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Gestur Olafsson

Louisiana State University at Baton Rouge

Shijun Zheng

Georgia Southern University, szheng@georgiasouthern.edu

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Recommended Citation

Olafsson, Gestur, Shijun Zheng. 2008. "Harmonic Analysis Related to Schrödinger Operators." *Contemporary Mathematics in AMS Book Series*, Gestur Ólafsson, Eric L. Grinberg, David Larson, Palle E. T. Jorgensen, Peter R. Massopust, Eric Todd Quinto and Boris Rubin (Ed.), 464: 213-230: American Mathematical Society. doi: 10.1090/conm/464 source: <http://arxiv.org/pdf/0711.3262v1.pdf> isbn: 978-0-8218-8143-9

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HARMONIC ANALYSIS RELATED TO SCHRÖDINGER OPERATORS

GESTUR OLAFSSON AND SHIJUN ZHENG

ABSTRACT. In this article we give an overview on some recent development of Littlewood-Paley theory for Schrödinger operators. We extend the Littlewood-Paley theory for special potentials considered in the authors' previous work. We elaborate our approach by considering potential in C_0^∞ or Schwartz class in one dimension. In particular the low energy estimates are treated by establishing some new and refined asymptotics for the eigenfunctions and their Fourier transforms. We give maximal function characterization of the Besov spaces and Triebel-Lizorkin spaces associated with H . We then prove a spectral multiplier theorem on these spaces and derive Strichartz estimates for the wave equation with a potential. We also consider similar problem for the unbounded potentials in the Hermite and Laguerre cases, whose $V = a|x|^2 + b|x|^{-2}$ are known to be critical in the study of perturbation of nonlinear dispersive equations. This improves upon the previous results when we apply the upper Gaussian bound for the heat kernel and its gradient.

1. INTRODUCTION

The purpose of this article is to review recent development of harmonic analysis for differential operators, in particular a Schrödinger operator $H = -\Delta + V$, where V is a real-valued potential function on \mathbb{R}^n . We are interested in developing the Littlewood-Paley theory for H in order to understand the associated function spaces and their roles in dispersive partial differential equations.

This subject has been drawing increasing attention in the area of harmonic analysis and PDE [29, 15, 13, 9, 17, 19, 21, 22, 26, 4, 33, 36, 23, 5], to name only a few. The function space theory for H was introduced in [19, 21, 15] for the Hermite and Laguerre operators. In [4, 33] the authors considered Littlewood-Paley theory for H with special potentials in an effort to extend the function space theory to the bounded potential case. In this paper we will summarize and develop the fundamental theory for general Schrödinger operators on \mathbb{R}^n . Furthermore we obtain a Littlewood-Paley decomposition for L^p spaces as well as Sobolev spaces using dyadic functions of H . We elaborate our approach by considering one dimensional H with V in $\mathcal{S}(\mathbb{R})$, the Schwartz class. We will give outlines of the proofs for some of the main results and refer to the references, either old or new, for the detailed proofs.

Date: October 10, 2007.

2000 Mathematics Subject Classification. Primary: 42B25; Secondary: 35J10.

Key words and phrases. Besov space, Littlewood-Paley theory, Schrödinger operator.

G. Olafsson is supported by NSF grant DMS-0402068. S. Zheng is supported by DARPA grant HM1582-05-2-0001.

1.1. Besov and Triebel-Lizorkin spaces. For a (Borel) measurable function $\phi: \mathbb{R} \rightarrow \mathbb{C}$ we define by functional calculus

$$\phi(H) = \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda$$

where $H = \int \lambda dE_\lambda$ is the spectral resolution of H .

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \subset C_0^\infty(\mathbb{R})$ be a dyadic system satisfying

- (i) $\text{supp } \varphi_j \subset \{x : 2^{j-2} \leq |x| \leq 2^j\}$,
- (ii) $|\varphi_j^{(k)}(x)| \leq c_k 2^{-kj}$, $\forall j \in \mathbb{Z}, k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$,
- (iii) $\sum_{j=-\infty}^{\infty} |\varphi_j(x)| \approx 1$, $\forall x \neq 0$.

Let $\alpha \in \mathbb{R}, 0 < p \leq \infty, 0 < q \leq \infty$ and $\{\varphi_j\}_{j \in \mathbb{Z}}$ be as above. The homogeneous Besov space associated with H , denoted by $\dot{B}_p^{\alpha,q}(H)$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the quasi-norm

$$(1.1) \quad \|f\|_{\dot{B}_p^{\alpha,q}(H)} = \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \|\varphi_j(H)f\|_{L^p}^q \right)^{1/q}.$$

Similarly, the homogeneous Triebel-Lizorkin space associated with H , denoted by $\dot{F}_p^{\alpha,q}(H)$, $\alpha \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ is defined by the quasi-norm

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q} |\varphi_j(H)f|^q \right)^{1/q} \right\|_{L^p}.$$

We are mainly concerned with the following three interrelated problems.

- Problem 1.1.**
- a. Littlewood-Paley theory for $B(H)$ and $F(H)$ using maximal function characterization.
 - b. Spectral multiplier theorem: Find sufficient condition for $\mu \in L^\infty(\mathbb{R})$ such that $\mu(H)$ is bounded on $L^p, B(H)$, and $F(H)$.
 - c. Strichartz estimates for $e^{-it\sqrt{H}}$ and e^{-itH} which measure the spacetime regularity of solutions to wave and Schrödinger equations.

The decay estimate in (1.2) has been known to be fundamental and useful in function space theory and spectral multiplier problem [19, 15, 4, 33, 34]. We will see that it can be applied to characterize $B(H)$ and $F(H)$ spaces with full range of parameters $0 < p, q < \infty$ and show Mihlin-Hörmander type multiplier result on $L^p, B(H)$ and $F(H)$ spaces; for the multiplier problem we actually formulate a more general condition as in (1.5).

Let $\phi(H)(x, y)$ denote the integral kernel of $\phi(H)$.

Assumption 1.2. Let $\phi_j \in C_0^\infty(\mathbb{R})$ satisfy the conditions in (i), (ii). Assume that for $\ell = 0, 1$ and for every $N \in \mathbb{N}_0$ there exists a constant $c_N > 0$ such that for all $j \in \mathbb{Z}$

$$(1.2) \quad |\nabla_x^\ell \phi_j(H)(x, y)| \leq c_N \frac{2^{(n+\ell)j/2}}{(1 + 2^{j/2}|x - y|)^N}.$$

We will outline the proof of the fact in Section 4 that on the real line H satisfies Assumption 1.2 for $V \in C_0^\infty(\mathbb{R})$. We discover that V being compactly supported and H having no resonance at zero are necessarily and sufficient for the gradient

estimate ($\ell = 1$) in (1.2) to hold in low energy $-\infty < j < 0$. The kernel decay for $\ell = 0$, $j = 0$ was an open question in [38, B.7].¹

Define the Peetre type maximal function for H as: for $j \in \mathbb{Z}$, $s > 0$

$$\varphi_{j,s}^* f(x) = \sup_{t \in \mathbb{R}^n} \frac{|\varphi_j(H)f(t)|}{(1 + 2^{j/2}|x - t|)^s}.$$

The following theorem gives a maximal function characterization of the *homogeneous* spaces.

