are connected to $C_4 + 3K_1$ (in blue). Because $pc(M) = 3$, $pc(G_5) \geq 3$.*

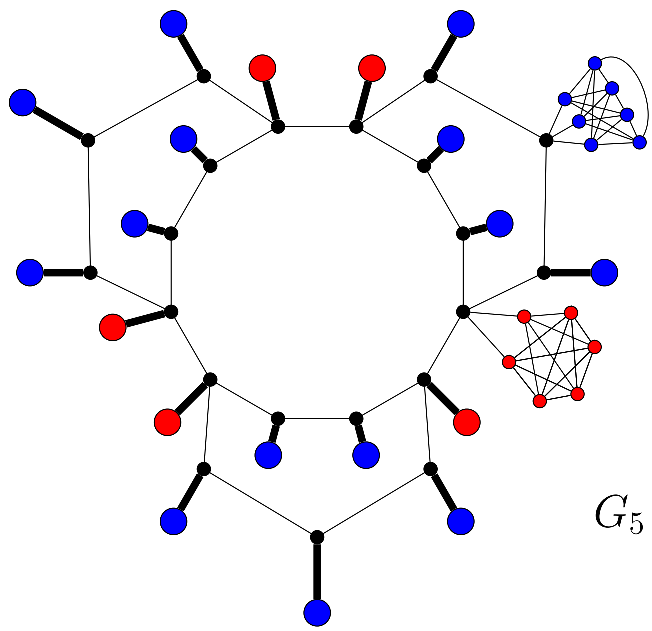

Figure 3: The graph G_5 is a 5-regular, 2-edge-connected graph with proper connection number greater than 2. To simplify the drawing, all large red nodes represent $K_6 - e$ and all large blue nodes represent $C_4 + 3K_1$. Thick edges between vertices of M and large nodes (red or blue) are used to indicated the 2 (or 3) edges between the vertex of M and the $K_6 - e$ (or $C_4 + 3K_1$).

Example 3.5. *The graph G_4, shown in Figure 4, is 4-regular, 2-edge-connected, and has proper connection number greater than 2. To see that G_4 cannot have proper connection number 2, observe that each triangle induced by vertices $\{u_i, v_i, w_i\}$ will have at least two monochomatic edges in any 2-coloring of the edges of G_4. Thus for each $i \in \{0, 1, 2, 3\}$, at least one of v_i or w_i will have the property that every path to u_i ends with an edge of a fixed color. That is, for one of v_i or w_i, every alternating path to u_i always ends on a red* *edge or a blue edge. Without loss of generality, assume that for each i, it is the v_iu_i-path that ends in some fixed color. In order for G_4 to contain an alternating v_0v_1-path, the v_0u_0- and v_1u_1-subpaths must end in the same color. To have an alternating v_0v_2-path, the v_0u_0- and v_2u_2-subpaths must end in different colors. But now there can be no properly colored v_1v_2-path in G.*

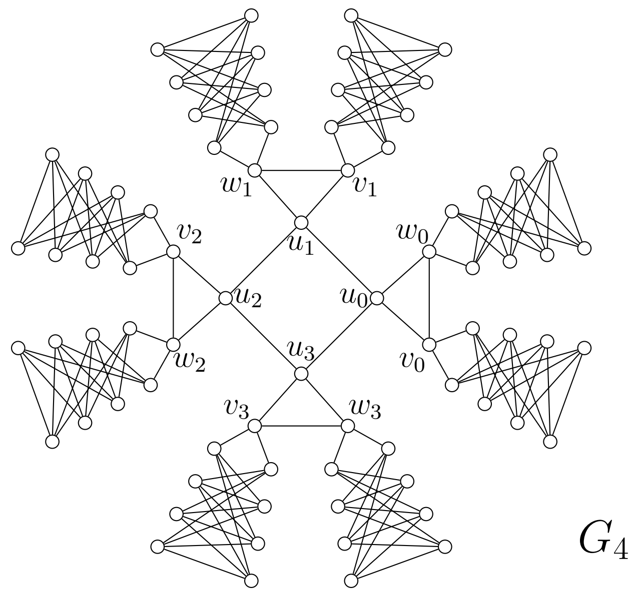

Figure 4: The graph G_4 is a 4-regular, 2-edge-connected graph with proper connection number greater than 2.

