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Applying Quantitative Reasoning to Clarify Arc Measurements

David Glassmeyer
Kennesaw State University, dglassme@kennesaw.edu

Daniel Barton
Lassiter High School, Daniel.Barton@cobbk12.org

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Abstract

The importance of reasoning quantitatively is reflected in both mathematics education research and mathematical standards for K-12 students. In this article, we detail how a quantitative reasoning framework can be used to help differentiate two quantities we have found students often struggle with: arc length and the measure of a central angle. We argue that taking the time to define all four components of a quantity can support students' understanding of theorems involving these quantities.

Keywords

quantitative reasoning, geometry

Quantitative reasoning is needed for students to understand phenomena in the world, and specifically mathematical concepts spanning the K-16 levels (Thompson, 2011). The importance of reasoning quantitatively is reflected as one of the eight standards for mathematical practices in the Common Core State Standards (MP2: Reason abstractly and quantitatively) and is required for understanding concepts beyond high school mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Extensive research has discussed an exact definition of quantitative reasoning, and what it means for a student to reason quantitatively (Moore, LaForest, & Kim, 2012; Moore, Carlson, & Oehrtman, 2009; Thompson, 2011). In particular, Thompson's (2011) framework of a quantity can be a useful tool for understanding and teaching students how to reason about quantities and quantitative relationships.

In this article, we apply Thompson's (2011) quantitative reasoning framework to unpack two quantities observed in high school mathematics standards: arc length and the measure of a central angle. As mathematics teachers at the high school and university levels, we (the authors) have noticed these two quantities can be particularly problematic for students to understand. Additionally, these quantities are essential to understanding more complex mathematical relationships such as trigonometric relationships (Moore, 2013; Moore & LaForest, 2012). We explain the context of our work, a brief overview of the quantitative reasoning framework, how we applied the framework to the quantities, and how applying the framework to instruction might help students understand the quantities.

Context

Georgia uses the Georgia Standards of Excellence, which align with the Common Core State Standards (CCSS). The CCSS High School – Geometry domain of "Circles" contains the cluster titled "Understand and apply theorems about circles", standard 2, which states students should "identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles…" (p. 77). These expectations are reflected in the Georgia Standards of Excellence for Analytic Geometry, unit 3, specifically MGSE9-12.G.C.2 (Georgia Department of Education, 2015a).

While the CCSS do not dictate curriculum or teaching methods, the Georgia Department of Education developed and made freely available a curriculum map, a comprehensive course overview, and a unit-by-unit explanation for all mathematics courses offered at for grades 1-12 (Georgia Department of Education, 2015b). Resources for each unit include the standards addressed, enduring understandings, essential questions, concepts and skills to maintain, selected terms and symbols, evidence of learning, formative assessment lessons, spotlight tasks, and tasks (Georgia Department of Education, 2018a). Relevant to this article is how the Analytic Geometry unit 3 explanation document addresses arc length and the measure of a central angle. Within the selected terms and symbols section, the Georgia Department of Education (2018a) defines and arc as "an unbroken part of a circle" (p. 5) and arc length as "a portion of the circumference of the circle" (p. 5). The central angle of a circle is defined as "an angle whose vertex is at the center of the circle. ∠APB is a central angle of ○P", and includes Figure 1 as an example.

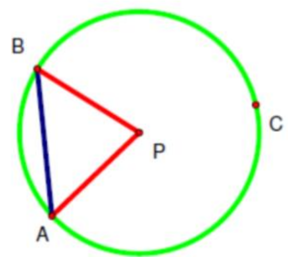


Figure 1: The depiction of a central angle given in Georgia Department of Education (2018a, p. 18).

The document acknowledges potential difficulties when teaching arc length and central angles within the list of common student misconceptions. Specifically, the document states "students sometimes confuse inscribed angles and central angles. For example, they will assume that the inscribed angle is equal to the arc like a central angle" (Georgia Department of Education, 2018a, p. 15). Later, the GDE authors of the document state "students may confuse the segment theorems" (p. 46).