Theorem 1.3. *Suppose H satisfies Assumption 1.2 and $\{\varphi_j\}$ is a system satisfying (i)–(iii). The following statements hold.*

a) *If $0 < p \leq \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $s > n/p$, then*

$$\|f\|_{\dot{B}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha} \varphi_{j,s}^*(H)f\}\|_{\ell^q(L^p)}.$$

b) *If $0 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $s > n/\min(p, q)$, then*

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha} \varphi_{j,s}^*(H)f\}\|_{L^p(\ell^q)}.$$

It is well-known that such a characterization implies that any two dyadic systems satisfying (i)–(iii) give rise to equivalent norms on $\dot{B}_p^{\alpha,q}(H)$ and $\dot{F}_p^{\alpha,q}(H)$. Another consequence is that H has the lifting property.

Corollary 1.4. *Suppose H satisfies Assumption 1.2. Let $s, \alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then H^s maps $\dot{B}_p^{\alpha,q}(H)$ isomorphically and continuously onto $\dot{B}_p^{\alpha-s,q}(H)$. Moreover, $\|H^s f\|_{\dot{B}_p^{\alpha-s,q}(H)} \approx \|f\|_{\dot{B}_p^{\alpha,q}(H)}$. The analogous statement holds for $\dot{F}_p^{\alpha,q}(H)$.*

The proofs of Theorem 1.3 and Corollary 1.4 are quite standard and can be found in [55, 33]; see also [19, 15].

Using Calderón-Zygmund decomposition and Assumption 1.2 we can show that $L^p(\mathbb{R}^n) = \dot{F}_p^{0,2}(H)$ if $1 < p < \infty$ and obtain the Littlewood-Paley inequality for L^p spaces. If in addition $V \in S$, we can use lifting property of H to characterize the Sobolev spaces $W_p^{2\alpha}(\mathbb{R}^n) = F_p^{\alpha,2}(H)$, the inhomogeneous versions of $\dot{F}_p^{\alpha,2}(H)$, with equivalent norms [55, 33].

Theorem 1.5. *Let $1 < p < \infty$. The following statements hold.*

a) *If H satisfies Assumption 1.2, then*

$$\|f\|_{L^p(\mathbb{R}^n)} \approx \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j(H)f(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$

b) *If, in addition to the condition in a), $V \in S(\mathbb{R}^n)$, then for all $\alpha \in \mathbb{R}$*

$$\|f\|_{W_p^{2\alpha}(\mathbb{R}^n)} \approx \|\Phi(H)f\|_{L^p(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} 2^{2j\alpha} |\varphi_j(H)f(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)},$$

where $\{\Phi, \varphi_j\}_{j=1}^{\infty}$ is an inhomogeneous system satisfying (i), (ii) and (iii') below.

¹For the sake of exposition we are not trying to pursue how singular $V \in L^1(\mathbb{R})$ can be. Indeed, for each N the proof of Theorem 4.1 shows that (1.2) holds for $j \geq 0$ if $|V^{(s)}(t)| \lesssim \langle t \rangle^{-N-2-\epsilon}$, $0 \leq s \leq N$ and that (1.2) holds for $j < 0$ if $V \in L^1$ with compact support and H has no resonance at zero.

2

The analogous results above also hold for the inhomogenous spaces $B_p^{\alpha,q}(H)$, $F_p^{\alpha,q}(H)$ if using the system $\{\Phi, \varphi_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R})$ with $\text{supp } \Phi \subset [-1, 1]$, φ_j satisfying (i), (ii) and instead of (iii)

$$(iii') \quad |\Phi(x)| + \sum_{j=1}^{\infty} |\varphi_j(x)| \approx 1, \quad \forall x.$$

However, the homogeneous space, which contains both high and low energy analysis of H , are essential and more useful in proving Strichartz inequality for wave equations; see e.g., [36, 30] and Section 2.

1.2. Spectral multipliers. For Problem 1.1 b, Mihlin-Hörmander type spectral multipliers for H have been considered in [26, 20, 16, 13] and more recently [36].

In the classical case $H_0 = -\Delta$, one can use Calderón-Zygmund lemma to prove the L^p Fourier multiplier theorem by showing that the kernel of $\mu(H_0)$ verifies the Hörmander condition

$$(1.3) \quad \int_{|x-y|>2|y-\bar{y}|} |K(x, y) - K(x, \bar{y})| dx \leq C$$

if μ satisfies certain smoothness condition; see e.g. [28, 41]. However, for a general elliptic operator (1.3) is not available. For the Schrödinger operator H with $V \geq 0$, Hebisch [26] used heat kernel estimates (h.k.e) to prove a spectral multiplier theorem on L^p . Later on the heat kernel approach has been further developed to deal with positive selfadjoint differential operators [13].

The question remains if the negative part of V is nonzero, in which case the upper Gaussian bound for e^{-tH} may *not* be valid. In [56] we are able to treat general V by replacing the h.k.e. with a (much) weaker condition for the pointwise decay of a spectral kernel, namely (1.5).

As in [56] the following hypothesis on H is the main ingredients in proving spectral multiplier theorem. Let $\phi_j(x) = \phi(2^{-j}x)$.

Assumption 1.6. *a. (Weighted L^2 estimate) There exists $s > n/2$ such that*

$$(1.4) \quad \sup_y \||x - y|^s \phi_j(H)(x, y)\|_{L_x^2} \leq c2^{(n/2-s)j/2} \quad \forall j \in \mathbb{Z},$$

where $c = c(\|\phi\|_{X_*^s})$.

b. (Weighted L^∞ estimate) There exists a finite measure ζ such that for all $j \in \mathbb{Z}$

$$(1.5) \quad |\phi_j(H)(x, y)| \leq c' \int_{\mathbb{R}^n} 2^{jn/2} (1 + 2^{j/2}|x - y - u|)^{-n-\epsilon} d\zeta(u),$$

where $c' = c'(\|\phi\|_{W_2^{n+\epsilon}})$.

Here $X_*^s = \{f \in X^s(\mathbb{R}_*) : \|f\|_{X_*^s} =: \sup_{t>0} \|f(t \cdot)\eta\|_{X^s} < \infty\}$, where η is a fixed function in C_0^∞ with support away from 0, $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ and X^s is either $W_2^s(\mathbb{R})$ or $C^s(\mathbb{R})$, the Hölder class [49, 16].

In one dimension Assumption 1.6 is satisfied for $X^s = C^1$ if $V \in L_2^1(\mathbb{R})$ or $V \in L_1^1(\mathbb{R})$ and H has no resonance at zero, where $L_\gamma^1 := \{V : (1 + |x|)^\gamma V \in L^1\}$. In three dimensions Assumption 1.6 is true for $X^s = W_2^s$, if $V = V_+ - V_-$ is in the

² Again we are not trying to seek optimal condition on V ; one can show that if $|\partial_x^k V(x)| \leq c_k$, $|k| \leq 2m_0 - 2$ for some $m_0 \in \mathbb{N}$, then b) is true for $|\alpha| \leq m_0$.