From graphs G_4 and G_5 in Examples 3.4 and 3.5 above, we can obtain r-regular, 2-edge-connected graphs with proper connection number greater than 2 for all $r \geq 6$, by appending the appropriate gadget to each vertex. Specifically, to construct a $2k$-regular graph with these properties for $k \geq 3$, start with G_4 and to each vertex v append the gadget H made from a K_{2k+1} from which a matching on $k-2$ edges has been deleted. Make v adjacent to the end vertices of the deleted matching. (See Figure 5.) To construct a $(2k+1)$-regular graph for $k \geq 3$, start with G_5 and to each vertex v append the gadget H' made from a K_{2k+2} from which a matching on $k-2$ edges has been deleted. Make v adjacent to the end vertices of the deleted matching.

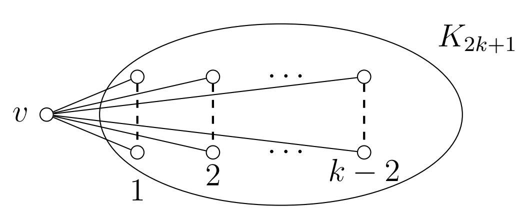

Figure 5: This is the gadget H used to construct a $2k$-regular, 2-edge-connected graph with proper connection number greater than 2 for all $k \geq 3$.

The preceding paragraph proves the existence of the following family of examples.

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Example 3.6. *For $k \geq 4$, there exist graphs G that are 2-edge-connected and k-regular with proper connection number greater than 2.*

Next we will demonstrate via an infinite family that there exist 2-connected almost-k-regular graphs with proper connection number greater than 2. By "almost-k-regular" we mean every vertex has degree k or $k+1$. Note that the graph M is such a graph where $k=2$.

In [3] it was conjectured that graphs with connectivity 2 and minimum degree at least 3 have proper connection number 2. This family demonstrates that the conjecture is false.

The construction is easier to understand using the following lemma.

Lemma 3.7. *Given a graph G with vertex cut $\{u_1, u_2\}$ and component C of $G - \{u_1, u_2\}$, let H be the graph induced by $V(C) \cup \{u_1, u_2\}$. Further assume H is bipartite and there exists a u_1u_2-path in H that contains at least three cut-edges of H. Let G' be a graph obtained from G by replacing C with a u_1u_2-path of the same parity as those through C in G. If $pc(G) = 2$, then $pc(G') = 2$.*

Intuitively, this Lemma asserts that if a graph with proper connection number 2 has cut-vertices u_1 and u_2 (see Figure 6) such that the component (shown in the large oval) is bipartite and such that a u_1u_2-path contains at least 3 cut-edges in the component (shown in bold), then the whole component can be replaced by a path without changing the proper connection number provided that path has the same parity as one through the component in G.

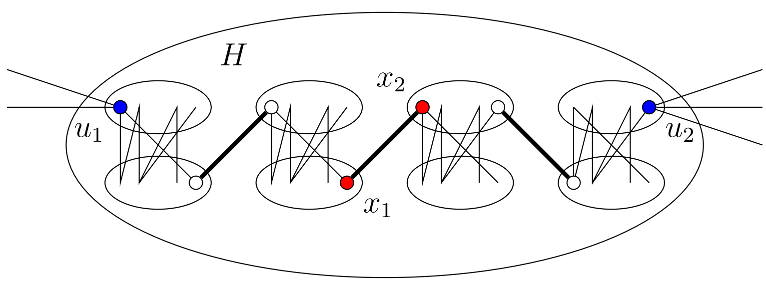

Figure 6: A component that can be replaced with a path without changing the proper connection number.

Proof. Assume the edges of G are 2-colored such that G is properly connected. Choose three cut-edges of H on a u_1u_2-path in H. Let x_1x_2 be the middle cut edge of these three edges. Let P and Q be two alternating x_iu_1-paths in H. Since both paths contain at least one cut edge, $end(P) = end(Q)$. Note that the observation of the previous sentence applies even when paths P and Q start at different x_i's. The same argument applies to any two alternating x_iu_2-paths in H.

Since G is properly connected, for every $z \in V(G) - V(H)$, there exists a properly colored x_iz-path. Moreover, for each z, the properly colored x_iz-path contains a first u_j. That is, while an alternating x_iz-path may contain both cut vertices, u_1 and u_2, one must appear first.

With respect to these alternating x_iz-paths, there are two possibilities.

Case 1: For at least one x_i, every properly colored path to every vertex $z \in V(G)-V(H)$ uses the same one of u_1 or u_2 first.