One task has students to draw circles and then "connect the center of each circle to the endpoints of the arcs, forming central angles" (Georgia Department of Education, 2018a, p. 16). Within the solution to this task is the note that:

> We say that the central angle ∠APB intercepts or has $\widehat{AB}$ [figure 1]. We also say that $\widehat{AB}$ subtends or has the central angle ∠APB. Note that when we refer to the arc of a central angle, we usually mean the minor arc unless otherwise stated. Arcs are measured in two different ways - using degree measure and using linear measure. Usually when we refer to the measure of an arc, we are referring to the degree measure. The measure of a minor arc is

> defined to be the measure of the central angle that intercepts the arc." (Georgia Department of Education, 2018a, p. 18).

The GDE authors later state "The length of an arc is different from its measure. The length is given in linear units (e.g., inches, centimeters, and feet)…Congruent arcs have equal degree measures and equal lengths. Equivalent arcs have equal degree measures" (Georgia Department of Education, 2018a, p. 19). Examples are given of this idea, specifically showing how the length of the arc, when measured in radians, remains the same regardless of the size of the circle. A task details how teachers can lead students to relate the central angle to the arc length of the circle, specifically $\frac{central\ angle}{360°} = \frac{arc\ length}{circumference}$. A second task details how teachers can lead student to relate the central angle to area of the sector. Finally, in a graphic organizer for the unit, the authors include two theorems related to arc length and the measure of a central angle (figure 2), which we will refer to as theorem 1: $m\angle A = arc$, and theorem 2: $m\angle A = \frac{1}{2}(arc)$.

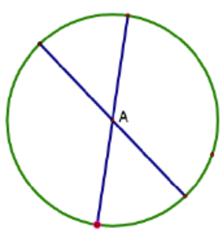
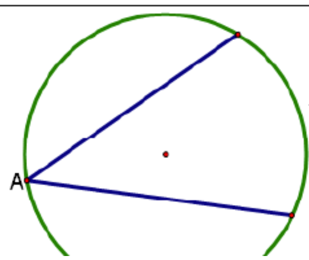
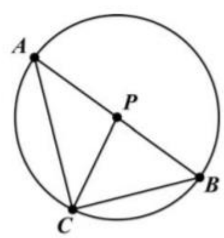
Picture	Theorem
	$m\angle A = arc$
	$m\angle A = \frac{1}{2}(arc)$

Figure 2. Theorems associated with arc measure and central angles, as shown in Georgia Department of Education (2018a, p. 19).

In our (the authors') experience teaching secondary mathematics, we have found students have difficulty distinguishing the quantities of arc length and the measure of a central angle. This difficulty becomes even more problematic when students must relate the quantities through theorems such as theorem 1 and 2. At a glance at figure 2, students have trouble distinguishing between the two theorems. We have seen students question how the two quantities can be equal to each other in theorem 1, then one quantity be equal to half the other in theorem 2. We have found that when students are given various pieces of information in random orders (i.e. given the arc length and asked to find the interior angle, or given the interior angle and asked to find the arc length), they often mix up the concepts of when to multiply by two or by one-half. For example, when we asked our high school students to find the measure of the arc on a circle with an inscribed angle measure of 92, common responses included: “divide 92 by two”, “yeah, the arc is 46”, and “no, you multiply 92 by two”. Furthermore, we have seen students incorrectly apply theorem 2 to problems when finding the intercepted arc length (figure 3).

2. In circle P below, $\overline{AB}$ is a diameter.



If $m\angle APC = 100°$, find the following:

a. $m\angle BPC$

b. $m\angle BAC$

c. $m\widehat{BC}$

d. $m\widehat{AC}$

Figure 3. When solving this problem, we have seen students incorrectly use theorem 2 to say $m\angle APC = 100° = \frac{1}{2}m\widehat{AC}$ and thus $m\widehat{AC} = 200$. Problem taken from the Georgia Milestones End of Course Study/Resource Guides for Analytic Geometry (Georgia Department of Education, 2018b; Georgia Milestones Assessment, 2017).

