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Drawing Numbers and Listening to Patterns

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Drawing Numbers and Listening to Patterns

An Honors Thesis submitted in partial fulfillment of the requirements for Honors in Mathematical Sciences.

By
Zo Haynes

Under the mentorship of Hua Wang and Jim Braselton

Abstract

The triangular numbers is a series of number that add the natural numbers. Parabolic shapes emerge when this series is placed on a lattice, or imposed with a limited number of columns that causes the sequence to continue on the next row when it has reached the $k^{th}$ column. We examine these patterns and construct proofs that explain their behavior. We build off of this to see what happens to the patterns when there is not a limited number of columns, and we formulate the graphs as musical patterns on a staff, using each column as a line or space on the treble staff. By listening to the pattern, we can pick up on elements of the pattern that are missed by simply glancing over the graphic or formulaic versions.

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Introduction

**Triangular Numbers**

The triangular numbers is a series of numbers that adds the natural numbers from 1 to \( n \). The natural numbers, \( n \in \mathbb{N} \), are the positive integers 1 through infinity. So the first four terms of the triangular numbers are 1, 1+2=3, 3+3=6, 6+4=10, 10+5=15, and so on. The first number (blue) is the previous triangular number, or the summation up to that point, and the second number (green) is the next natural number in order from least to greatest.

\[
s_1 = 1 \quad s_2 = 3 \quad s_3 = 6 \quad s_4 = 10
\]

Let \( 1 + 2 + 3 + 4 + \cdots + n = A \), where \( A \) is a triangular number.

So we add the sequence to itself and arrange the numbers in the opposite order on the second line so that the largest number is added to the smallest, the second largest number to the second smallest and so on,

\[
\begin{align*}
    s &= 1 + 2 + 3 + 4 + \cdots + n \\
    s &= n + (n-1) + \cdots + 2 + 1 \\
    2s &= (1 + n) + (2 + (n-1)) + \cdots + (n + 1)
\end{align*}
\]

which simplifies to

\[
2s = n(1 + n)
\]

When solving for \( s \), the sequence looks like

\[
s = \frac{n(n + 1)}{2}
\]
so the summation of the series is

\[ \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \frac{n(n + 1)}{2} \]

and every value of the series can be found from plugging in the natural numbers into this formula. [3]

Central Polygonal Numbers

A modified version of this sequence is the central polygonal numbers. [5] This sequence works exactly the same way, but has an extra point at the very beginning and adds 1 to the whole sequence. The first five terms of the central polygonal numbers are 1, 1+1=2, 2+2=4, 4+3=7, 7+4=11, and so on. The first number is the previous central polygonal number, or the summation up to that point, and the second number is the next natural number in order from least to greatest.

Let \( A_n \) denote this sequence; we have

\[ A_0 = 1 \]

and

\[ \sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{n(n + 1)}{2} + 1 \]

as the series of central polygonal numbers.

This sequence is also known as the lazy caterer’s sequence. It is called the lazy caterer’s sequence because each cut of the cake (or each number of the sequence) gives you the most pieces for that many cuts.
**Modulo**

The division of a number has a divisor, dividend, quotient, and residue. Every dividend can be written as the multiplication of the divisor and quotient with the addition of the residue, or what is left over. For example:

\[
\frac{38}{12} = 3 + \frac{2}{12}
\]

In this case, 38 is the dividend, 12 is the quotient, 3 is divisor, and \(\frac{2}{12}\) is the residue. This equation can be rewritten as a multiplication equation in the form:

\[
38 = (3 \times 12) + 2
\]

Another way to write this equation is using modulo \(n\) or mod \(n\). Modular arithmetic looks at the remainder that is left over when dividing by a base number. Telling time is an everyday example of modular arithmetic. Standard time has hours from one to twelve. Every time we reach 12:00, the next hour is 1:00 instead of 13:00. This is an example of modular division base 12 or mod 12. 1:00 is actually 13:00 divided by 12 and written as just the remainder 1:00. Military time does the same thing, but with modulo base 24 instead of base 12. Instead of reading the time 38:00, the time reads as 2:00. Mathematically, this can be structured:

\[
38 \equiv 2 \pmod{12}
\]

In this case, it does not matter how many multiples of 12 are a part of 38. The result of modular division is only the remainder.
Observations

We plotted the triangular numbers on a number line, we sectioned the number line using modulo $k$, and then we arranged these sections as rows of a graph.