Kato class with small Kato norm, namely $\|V_{\pm}\|_K < 2\pi$ and H has no resonance at zero [57, 9].

Under Assumption 1.6 for H and the condition $\mu \in X_*^s$ for some $s > n/2$, the boundedness of $\mu(H)$ on $\dot{B}_p^{\alpha,q}(H)$, $1 < p < \infty$ is an immediate consequence of the L^p result in [56]. To prove that $\mu(H)$ is bounded on $\dot{F}_p^{\alpha,q}(H)$ we use an $L^p(\ell^q)$ vector-valued version of the proof of the L^p result by applying Calderón-Zygmund decomposition and the dyadic estimates (1.4) and (1.5).

Theorem 1.7. [34] *Suppose H satisfies Assumption 1.6. Let $X^s = C^s$ or W_2^s and η be a fixed function in $C_0^\infty(\mathbb{R}_*)$. If there exists some $s > \frac{n}{2}$ so that*

$$\sup_{t>0} \|\mu(t \cdot) \eta\|_{X^s} < \infty,$$

then $\mu(H)$ is bounded on $\dot{B}_p^{\alpha,q}(H)$ and $\dot{F}_p^{\alpha,q}(H)$, $1 < p, q < \infty$, $\alpha \in \mathbb{R}$.

It is easy to observe that $X = C^s$ corresponds to the usual Mihlin condition and $X = W_2^s$ the Hörmander condition. That the the exponent $\frac{n}{2}$ is sharp has been noted in e.g., [41, 13]. Note that under the conditions in Assumption 1.6, which is an alternative condition than Assumption 1.2, Theorem 1.7 implies

$$\begin{aligned} \|f\|_{\dot{B}_p^{\alpha,q}(H)}^\phi &\approx \|f\|_{\dot{B}_p^{\alpha,q}(H)}^\psi \\ \|f\|_{\dot{F}_p^{\alpha,q}(H)}^\phi &\approx \|f\|_{\dot{F}_p^{\alpha,q}(H)}^\psi \end{aligned}$$

given any two system $\{\phi_j\}_{j \in \mathbb{Z}}$, $\{\psi_j\}_{j \in \mathbb{Z}}$, which is also a corollary of Theorem 1.3 as we have mentioned. Moreover, by interpolation and duality we obtain from the proof of Theorem 1.7 that

$$\dot{F}_p^{0,2}(H) = L^p, \quad 1 < p < \infty$$

(see [33] for the inhomogeneous case), which is part a) of Theorem 1.5 while under somehow more general conditions in Assumption 1.6.

Remark on Assumption 1.6 Assumption 1.6 is *intrinsic* in the sense that it only relies on the property of H and is independent of the multiplier μ . As can be seen from the proofs in [56, 34], Inequalities (1.4) and (1.5) are to control higher and lower energy estimates of $\mu(H)$ respectively. If in (1.4) letting $\zeta = \delta$, the Dirac measure, then we obtain the following pointwise decay

$$|\phi_j(H)(x, y)| \leq c_{n,\epsilon} 2^{jn/2} (1 + 2^{j/2} |x - y|)^{-n-\epsilon}$$

which is only valid, in general, for nonnegative potentials. This is the reason why we call (b) a “weighted” pointwise estimate.

The remaining of the paper is organized as follows. In Section 2, we apply the interpolation properties of $B(H)$, $F(H)$ to obtain Strichartz estimates for $e^{-it\sqrt{H}}$. In Section 3 we provide the outlines of the proofs of Theorem 1.3 and Theorem 1.5 under Assumption 1.2. In Section 4 we show that for any V in $\mathcal{S}(\mathbb{R})$, Assumption 1.2 is verified for high energy and for any V in $C_0^\infty(\mathbb{R})$, the low energy estimates holds in the absence of resonance. The proofs are based on certain new and refined estimates for the modified Jost functions and its Fourier transforms whose details are quite lengthy and will appear elsewhere. For unbounded potentials we consider in Section 5 the analogue of Theorem 1.3 for Hermite and Laguerre operators, where $V = a|x|^2 + b|x|^{-2}$, by using gradient estimates for e^{-tH} . Further, we would like

to mention that the literature in the area suggests that it is possible to consider analogous problems for H with (degenerate) magnetic potentials, cf. [48, 44, 57]

$$-\frac{1}{2} \sum_{j=1}^n (\partial_{x_j} + ia_j y_j)^2 + (\partial_{y_j} - ia_j x_j)^2, \quad a_j \in \mathbb{R},$$

by following a similar approach developed here.

2. STRICHARTZ ESTIMATES FOR H

It is well known that Strichartz estimates have useful applications in wellposedness problem for nonlinear dispersive equations [30, 42, 37]. Consider the following perturbed wave equation with a potential on \mathbb{R}^{1+n}

$$(2.1) \quad \begin{cases} u_{tt} + Hu = F(t, x) \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \end{cases}$$

whose solution is given by

$$u(t, x) = \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}F(s, \cdot)ds.$$

When $n \geq 2$, the original Strichartz estimate for (2.1) with $V = 0$ reads [43]

$$(2.2) \quad \|u\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^{n+1})} \leq c\|f\|_{\dot{W}_2^{-\frac{1}{2}}}$$

if $u_0 = 0$, $u_1 = f$ and $F = 0$, where $\dot{W}_2^s := \dot{W}_2^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space. We are interested in proving Strichartz estimates for (2.1) in the Besov space scale, assuming that the dispersive estimate (2.3) is true. In this section for convenience we will use $\dot{B}_p^{\alpha, q}(\sqrt{H})$ instead of $\dot{B}_p^{\alpha, q}(H)$, whose norm is given by (1.1) with H replaced by \sqrt{H} . Note that $\dot{B}_p^{2\alpha, q}(\sqrt{H}) = \dot{B}_p^{\alpha, q}(H)$.

Assumption 2.1. *Assume that $u(t, x)$, the solution to (2.1) with $u_0 = 0$, $u_1 = f$ and $F = 0$, satisfies for all $t \neq 0$*

$$(2.3) \quad \|u(t, \cdot)\|_\infty \leq c|t|^{-(n-1)/2}\|f\|_{\dot{B}_1^{\frac{n-1}{2}, 1}(\sqrt{H})}.$$

Dispersive estimates in (2.3) were obtained, for instance, in [3, 7] for V being smooth and in [9] for V in the Kato class.

The idea to treat (2.1) is to combine the arguments in [30] and [25] for a free wave equation. The Littlewood-Paley decomposition seems efficient in dealing with this type of estimates although we do not have available the scaling invariance for H or $\varphi_j(H)$, as is important and crucial in the classical case.

For this purpose we will need some interpolation, duality and embedding properties for $\dot{B}_p^{\alpha, q}(\sqrt{H})$, which are analogues of $\dot{B}_p^{\alpha, q}(\mathbb{R}^n)$ and can be derived as corollaries of Theorem 1.7 [33, 49]. Directly using the classical Besov spaces would encounter commuting problem.