These kinds of problems have also been documented by the authors of Georgia Department of Education (2018a), as indicated in their list of common misconceptions, as well as larger mathematics literature on the problems of student understanding of circle relationships (Moore et al., 2012). Given the importance and usefulness of quantitative reasoning in understanding mathematical concepts (Moore et al., 2009), we decided to use a quantitative reasoning framework to clarify each quantity and see how this explanation might help a student distinguish theorems 1 and 2.

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Quantitative Reasoning Framework

The quantitative reasoning framework defines quantities and what reasoning about quantities entails. A quantity is a conceptualization of four components: (1) an object, (2) a measurable attribute of the object, (3) a way of measuring the attribute, including a unit of measurement, and (4) a conceivable numerical value, or values, associated through a proportional relationship with the unit of measurement (Thompson, 2011). Quantitative relationships relate quantities based on mutual constraints on the measurable attributes involved and considering how the quantities covary together in the relationship (Moore, Carlson, & Oehrtman, 2009). Difficulties modeling or analyzing a situation mathematically often result from conflating the four components of a quantity, such as not distinguishing the units from the attribute, or by focusing on relationships between sets of numbers rather than the measurement values derived from varying quantities (Thompson, 2011). Quantitative reasoning entails attending to and identifying quantities, constructing new quantities, and identifying and representing relationships between quantities (Moore et al., 2009).

Applying the Quantitative Reasoning Framework

We can apply the quantitative reasoning framework to each of our quantities, arc length and the measure of a central angle. For both quantities, we will clearly identify the four components, and note where these components are similar and different.

Quantity 1: Arc Length

- Object: an unbroken portion of a circle’s circumference. This portion is often indicated by a portion of the circle circumference between points where two segments intersect the circle. For example, circle O in figure 4 has an unbroken portion of its circumference between points A and B, designated as the minor arc between the points.
- Attribute: the length of the object. Specifically, the distance along the circumference from the start to the end of the designated portion of the circle’s circumference.
- Way of measuring the attribute: we can measure the length of the object in a few ways, all of which results in a linear measurement. For example: we could measure the length using:
    - A measuring tape that is curved against the portion of the circumference, using whatever length units we like, such as inches.
    - String that is curved against the portion of the circumference, then straightening the string against a ruler that measures whatever length units we like, such as centimeters.

    - Either of the two methods above, except making the units be radius lengths of the circle being measured.
- Possible values and representation: the length of the object will always be a positive real number and is represented using the "arc" symbol above the points designating the beginning and end of the unbroken portion of the circle's circumference. For example, we can refer to the arc length depicted in figure 4 as $\widehat{AB}$.

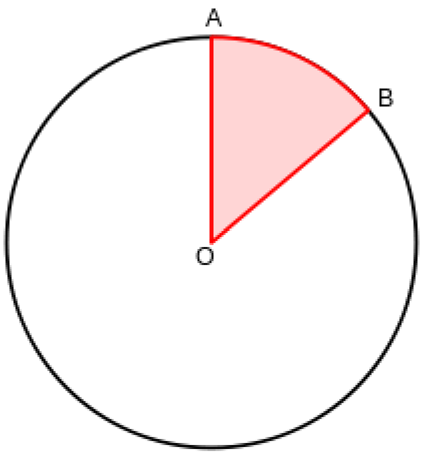

Figure 4. Within circle O, the quantity of arc length between points A and B is written as $\widehat{AB}$.

Now let's examine our second quantity.