As you can see, parabolic shapes were created by this arrangement of the series. The vertexes are in the first and middle columns in the case of $K=24$, but the parabolic shapes were not always clean. In the second parabolic shape in $K=24$, there are two points that show up in the last column, which breaks the pattern of mirrored points. This happens in every even numbered parabolic shape in cases where $K$ is even.

In order to clean up the parabolic shape, we added an extra point, an $A_0 = 1$, to the beginning of the sequence to shift the sequence over by one column. This changed the sequence from the triangular numbers to the central polygonal numbers. Also, the points closest to the vertex of the parabolic shape shifted from the $\frac{k}{2}$ and $k$ columns to the 1 and $\frac{k}{2} + 1$ columns.
We studied the first parabolic shape and came up with a proof that shows why sets of points line up in the same columns in a mirrored fashion to create these parabolic shapes. These shapes form when \( k \) is equal to \( n \) or when the number of columns is equal to the number of blocks between two points. This makes intuitive sense because if the number of spaces between points is equal to the number of columns, the space will be exactly one row, so the points on either side will line up in the same column.

The number of columns affects how the pattern will behave. When \( k \) is odd, all parabolic shapes start in the first column, or when \( A_n \equiv 1 \pmod{k} \). When \( k \) is even and when \( n \) is even, the vertex is in the first column, or when \( A_n \equiv 1 \pmod{k} \). When \( k \) is even and \( n \) is odd, the vertex is in the middle column, or when \( A_n \equiv \frac{k}{2} + 1 \pmod{k} \).

There is also a noticeable symmetry in the points between the most noticeable cases. These points are repeated between each parabola and are also parabolic in shape. When the \( k \) is odd, these secondary parabolic shapes are more noticeable as the sets of points line up in the same columns to make up these secondary parabolic shapes.
Once the set of points is established, the same points are used in the same pattern for each of the following parabolic shapes. The same points are used, the same columns are used, and in the same order. The only difference between these sections of the pattern is that each time the pattern is repeated, a row is added in between points. For the first parabolic shape, there is a point in each row. For the second parabolic shape, there is a row in between points. There are two rows in between the points of the third parabolic shape, and this pattern continues as these shapes get repeated.

Another noticeable trait about these graphs is that some of the columns are never used. All of the columns that will be used are used within the first few rows. After looking into why this might be, we found that this was not an elementary proof and looked to see if others might be looking into this same problem. We found that other mathematicians such as Ken Ono and his collaborators were working on proofs to these questions and others like it. [1]

The next page contains five different examples that show how the number of columns affects the pattern.
Mathematical Proofs

**Theorem 1** [The Construction of the First Parabolic Shape]:

If $A_{n+1} - A_n = k$ for some $n$, then $A_{n+1+c} - A_{n-c} \equiv 0 \mod k$.

**Proof:**

Recall: $A_n = \frac{(n + 1)n}{2} + 1$

Plug in the formula of $A_n$ into the left hand side equation and simplify.

\[
A_{n+1} - A_n = \left(\frac{([n + 1] + 1)[n + 1]}{2} + 1\right) - \left(\frac{[n + 1]n}{2} + 1\right)
\]

\[=rac{(n + 2)[n + 1] + 2}{2} - \frac{[n + 1]n + 2}{2}\]

\[= \frac{(n + 2)[n + 1] + 2 - n(n + 1) - 2}{2}\]

\[= \frac{(n + 2 - n)[n + 1]}{2}\]

\[= n + 1 = k\]

as given by the statement.