From the expression of $u(t, x)$ we see that it is essential to estimate $e^{\pm it\sqrt{H}}f$. Note that (2.3) is equivalent to

$$(2.4) \quad \|u_j(t, \cdot)\|_\infty \leq c|t|^{-(n-1)/2}2^{j(n-1)/2}\|\varphi_j(\sqrt{H})f\|_1,$$

where $u_j(\cdot) = \varphi_j(\sqrt{H})u(t, \cdot)$. Using (2.4) and TT^* argument in [25, 30] we obtained in [57] the following theorem.

Theorem 2.2. *Let $n \geq 2$, $q, r \in [2, \infty]$ and $s \in \mathbb{R}$. Suppose H satisfies the estimate in (2.3). Then the following estimates hold.*

a)

$$(2.5) \quad \|e^{it\sqrt{H}}f\|_{L^q \dot{B}_r^{s,2}(\sqrt{H})} \leq c \|f\|_{\dot{B}_2^{s+\sigma,2}(\sqrt{H})},$$

where $q = 2$ and $\sigma = \sigma(q, r)$ verifies the gap condition $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \sigma$.

b) Let $I \subset \mathbb{R}$ be an interval.

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} F(s, \cdot) ds \right\|_{L^q \dot{B}_r^{s,2}(\sqrt{H})} \leq c \|F\|_{L^{q'} \dot{B}_{r'}^{s+2\sigma-1,2}(\sqrt{H})},$$

where q', r' denote the usual Hölder conjugate exponents of q, r , $(q, r) \neq (2, \frac{2n-2}{n-3})$ and (q, r) are wave-admissible, that is, $\frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}$.

The estimates in Theorem 2.2 are analogous to those in the case of zero potential [25]. Under additional condition, e.g., assuming H satisfies (1.2), we can release the restriction $q = 2$ in (2.5) by applying Besov embedding inequality. Observe that then the Besov space method yields a sharper estimate than (2.2) if in (2.5) taking $q = r = (2n+2)/(n-1)$, $\sigma = 1/2$, $s = 0$ and substituting $H^{-\frac{1}{2}}f$ for f , noticing that $\dot{B}_r^{0,2}(\sqrt{H}) \hookrightarrow \dot{F}_r^{0,2}(\sqrt{H}) = L^r$ provided $r \geq 2$.

Problem: Does the endpoint estimate in Theorem 2.2 hold with $(q, r) = (2, \frac{2n-2}{n-3})$ for $n \geq 4$?

The endpoint estimates involving $L_t^q L_x^r$ -norm were proved in [30] in dimensions ≥ 4 (In dimensions 2 and 3 the endpoint estimates fail). We do not know the answer to the question in the problem for $L^q \dot{B}_r^{s,2}$. Such a result would not only be sharper but technically might involve bilinear estimates that provide insight and deeper understanding of the non-scaling-invariant case when $V \neq 0$. We can also formulate similar result and problem for the Schrödinger equation with a potential; here we would rather refer to [23, 37, 33, 57] for further discussions.

3. OUTLINE OF PROOFS

3.1. Proof of Theorem 1.3. As in [33, 19] or [49], Theorem 1.3 is a consequence of Peetre type maximal equality (Lemma 3.2) and the well-known $L^p(\ell^q)$ -valued Fefferman-Stein maximal inequality. The proof of Lemma 3.2 is standard and follows from Bernstein type inequality (Lemma 3.1). Let

$$\varphi_{j,s}^{**}f(x) = \sup_{t \in \mathbb{R}^n} \frac{|(\nabla_t \varphi_j(H)f)(t)|}{(1 + 2^{j/2}|x-t|)^s}.$$

Lemma 3.1. *For $s > 0$, there exists a constant $c_{n,s} > 0$ such that for all $j \in \mathbb{Z}$*

$$\varphi_{j,s}^{**}f(x) \leq c_{n,s} 2^{j/2} \varphi_{j,s}^* f(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Similar to [33, 49] Lemma 3.1 can be easily proved using (1.2) with $N > n + s$. Let M denote the Hardy-Littlewood maximal function

$$(3.1) \quad Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy$$

where the supreme is taken over all balls B in \mathbb{R}^n centered at x .

Lemma 3.2. *Let $0 < r < \infty$ and $s = n/r$. Then for all $j \in \mathbb{Z}$*

$$(3.2) \quad \varphi_{j,s}^* f(x) \leq c_{n,r} [M(|\varphi_j(H)f|^r)]^{1/r}(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Thus we have seen that the proof of Theorem 1.3 relies on the decay estimates in Assumption 1.2. In Section 4 and Section 5 we will prove such estimates for $H = -\Delta + V$ with $V \in \mathcal{S}$ on \mathbb{R} , and $V = |x|^2$ and $V = |x|^2 + b|x|^{-2}$, $b \geq 0$ on \mathbb{R}^n , where for the latter, the Laguerre operator, initially defined on \mathbb{R}_+^n , can be regarded as an operator acting on $L^2(\mathbb{R}^n)$ for “even” functions.

3.2. Proof of Theorem 1.5. The same proof of the identification $F_p^{0,2}(H) = L^p$ in the inhomogeneous case [33, Theorem 5.1] gives us

$$(3.3) \quad \|f\|_{\dot{F}_p^{0,2}(H)} \approx \|f\|_{L^p}, \quad 1 < p < \infty$$

for any system $\{\varphi_j\}_{j \in \mathbb{Z}}$ satisfying (i), (ii), (iii), by applying $L^p(\ell^2)$ -valued Calderón-Zygmund decomposition. \square

For $p = 1$, Dziubański and Zienkiewicz recently obtained a characterization of Hardy space associated with H using the heat operator.

Theorem 3.3. ([17]) *Let $V \in L^{\frac{n}{2}+\epsilon}(\mathbb{R}^n)$, $n \geq 3$, with $V \geq 0$ being compactly supported. Then*

$$\|f\|_{\mathcal{H}_V^1} \approx \|f\|_{\mathcal{H}_{atom}^1} \approx \|wf\|_{H^1(\mathbb{R}^n)},$$

where $\mathcal{H}_V^1 = \{f \in L^1 : \sup_{t>0} |e^{-tH}f(\cdot)| \in L^1\}$ and the weight w is defined by $w(x) = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^n} e^{-tH}(x, y) dy$. The norm in the atomic decomposition is defined as

$$\|f\|_{\mathcal{H}_{atom}^1} := \inf \sum_j |\lambda_j|,$$

where the infimum runs over all representations $f = \sum_j \lambda_j a_j$, a_j being \mathcal{H}_V^1 atoms satisfying (i) $\text{supp } a \subset B(x_0, r) := \{x : |x - x_0| < r\}$, (ii) $\|a\|_\infty \leq |B(x_0, r)|^{-1}$, (iii) $\int a(x)w(x)dx = 0$.