Without loss of generality, assume that for every $z \in V(G)-V(H)$, every alternating x_1z-path exits H via vertex u_j for $j = 1$ or $j = 2$. It follows that for every $z_1, z_2 \in V(G)-V(H)$, there does not exist an alternating z_1z_2-path through C.

Let Q_1, Q_2 be alternating x_1z_1- and x_1z_2-paths, respectively. Then each of Q_1 and Q_2 contain alternating x_1u_j-subpaths in H. From our earlier observations $end(x_1Q_1u_j) = end(x_1Q_2u_j)$, where $x_1Q_1u_j$ is the subpath of Q_1 starting at x_1 and ending at u_j and $x_1Q_2u_j$ is defined similarly.

We can now demonstrate how to color G' such that $pc(G') = 2$. Color all edges in $G' - V(P - \{u_1, u_2\})$ the same as in G. Alternately color the edges of P, starting at u_j and beginning with color $end(x_1Q_1u_j)$.

To demonstrate that $pc(G') = 2$, we check pairs of vertices. Since P is alternately colored, any pair of vertices on P are properly connected. Given any two vertices in $G' - (V(P - \{u_1, u_2\}))$, the alternating path that existed in G must remain in G' since, by assumption in this case, no properly colored path could have used a u_1u_2-path through C. Finally, if one vertex, v is in P and one, z, is in $G' - P$, observe that by construction, there exists an alternating path from v to u_j. This path arrives at u_j using the same color as was used in the alternating x_1z-path in G. Thus, the path from v to z in G' can be completed by continuing along the same u_1z-path used by x_1 in G.

Case 2: For all i, j there exists some $z \in V(G)-V(H)$ such that there exists a properly colored x_iz-path through u_j.

By assumption, there exists in H an alternating x_iu_j-path for every $i \in \{1, 2\}$ and every $j \in \{1, 2\}$. Moreover, for every alternating x_1u_1-path Q_1 and every alternating x_2u_1-path Q_2, $end(Q_1) = end(Q_2)$. The previous statement holds if u_1 is replaced with u_2. This implies that there exists an alternating u_1u_2-path in H, say Q', and $start(Q') = end(Q_1) = end(Q_2)$ and $end(Q')$ must have the same color as every alternating x_1u_2- or x_2u_2-path in H.

Hence, $start(Q')$ must be the same color as the edge incident to u_1 in H on every x_iz-path for every $z \in V(G) - V(H)$.

We can now demonstrate how to 2-color the edges of G' such that G' is properly connected. Color all edges in $G' - V(P - \{u_1, u_2\})$ the same as in G. Then, alternately color the edges of P starting (and necessarily ending) with the same color as Q'.

To demonstrate that $pc(G') = 2$, we check pairs of vertices. Since P is alternately colored, any pair of vertices on P are properly connected. Given any two vertices in $G' - (V(P - \{u_1, u_2\}))$, if the alternating path in G didn't use vertices from H, the same path exists in G'. If the alternating path did use vertices from H, the path must have passed through H. Since every alternating u_1u_2-path in H began and ended with the same colors and those colors are used to color P, the path through H in G can be replaced with P in G'. Finally, if $v \in P$ and $z \in G' - P$, recall that an alternating zx_1-path existed in G. Follow this path from z to u_i. By construction, this path must arrive at u_i via a color that is compatible with proceeding along P to v. □

Example 3.8. *For $k \geq 3$, there exist graphs G that are 2-connected, such that for every vertex v, $d(v) \in \{k, k+1\}$, and $pc(G) > 2$.*

Observe that graph M is 2-connected, has proper connection number greater than 2, and