Quantity 2: Measure of a central angle

- Object: two rays or segments sharing a common endpoint at the center of the circle with a designated orientation, usually the smaller of the two angles created. For example, in Figure 4 there are two segments ($\overline{OA}$ and $\overline{OB}$) sharing a common endpoint at the center of the circle that intersect the circle at the beginning and end of the arc on the circle.
- Attribute: the openness/angle between the two rays or segments of the object, usually defined to be the smaller of the two possible angles unless otherwise noted.
- Way of measuring the attribute, with units: the fractional portion of the arc length cut off in comparison to the entire circumference (this portion could be calculated using the circle given or of any circle other centered created at the angle's vertex). The attribute could be measured in a number of ways, including:
    - As a percentage, which is calculated by measuring the arc length using a unit (inches, meters, radius lengths, etc.), then dividing that arc length by the length of the entire circumference, measured it the same unit (inches, meters, radius lengths, etc.). Since the same unit is used

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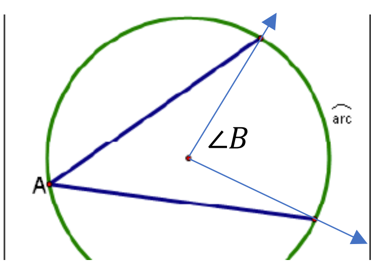
(inches, meters, radius lengths, etc.) to measure both the arc length and circumference length, thus the quotient of these two values is a unitless quantity.

  - Using degrees, by multiplying the percentage by 360. A protractor manually computes the fractional portion of the arc length (of a circle with a radius the size of your protractor's measurement) cut off by two rays, multiplied by 360.
  - Using radians, by multiplying the percentage by 2π. As Moore and LaForest (2014) detail, this is the same as conceptualizing an angle as measuring the arc length as a particular number of radii lengths, where again the size of the circle does not matter.
- Possible values and representation: The possible values are typically positive real numbers, and the measure of the central angle is designated $m\angle AOB$.
-

Now that the quantities have been defined clearly using the quantitative reasoning framework, we can examine the relationship given in theorems 1 and 2. Theorem 1, $m\angle A = arc$, can now be put into words: Given that angle A is a central angle that creates an arc, the measure of the central angle A is the same value as the arc measurement because this is how we defined arc measure.

The second theorem, $m\angle A = \frac{1}{2}(arc)$, can also be understood when considering the quantities involved. In the left-hand side of the equation, $m\angle A = \frac{1}{2}arc$, the quantity $m\angle A$ does not refer to the measure of a central angle, but instead the measure of an inscribed angle. The right-hand side of the equation is one-half times the value of the arc measure, which defined to be one-half the measure of the central angle that creates the arc (theorem 1). Since no central angle is given in the picture, it may be helpful draw one, and label this central angle B (Figure 4). Now we can rewrite $m\angle A = \frac{1}{2}arc$ as $m\angle A = \frac{1}{2}(m\angle B)$ because of theorem 1.



$$m\angle A = \frac{1}{2}(arc)$$

$$m\angle A = \frac{1}{2}(m\angle B)$$

Figure 5. Drawing the central angle can help student recall the meaning of arc measure.

To help explain why the left hand side of the equation is the same as the right hand side (e.g. the measure of the inscribed angle is one-half times the measure of central angle creating the same arc), we have students explore dynamic geometry sketches similar to Figure 5 and what is described in Baccaglini-Frank (2012). These sketches automatically measure central angles and arc lengths for students. Allowing students to adjust the points and vary the quantities allow students to record multiple values $m\angle A$ and $m\angle B$ and thus generate this relationship on their own. We have used this activity lead students to find this theorem rather than present it directly.

Conclusions

Having students memorize the theorems that are often presented in textbooks and curricular documents can lead to problems in students' understanding of the associated mathematical concepts. In the geometric context of arc measures, applying mathematical notation can be difficult, as arc measure and arc length notation is confusing, especially since arc measure discretely relies on an object, the central angle that is not always drawn.

In considering how to prevent and correct student misconceptions, we turned to quantitative reasoning literature. Using this approach has helped our students distinguish and make sense of the quantities arc length and the measure of a central angle, and the relationships between these quantities. Being explicit and attending to the components of quantities, or going through the process of quantification, aligns with best practices in mathematics education research (Moore, 2013; Thompson, 2011). Quantitative reasoning can help students recognize the way we measure these quantities and can build towards a more robust understanding of these mathematical relationships. We also recommend the use of dynamic geometry sketches to have students create these theorems on their own.

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