Remark. It would be interesting to obtain a norm characterization for \mathcal{H}_V^1 of Littlewood-Paley type in the sense of Theorem 1.5. Note that if $V \neq 0$, then $\mathcal{H}_V^1 \neq H^1(\mathbb{R}^n)$, $n \geq 3$. Also, in the 1D case and unbounded potential case (e.g. V being a (nonnegative) polynomial), the p -atoms ($p = 1$) for \mathcal{H}_V^1 ($V \neq 0$) are not variant of the local atoms, cf. [17, 18].

$$4. \quad H = -\frac{d^2}{dx^2} + V, \quad V \text{ IN } \mathcal{S}(\mathbb{R})$$

Associated to H there exists a decomposition $L^2(\mathbb{R}) = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp}$, where \mathcal{H}_{ac} is the absolute continuous subspace and \mathcal{H}_{pp} the pure point subspace of $L^2(\mathbb{R})$. Let E_{ac}, E_{pp} be the corresponding orthogonal projections. If $\sigma_{ac}(H)$ denotes the absolute continuous spectrum and $\sigma_{pp}(H)$ the pure point spectrum of H , then $\sigma_{ac}(H) = [0, \infty)$ and $\sigma_{pp}(H) = \{-\lambda_k^2\}$ is a finite set of eigenvalues of H in $(-\infty, 0)$.

4.1. Decay estimates of $\phi_j(H)$. For $\lambda \in \mathbb{R}$ let $e(x, \lambda) = (1 + R_0(\lambda^2 + i0)V)^{-1} e^{i\lambda x}$ be the Lippman-Schwinger scattering eigenfunction and $e_k(x)$ the L^2 eigenfunction of H with eigenvalue $-\lambda_k^2$, where $R_0(z) = (H_0 - z)^{-1}$ is the resolvent of $H_0 = -d^2/dx^2$ with $z \in \mathbb{C} \setminus [0, \infty)$. Then if $\phi \in C_0$, a continuous function with compact support, we have

$$\phi(H)f(x) = \int K(x, y)f(y)dy$$

where $K = K_{ac} + K_{pp}$,

$$(4.1) \quad K_{ac}(x, y) = (2\pi)^{-1} \int \phi(\lambda^2) e(x, \lambda) \bar{e}(y, \lambda) d\lambda$$

is the kernel of $\phi(H_{ac})$, $H_{ac} = HE_{ac}$ and $K_{pp}(x, y) = \sum_k \phi(-\lambda_k^2) e_k(x) e_k(y)$ is the kernel of $\phi(H)E_{pp}$. Let u_1, u_2 be two linear independent solutions of (4.3). H is said to have resonance at zero provided that the Wronskian of u_1, u_2 vanishes at zero.

Theorem 4.1. *Let $\{\phi_j\} \subset C_0^\infty(\mathbb{R})$ satisfy (i), (ii).*

a) *If $V \in \mathcal{S}$, then for each $\ell \geq 0$ and $N \geq 0$ there exists a constant $c_{N,\ell}$ so that for all $j \geq 0$*

$$(4.2) \quad |\partial_x^\ell \phi_j(H)(x, y)| \leq c_{N,\ell} 2^{(\ell+1)j/2} (1 + 2^{j/2} |x - y|)^{-N}.$$

b) *If $V \in C_0^\infty$ and H has no resonance at zero, then (4.2) holds for each $\ell = 0, 1$, $N \geq 0$ and all $-\infty < j < 0$.*

Denote by K_j the kernel of $\phi_j(H)$ and write $K_j = K_{j,ac} + K_{j,pp}$. Since σ_{pp} is finite and according to e.g. [38, Theorem C.3.4], eigenfunctions of H belonging to $\cap_{m,N=0}^\infty (x)^{-N} W_2^{2m}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$, K_{pp} satisfies (4.2) trivially. Hence it is sufficient to deal with $K_{j,ac}$.

Let $R_V(z) = (H - z)^{-1}$. Let $W(\lambda)$ be the Wronskian of f_+, f_- , then for $\lambda \neq 0$

$$R_V(\lambda^2 \pm i0)(x, y) = \begin{cases} \frac{f_+(x, \pm\lambda) f_-(y, \pm\lambda)}{W(\pm\lambda)} & x > y \\ \frac{f_+(y, \pm\lambda) f_-(x, \pm\lambda)}{W(\pm\lambda)} & x < y \end{cases}$$

where $f_\pm(x, z)$ are the Jost functions that solve for $\Im z \geq 0$

$$(4.3) \quad -f_\pm''(x, z) + V(x) f_\pm(x, z) = z^2 f_\pm(x, z)$$

and satisfy

$$f_\pm(x, z) \rightarrow \begin{cases} e^{\pm izx} & x \rightarrow \pm\infty \\ \frac{1}{i(z)} e^{\pm izx} + \frac{r_\mp(z)}{i(z)} e^{\mp izx} & x \rightarrow \mp\infty \end{cases}$$

where $t(z), r_\pm(z)$ are called the transmission and reflection coefficients [12].

Let $m_\pm(x, z) = e^{\mp izx} f_\pm(x, z)$ be the modified Jost function. We obtain, from the resolvent formula of the spectral measure of H_{ac} , that if $x > y$

$$(4.4) \quad \phi(H_{ac})(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\lambda^2) m_+(x, \lambda) m_-(y, \lambda) t(\lambda) e^{i\lambda(x-y)} d\lambda$$

where $t(\lambda) = -2i\lambda/W(\lambda)$; see e.g. [23]. One can prove that the above formula and (4.1) coincide. As a result the restriction $x > y$ can be dropped.

To analyze the kernel we need to study the analytic asymptotics of $m_\pm(x, k)$, namely the higher order mixed differentiability on m_\pm concerning the smoothness and decay in x and k , the space and spectral parameters. The asymptotics of m_\pm and their derivatives of first order were originally studied in [12]. However our estimates and certain formulas, including those of $t(k)$ and the Fourier transforms of m_\pm , are more refined and delicate.

4.2. Derivatives of $m(x, k)$.

Lemma 4.2. *Let $V \in \mathcal{S}$ and $\ell, n \geq 0$. Then $m \in C^\infty(\mathbb{R} \times \mathbb{R})$ and for all $x, k \in \mathbb{R}$*

$$|\partial_x^\ell \partial_k^n m_\pm(x, k)| \leq c_{n,\ell} \begin{cases} (1 + \max(0, \mp x))^n & \ell \geq 1, \\ (1 + \max(0, \mp x))^{n+1} & \ell = 0. \end{cases}$$

If $\ell \geq 0, n \geq 1$ or $\ell \geq 1, n \geq 0$, then

$$|\partial_x^\ell \partial_k^n m_\pm(x, k)| \leq c_{n,\ell} \frac{(1 + \max(0, \mp x))^n}{|k|^n}.$$

The proof of the lemma are based on the integral equation [12]

$$(4.5) \quad m_+(x, k) = 1 + \int_x^\infty h(t-x, k)V(t)m_+(t, k)dt,$$

where $h(t-x, k) = \int_0^{t-x} e^{2iku} du$, and the equation for its mixed partial derivatives: for $\ell = 1, 2, 3, \dots, n \geq 0$

$$(4.6) \quad \partial_x^\ell \partial_k^n m_+(x, k) = - \sum_{j=0}^n \binom{n}{j} (2i)^j \int_x^\infty e^{2ik(t-x)} (t-x)^j \partial_t^{\ell-1} (Vm_+^{(n-j)}) dt,$$

where $m_+^{(i)} = \partial_k^i m_+(t, k)$.