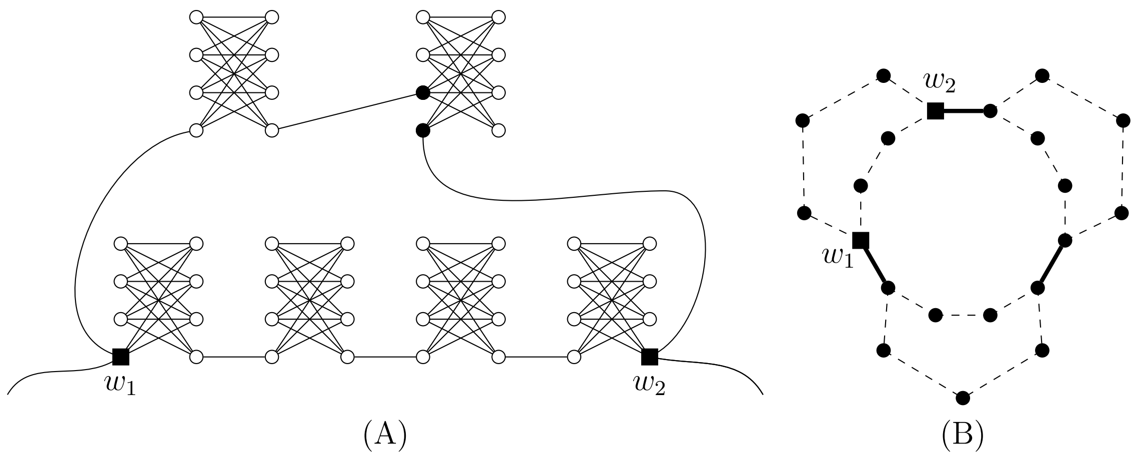

Figure 7: Illustration (A) is the subgraph (or gadget) used to construct a nearly-4-regular, 2-connected graph with proper connection number greater than 2. Each of the three dashed 7-cycles from graph (B) would be replaced by the gadget from (A). The square vertices indicate identified vertices. Once all three 7-cycles are replaced by the gadget, the resulting graph is 2-connected with proper connection number greater than 2 and all vertices have degree 4 or 5.

every vertex has degree 2 or 3. To construct a 2-connected graph G with $pc(G) > 2$ and such that every vertex has degree 4 or 5, replace each 7-cycle in the graph M with the subgraph in Figure 7 (A) such that w_1 and w_2 in the gadget replace the vertices of degree 3 on each 7-cycle. The result is a graph in which every vertex has degree 4 (white) or degree 5 (black). Two applications of Lemma 3.7 imply that, since M has proper connection number greater than 2, so does G. To construct the general family described in Example 3.8 in Figure 7, each $K_{4,4} - e$ can be replaced by $K_{k,k} - e$.

Examples 3.6 and 3.8 indicate that Conjecture 1.1 is sharp. The conjecture certainly holds if G is 3-connected (by Corollary 2 in [3]) or the regularity is sufficiently large (by Lemma 4.4 in [8]).

4 Main Results

In Theorem 3 of [3], the authors describe an additional "strong property" that we defined as being flexibly connected (Definition 1.1). Restated using this new terminology, Borozan et al. prove that if G is bipartite and 2-connected, then $fpc(G) = 2$. In the notes after the proof, the authors mention that their proof strategy will still work if the hypothesis "2-connected" is replaced by the hypothesis "2-edge-connected."

Theorem 4.1. *(Theorem 3 in [3]) Let G be a graph. If G is bipartite and 2-connected (2-edge-connected), then $fpc(G) = 2$.*

We give an abbreviated proof of the stronger, 2-edge-connected result from [3] and prove the converse.

Theorem 4.2. *Let G be a bipartite graph. Then $fpc(G) = 2$ if and only if G is 2-edge-connected.*

Proof. If G is 2-edge-connected, then G has a closed ear decomposition. Since G is bipartite, the cycle that starts the decomposition is even and therefore any alternate coloring results in a flexibly connected graph. Construction 2.5 implies that each time a new ear is added, there exists a coloring such that the new graph is flexibly connected.

If G is not 2-edge-connected, then either G is disconnected or G has a cut-edge, say $e = uv$. Since G is bipartite, given any pair of vertices x and y, every xy-path has the same parity. Thus, if e is a cut-edge, every alternating path that uses edge e must begin on the same color, namely that color which parity requires in order to be consistent with the color of edge e. Thus, it is not possible to color the edges of G such that two vertices in different components of $G - e$ are flexibly connected. □

Theorem 4.3. *Let G be a bipartite graph. Then $pc(G) = 2$ if and only if there exists a path containing every cut-edge of G and G is connected.*

Proof. Let G be a bipartite graph such that $pc(G) = 2$. Assume G is 2-colored such that it is properly connected. Suppose there is not a path containing every cut-edge of G. Then, there exists a block, B, with three edges incident to blocks different from B. Label these edges b_1a_1, b_2a_2 and b_3a_3 where $b_i \in V(B)$ and a_i are each in distinct blocks different from B.

Since G is properly connected, there exists an alternating a_ia_j-path, P. If a_i and a_j are in the same partite set, $start(P) \neq end(P)$. If a_i and a_j are in different partite sets, $start(P) = end(P)$. Since there are only two colors, it is not possible for all three of a_1, a_2 and a_3 to be in the same partite set. Without loss of generality, assume a_1 and a_2 are in the same partite set and a_3 is in the other partite set. So edges b_1a_1 and edge b_2a_2 must be different colors, yet edge b_3a_3 must be the same color as both of them, a contradiction. Thus, there must be a path through all cut-edges of G.