The first estimate in the lemma is also a consequence of Lemma 4.5 concerning the weighted L^1 bound for $B_\pm(x, y)$, the Fourier transforms of $m_\pm - 1$, which are especially needed for low energy estimates. The Marchenco functions $B_\pm(x, y)$ are related to m_\pm via

$$m_\pm(x, k) = 1 \pm \int_0^{\pm\infty} B_\pm(x, y)e^{2iky} dy.$$

4.3. Analyticity and asymptotics of $t(k)$ and $r_\pm(k)$. The kernel of $\phi(H_{ac})$ given by (4.4) also requires estimates of the coefficients $t(k)$ and $r_\pm(k)$.

Lemma 4.3. *Let $V \in \mathcal{S}$. Then $t(k), r_\pm(k) \in C^\infty$. If $|k| \geq 1$, then*

$$(4.7) \quad \frac{d^n}{dt^n} t(k) = \begin{cases} 1 + O(k^{-1}) & n = 0 \\ O(k^{-n-1}) & n \geq 1 \end{cases}$$

and

$$(4.8) \quad \frac{d^n}{dt^n} r_\pm(k) = O(k^{-n-1}) \quad n \geq 0.$$

For the proof of high energy $|k| \geq 1$, we use [12, p.145]

$$(4.9) \quad \begin{aligned} t(k)^{-1} &= 1 - \frac{1}{2ik} \int_{-\infty}^\infty V(t)m_+(t, k)dt \\ (r_\pm(k) + 1)t(k)^{-1} &= 1 \mp \int_{-\infty}^\infty h(t, \mp k)V(t)m_\mp(t, k)dt. \end{aligned}$$

For low energy to show $t \in C^\infty$ the formulas we use are more delicate.

Lemma 4.4. *Let $\nu := W(0)$. The following formulas hold.*

a) *If $\nu \neq 0$, then*

$$t(k) = \frac{-2ik}{W(k)},$$

where $W(k) = -2ik + \nu + \int_{-\infty}^{\infty} V(t)dt \int_0^{\infty} B_+(t, y)(e^{2iky} - 1)dy$.
 b) If $\nu = 0$, then

$$t(k)^{-1} = 1 - \int_{-\infty}^{\infty} V(t)dt \int_0^{\infty} \left(\int_{\xi}^{\infty} B_+(t, \eta)d\eta \right) e^{2ik\xi} d\xi.$$

Based on Lemma 4.4 we give the proof of the statement $t \in C^{\infty}$ as in Lemma 4.3. The statement $r \in C^{\infty}$ follows from the fact that $t \in C^{\infty}$ and (4.9). We divide the discussions into two cases.

Case (a) $\nu \neq 0$. From the formula in Lemma 4.4 (a) it is easy to see that $W(k)$ is C^s if $\int_0^{\infty} y^s |B(t, y)|dy \leq c\langle t \rangle^{s+1}$, but this is true provided $V \in L_{s+1}^1$ according to Lemma 4.5. Now since $W(0) \neq 0$, it follows that $t \in C^s$ as long as $V \in L_{s+1}^1$.

Case (b) $\nu = 0$. From Lemma 4.4 (b) we have if $s \geq 1$

$$(t^{-1})^{(s)}(k) = - \int_{-\infty}^{\infty} V(t)dt \int_0^{\infty} (2i\xi)^s e^{2ik\xi} d\xi \left(\int_{\xi}^{\infty} B_+(t, \eta)d\eta \right),$$

which is the Fourier transform of the function

$$\xi \mapsto -\chi_{(0, \infty)}(\xi) \int_{-\infty}^{\infty} V(t)dt (2i\xi)^s \int_{\xi}^{\infty} B_+(t, \eta)d\eta.$$

Observe that in view of Lemma 4.5 this function is in L^1 if $V \in L_{s+2}^1$. Hence t^{-1} is C^s . We conclude that

$$t(k) = \frac{1}{t(k)^{-1}}$$

is also in C^s whenever $V \in L_{s+2}^1$, since it is a basic fact that $\nu = 0 \iff |t(k)| \geq c_0 > 0, \forall k$; cf. e.g., [12, Theorem 1]. \square

4.4. Fourier transform of modified Jost function. The estimates for $m_{\pm}(x, k)$, $t(k)$ and $r_{\pm}(k)$, especially in the low energy, depend on the weighted L^1 inequalities for the Marchenko functions $B_{\pm}(x, y)$ and their derivatives. Recall that $L_{\gamma}^1 = \{V : \int(1 + |y|)^{\gamma}|V(y)|dy < \infty\}$ and $W_{1, \gamma}^n = \{V : \int(1 + |y|)^{\gamma}|V^{(i)}(y)|dy < \infty, i = 0, \dots, n\}$.

Lemma 4.5. *Let $s \in \mathbb{N}_0$. a) If $V \in L_{s+1}^1(\mathbb{R})$, then there exists $c = c(\|V\|_{L_{s+1}^1})$ so that for all $x \in \mathbb{R}$*

$$\int_{-\infty}^{\infty} |y|^s |B_{\pm}(x, y)|dy \leq c(1 + \max(0, \mp x))^{s+1}.$$

b) If $V \in W_{1, s+1}^{n-1}$, $n \geq 1$, then there exists $c = c(\|V\|_{W_{1, s+1}^{n-1}})$ so that for all $x \in \mathbb{R}$

$$\begin{aligned} \int_{-\infty}^{\infty} |y|^s |\partial_y^n B_{\pm}(x, y)|dy &\leq c(1 + \max(0, (\mp x)))^s \\ \int_{-\infty}^{\infty} |y|^s |\partial_x^n B_{\pm}(x, y)|dy &\leq c(1 + \max(0, (\mp x)))^s. \end{aligned}$$

The proof exploits careful iterations of the Marchenko equations

$$\begin{aligned} B_+(x, y) &= \int_{x+y}^{\infty} V(t)dt + \int_0^y dz \int_{t=x+y-z}^{\infty} V(t)B_+(t, z)dt \\ B_-(x, y) &= \int_{-\infty}^{x+y} V(t)dt + \int_y^0 dz \int_{-\infty}^{t=x+y-z} V(t)B_-(t, z)dt, \end{aligned}$$

where we note that $B_{\pm}(x, \cdot)$ are supported in \mathbb{R}_{\pm} .

Remark. The inequality in a) is an improvement of [8, Lemma 3.2, Lemma 3.3], where the cases $s = 0, 1$ were proved via Gronwall's inequality.

4.5. High and low energy estimates for $\phi_j(H)(x, y)$. Applying Lemmas 4.2 and Lemma 4.3 we can easily prove Theorem 4.1 (a) in high and low energy via integration by parts; here the high energy refers to ϕ_j , $j \geq 0$ and local energy refers to $\Phi \in C_0^\infty$ with $\text{supp } \Phi \subset [-1, 1]$.

For low energy $j < 0$, (b) of Theorem 4.1, we observe that the condition $\partial_x m_{\pm}(x, 0) = 0$ for large $|x|$ together with H having no resonance at zero is necessary and sufficient for $\partial_x \phi_j(H)(x, y)$ to satisfy (4.2). Therefore Theorem 4.1 (b) is true if and only if H has no resonance at zero and $m_{\pm}(x, 0) \equiv \text{constant} = 1$ for large $|x|$. The latter can occur *only when V has compact support* in view of $\partial_x^2 m + 2ik\partial_x m = Vm$.

Remark. Let $V \in \mathcal{S}$. If $-\infty < j < 0$, then (4.2) still holds for $\phi_j(H)(x, y)$ with $\ell = 0$. However, if H has resonance at zero or V is not compactly supported, then (4.2) fails for $\partial_x \phi_j(H)(x, y)$ in the low energy as we mentioned above. A counterexample can be found in [33] for $V = -\nu(\nu + 1)\text{sech}^2 x$. For non-smooth potentials, in [4] we are able to obtain an appropriate variant of the kernel decay (4.2) with $j \in \mathbb{Z}$, $\ell = 0, 1$ for $V = c\chi_{[a,b]}(x)$, $c > 0$, χ_E being a characteristic function of the set E .

Problem For non-smooth potentials, it is still open as to determine the class of V for which (4.2) remains valid especially in the low energy case.

5. H SATISFYING UPPER GAUSSIAN BOUND

In this section we prove the analogy of Theorem 1.3 for H with unbounded potentials, namely, the Hermite and Laguerre operators. We see that using the upper Gaussian bound for ∇p_t of H we can prove the decay in (1.2) in a *simple* way, which generalize the results of [19, 21] and [15]. Previously Epperson studied the Hermite and Laguerre cases in one dimension using oscillatory integral method; later Dziubanski used Heisenberg group technique to prove the kernel decay for the Hermite expansion in n -dimension and the Laguerre expansion of integral order in 1D.

[55] showed that Assumption 1.2 is verified when H satisfies the upper Gaussian bound (5.1) for its heat kernel. The proof is based on a weighted L^1 inequality which is a scaling version of [26, Lemma 8].

Proposition 5.1. *Let $\ell = 0, 1$. Suppose $V \geq 0$ and H satisfies the upper Gaussian bound*

$$(5.1) \quad |\nabla_x^\ell e^{-tH}(x, y)| \leq c_n t^{-(n+\ell)/2} e^{-c|x-y|^2/t}, \quad \forall t > 0.$$

If $\{\varphi_j\}_{j \in \mathbb{Z}}$ is a dyadic system satisfying (i), (ii), then for each $N \geq 0$

$$|\nabla_x^\ell \varphi_j(H)(x, y)| \leq c_N 2^{j(n+\ell)/2} (1 + 2^{j/2}|x-y|)^{-N}, \quad \forall j.$$

Remark. The long time gradient estimates for ∇p_t ($t > 1$ corresponding to low energy) is, in general, not valid for bounded V , not even for positive $V \in \mathcal{S}(\mathbb{R}^n)$.

5.1. Hermite operator $H = -\Delta + |x|^2$. To verify Assumption 1.2 it is sufficient to show that H satisfies the upper Gaussian bound in (5.1), according to Proposition 5.1.

For $k \in \mathbb{N}_0$, let h_k be the k^{th} Hermite function with $\|h_k\|_{L^2(\mathbb{R})} = 1$ such that

$$\left(-\frac{d^2}{dx^2} + x^2\right)h_k = (2k + 1)h_k.$$

Then $\{h_k(x)\}_0^\infty$ forms a complete orthonormal basis (ONB) in $L^2(\mathbb{R})$. Let $\Phi_k(x) := h_{k_1} \otimes \cdots \otimes h_{k_n}$, $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$. Then $\{\Phi_k\}$ is an ONB in $L^2(\mathbb{R}^n)$.

By Mehler's formula [46, Ch.4], the heat kernel has the expression

$$\begin{aligned} e^{-tH}(x, y) &= \sum_{k \in \mathbb{N}_0^n} e^{-t(n+2|k|)} \Phi_k(x) \Phi_k(y) \\ &= (2\pi \sinh(2t))^{-n/2} e^{-\frac{1}{2} \coth(2t)(|x|^2 + |y|^2) + \text{cosech}(2t)x \cdot y} \end{aligned}$$

for all $t > 0$, $x, y \in \mathbb{R}^n$.

One can easily calculate (cf. [55]) that for $\ell = 0, 1$ there exist constants $c' > 0$, $0 < c < 1$ such that

$$|\nabla_x^\ell p_t(x, y)| \leq c' \begin{cases} t^{-(n+\ell)/2} e^{-c|x-y|^2/t} & 0 < t < 1 \\ e^{-nt} e^{-c|x-y|^2} & t \geq 1, \end{cases}$$

where $p_t(x, y) := e^{-tH}(x, y)$. Hence (5.1) holds.

5.2. Laguerre operator $H = L_\alpha = -\frac{d^2}{dx^2} + x^2 + (\alpha^2 - \frac{1}{4})x^{-2}$, $\alpha \geq -1/2$ in $L^2(\mathbb{R}_+)$. The Laguerre is basically a generalization of the Hermite. We will show that the heat kernel estimates (5.1) are valid for L_α with $\alpha \geq 1/2$.

For $k \in \mathbb{N}_0$ define the Laguerre function

$$M_k^\alpha(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} L_k^\alpha(x^2),$$

where $L_k^\alpha(x)$ are the Laguerre polynomials. Then

$$L_\alpha M_k^\alpha(x) = (4k + 2\alpha + 2)M_k^\alpha(x)$$

and $\{M_k^\alpha(x)\}_0^\infty$ is an orthonormal basis in $L^2(\mathbb{R}_+)$; see e.g. [46].