Now suppose G is a connected graph such that there exists a path, $P = x_1, x_2, \cdots, x_r$, containing all of the cut-edges of G. We will demonstrate a coloring of the edges such that G is properly connected.

By Theorem 4.2, every 2-connected block can be flexibly 2-colored, so color these in this manner. Starting at x_1, proceed to the first induced path consisting of cut-edges. Alternately color this path. Proceed to the next induced path consisting of cut-edges and alternately color this path such that it is consistent with a path through the block connecting them. Continue in this manner until all cut-edges are colored.

Observe that by construction, there exists an alternately colored path through any 2-connected component that is consistently colored with cut-edges incident to it. Since each block is flexibly connected, for vertices u and v in a 2-connected component such that v is incident to cut-edge vw, there exists an alternating uv-path in the block that ends with a color different from the color of vw. These two observations show that G is properly connected. □

In [13], the authors define $M(G)$ to be the spanning subgraph that results from deleting all bridges from G. The Theorem below is a combination of one stated in [13] and the comments following that theorem on pages 1273 and 1274:

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Theorem 4.4. *[13] Let G be a connected bipartite graph with order at least 3. Then $pc(G) = 2$ if and only if every component of $M(G)$ is incident with at most two bridges.*

Assuming a graph is connected, the property that every component of $M(G)$ is incident to at most two bridges is equivalent to the property that there exists a path through every bridge.

Recall that a graph is **Class 1** if $\chi'(G) = \Delta(G)$.

Theorem 4.5. *Let $k \geq 2$. If G is a connected, k-regular Class 1 graph on at least three vertices, then $fpc(G) = 2$.*

Proof. Suppose G is connected, k-regular and Class 1. Partition the edges of G into k color classes according to a proper k-edge coloring: $C_1, C_2, \cdots C_k$. Give the edges of G a new coloring by assigning edges in C_1 the color red and assigning all remaining edges the color blue. Note that this new coloring may not be a proper edge coloring.

For each $i \in \{2, 3, \cdots, k\}$, the graph $C_1 \cup C_i$ forms a spanning subgraph of G consisting of even cycles, alternately colored red and blue. We call this collection of even cycles D_i and let $\mathcal{D} = \cup D_i$. Observe that $E(G) = E(\mathcal{D})$. We will show that with this edge coloring, G is properly connected.

Let u and v be arbitrary vertices in G. We will demonstrate an alternating path from u to v that begins with a red edge. Find the fewest number of cycles in $\mathcal{D}$ whose union is connected and that contains both u and v. For convenience, label these cycles $F_1, F_2, \cdots, F_r$ where $u \in V(F_1)$, $v \in V(F_r)$, and $V(F_i) \cap V(F_{i+1}) \neq \emptyset$, for $i \in \{1, 2, \cdots, r-1\}$. As $\{F_1, F_2, \cdots, F_k\}$ is a minimum set of cycles, if $i + 1 < j$, then $V(F_i) \cap V(F_j) = \emptyset$.

We build a uv-path as follows. Start at u and include the red edge of F_1 incident to u. Proceed with this edge around the edges of F_1 until the first vertex of F_2 is encountered. Call this vertex w. If the last edge used from F_1 was red (blue), then we now add the edge of F_2 incident to w that is blue (red).

Note that F_1 and F_2 are not necessarily edge-disjoint and may indeed share red edges. However, because w is the first vertex the two cycles share, we know that none of the edges of F_2 have, thus far, been used on our path.

We proceed in this manner along F_2 until the first vertex of F_3 is encountered and again the preceding edge effectively determines the direction the path goes around cycle F_3. We continue in a similar fashion until v is reached. As desired, an alternating uv-path beginning with a red edge is obtained.

Since v is also incident to a blue edge on F_1, using the same argument we can obtain an alternating uv-path beginning with a blue edge. By Lemma 2.1, $fpc(G) = 2$. □

Corollary 4.6. *If G is a connected, regular, bipartite graph on at least three vertices, then $fpc(G) = 2$.*

Proof. König's Line Coloring Theorem ([9]) states that every bipartite graph is Class 1. □

Corollary 4.7. *Any graph G on at least three vertices that contains a spanning, connected, regular, Class 1 subgraph has $fpc(G) = 2$.*

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