Lemma 5.2. *Let $\alpha \geq 1/2$. Then the heat kernel of L_α satisfies for $\ell = 0, 1$ there exists constants $c' > 0$, $0 < c < 1$ such that with $n = 1$*

$$(5.2) \quad \partial_x^\ell e^{-tL_\alpha}(x, y) \leq c' \begin{cases} t^{-(n+\ell)/2} e^{-c|x-y|^2/t} & 0 < t < 1 \\ e^{-nt} e^{-c|x-y|^2/t} & t \geq 1. \end{cases}$$

Proof. Using the following Mehler type formula: if $0 < r < 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} r^k M_k^\alpha(x) M_k^\alpha(y) &= 2e^{-i\frac{\pi}{2}\alpha}(xy)^{1/2} (1-r)^{-1} r^{-\alpha/2} e^{-\frac{1}{2}\frac{1+r}{1-r}(x^2+y^2)} \\ &\quad \times J_\alpha(2ixyr^{1/2}(1-r)^{-1}) \end{aligned}$$

(see e.g. (11) of [21]), we can obtain the heat kernel formula for L_α , $\alpha \geq -1/2$

$$(5.3) \quad \begin{aligned} e^{-tL_\alpha}(x, y) &= \sum_{k=0}^{\infty} e^{-t(4k+2\alpha+2)} M_k^\alpha(x) M_k^\alpha(y) \\ &= (-i)^\alpha (\sinh 2t)^{-1} e^{-\frac{x^2+y^2}{2} \coth 2t} (xy)^{1/2} J_\alpha \left(\frac{ixy}{\sinh 2t} \right), \end{aligned}$$

where J_α is the Bessel function of order α . Now if $\alpha \geq 1/2$, (5.2) follows from the usual asymptotics of J_α (see [51] or [41, VIII.5]):

$$\begin{aligned} J_\alpha(z) &= \begin{cases} z^\alpha & z \rightarrow 0 \\ z^{-1/2} & z \rightarrow \infty, \end{cases} \\ J'_\alpha(z) &= \begin{cases} z^{\alpha-1} & z \rightarrow 0 \\ z^{-1/2} & z \rightarrow \infty. \end{cases} \end{aligned}$$

□

5.3. Laguerre operator in \mathbb{R}_+^n . A natural extension of L_α to n -dimensions is for $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \geq -\frac{1}{2}$

$$\mathcal{L}_\alpha = \sum_{j=1}^n (-\partial_{x_j}^2) + x_j^2 + (\alpha_j^2 - \frac{1}{4}) x_j^{-2}$$

in $L^2(\mathbb{R}_+^n)$. It can be defined as the Friedrich's extension of the form

$$(\mathcal{L}_\alpha f, f) = \sum_{j=1}^n \int_{\mathbb{R}_+^n} |\partial_{x_j} f(x)|^2 + (x_j^2 + (\alpha_j^2 - \frac{1}{4}) x_j^{-2}) |f(x)|^2 dx$$

for $f \in C_0^\infty(\mathbb{R}_+^n)$; hence it's domain is a subspace of the Sobolev space $W_2^1(\mathbb{R}^n)$.

The ONB in $L^2(\mathbb{R}_+^n)$ consists of $\{M_k^\alpha(x) = M_{k_1}^{\alpha_1}(x_1) \otimes \dots \otimes M_{k_n}^{\alpha_n}(x_n), x = (x_1, \dots, x_n) \in \mathbb{R}_+^n, k = (k_1, \dots, k_n) \in \mathbb{N}_0^n\}$. Hence the heat kernel of \mathcal{L}_α is given by

$$\begin{aligned} e^{-t\mathcal{L}_\alpha}(x, y) &= \sum_{k=(k_1, \dots, k_n)} e^{-t\lambda_k^\alpha} M_k^\alpha(x) M_k^\alpha(y) \\ &= (-i)^{|\alpha|} (\sinh 2t)^{-n} e^{-\frac{|x|^2+|y|^2}{2} \coth 2t} \prod_{j=1}^n (x_j y_j)^{1/2} J_{\alpha_j} \left(\frac{ix_j y_j}{\sinh 2t} \right), \end{aligned}$$

where $\lambda_k^\alpha = \sum_{j=1}^n (4k_j + 2\alpha_j + 2)$, $x, y \in \mathbb{R}_+^n$. Therefore applying the 1D result we easily obtain the estimates in (5.2) for \mathcal{L}_α provided $\alpha_j \geq 1/2$, $j = 1, \dots, n$.

Remark. Thangavelu [47] obtained a heat kernel formula in the special case $\alpha \in \mathbb{N}_0$ via different method. We observe that the formula in [47, Theorem 2.17] for $L(\alpha) = -d^2/dx^2 - (2\alpha + 1)x^{-1}d/dx + x^2/4$ acting in the weighted space $(L^2(\mathbb{R}_+), x^{2\alpha+1})$ is equivalent with (5.3) for L_α acting in the unweighted space $L^2(\mathbb{R}_+)$.

5.4. Schrödinger operator and associated heat kernel. Proposition 5.1 shows that the upper Gaussian bound estimates of $e^{-tH}(x, y)$ imply the decay estimates (1.2) in Assumption 1.2. However for bounded $V \neq 0$ the gradient estimates for $\nabla_x p_t$ are not valid in general, this is one reason we work in a more direct way to deal with the decay of $\nabla_x \varphi_j(H)(x, y)$ for all $j \in \mathbb{Z}$ as illustrated in the one

dimensional case. The heat kernel approach seems to work more efficiently for unbounded potentials. For bounded potentials, when the Gaussian bounds are not available, we can consider, for instance, the radial case in three dimensions using Volterra type equation for the eigenfunction of H , or the non-radial case using stationary phase method [36, 7, 9, 57].

6. CONCLUSION

The Littlewood-Paley theory of $\dot{B}_p^{\alpha,q}(H)$ and $\dot{F}_p^{\alpha,q}(H)$ surveyed in this paper leaves open other problems in the area of harmonic analysis and PDE. Here we would like to mention a few problems related to our subject.

a. The further characterization and identification of $\dot{B}_p^{\alpha,q}(H), \dot{F}_p^{\alpha,q}(H)$, $0 < p, q \leq \infty$ as well as other elements of fundamental theory of function spaces, including distribution theory, have yet to be understood, compared with classical Fourier analysis [49, 50, 22]. It may involve semigroup method, Riesz transform as well as singular integrals for certain class of rough potentials.

b. Applying the Littlewood-Paley decomposition to establish dispersion and Strichartz estimates for the perturbed wave, Klein-Gordon and Schrödinger equations with potentials. Although there has been quite extensive work in this area, cf. [3, 6, 7, 37, 52, 53], regularities of these estimates involving the associated function spaces have not received comparable attention. We hope the development of the Littlewood-Paley theory in our continuing investigation could give a systematic treatment of the regularity problems, which are related to one of the open problems concerning Strichartz type estimates for wave equations with potentials in the presence of resonance or eigenvalue at zero energy, cf. [31, 36, 34].

c. For the Laplacian-Beltrami operator $-\Delta_g$ on a (complete) Riemannian manifold (M, g) , in many cases the generic heat kernel estimates are valid [10, 11, 24]. However for a Schrödinger operator $H_V = -\Delta_g + V$ on M , the heat kernel estimates is in general not valid, this makes it unique and more difficult especially for the high and low energy analysis of H_V . The treatment here might throw a light on considering the analogous problems in a, b on M ; in particular we are interested in obtaining the analogous results for H_V on Riemannian symmetric spaces. We refer to [1, 32, 39, 40, 45, 2, 35] for some of the recent development in this direction.

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(Gestur Ólafsson) DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803

E-mail address: olafsson@math.lsu.edu

URL: <http://www.math.lsu.edu/~olafsson>

(Shijun Zheng) DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460

AND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208

E-mail address: szheng@georgiasouthern.edu

URL: <http://math.georgiasouthern.edu/~szheng>