Then, plug in the formula for $A_n$ into the right hand side and simplify.

\[
A_{n+1+c} - A_{n-c} = \left(\frac{([n + 1 + c] + 1)[n + 1 + c]}{2} + 1\right) - \left(\frac{([n - c] + 1)[n - c]}{2} + 1\right)
\]

\[= \frac{(n + 2 + c)[n + 1 + c] + 2}{2} - \frac{(n - c + 1)[n - c] + 2}{2}\]

\[= \frac{(n + 2 + c)[n + 1 + c] + 2 - (n - c + 1)[n - c] - 2}{2}\]
\[
\begin{align*}
&= \frac{(n + 2 + c)[n + 1 + c] - (n - c + 1)[n - c]}{2} \\
&= \frac{(n^2 + 3n + 2cn + 2 + 3c + c^2) - (n^2 - cn - cn + c^2 + n - c)}{2} \\
&= \frac{n^2 + 3n + 2cn + 2 + 3c + c^2 - n^2 + 2cn - c^2 - n + c}{2} \\
&= \frac{2n + 4cn + 2 + 4c}{2} \\
&= n + 2cn + 1 + 2c = (n + 1) + (2cn + 2c) \\
&= (n + 1) + 2c(n + 1)
\end{align*}
\]

and because \(n+1=k\),

\[(n + 1) + 2c(n + 1) = k + 2c(k) \equiv 0(\text{mod } k) \quad \square\]

**Theorem 2** [Vertexes of each Parabolic Shape]:

Recall: 
\[
A_n = \frac{n(n + 1)}{2} + 1
\]

\[
A_n - A_{n-1} = n
\]

If \(A_n - A_{n-1} = ck\) for \(c \in \mathbb{Z}\), then 
\[
A_n = \frac{ck(ck + 1)}{2} + 1.
\]

**Proof:**

We want to examine where the vertex falls. There are two cases: \(k\) is either odd or even.

I. If \(k\) is odd, then \(2|c(ck + 1)\).

If \(c\) is even, then the \(c\) on the outside of the parentheses ensures that the whole is divisible by 2.

If \(c\) is odd, then \((ck + 1)\) is \((\text{odd} + 1)\), which is even and therefore insures that the whole is divisible by 2.
Therefore, \( \frac{c(ck+1)}{2} \) is a whole number.

So, \( k | \frac{c(ck+1)}{2} \)

and \( \frac{c(ck+1)}{2} + 1 \equiv 1 \mod k \)

Therefore, the vertex is always in the first column when \( k \) is odd.

II. If \( k \) is even

a. And if \( c \) is even, then

\( 2 | c(ck + 1) \)

and by the logic of case I, \( \frac{c(ck+1)}{2} + 1 \equiv 1 \mod k \)

Therefore, the vertex is always in the first column when \( k \) is even and \( c \) is even.

b. And if \( c \) is odd, then

\( 2 \not| c(ck + 1) \)

Let \( c(ck + 1) = 2m + 1 \).

Therefore \( A_n = \frac{k(2m+1)}{2} + 1 = \frac{2km+k}{2} + 1 = km + \frac{k}{2} + 1 = \frac{k}{2} + 1 \mod k \)

Therefore, the vertex is always in the column after the midpoint of \( k \).  \( \square \)

**Theorem 3** [Secondary parabolic odd case]:

Recall: \( A_n = \frac{n(n+1)}{2} + 1 \)

If \( A_n - A_{n-2} = \frac{n(n+1)}{2} - \frac{(n-1)(n-2)}{2} = \frac{4n-2}{2} = 2n - 1 \) and if \( k | 2n - 1 \) (\( k \) is odd), then \( 2n - 1 = kq \) for an odd \( q \).

**Proof:**

\( q = (2p - 1) \) for \( p \in \mathbb{Z} \geq 1 \)

So \( 2n - 1 = k(2p - 1) \)
Solve for $n$:

$$2n = k(2p - 1) + 1$$

$$n = \frac{k(2p - 1) + 1}{2} = \frac{2kp - k + 1}{2} = kp - \frac{k - 1}{2}$$

Plugging $n$ back into the original definition, apply to the second set of points in the parabolic shape:

$$A_{n+2} - A_{n-4} = \frac{(n + 3)(n + 2)}{2} - \frac{(n - 4)(n - 3)}{2}$$

$$= \frac{n^2 + 2n + 3n + 6 - (n^2 - 3n - 4n + 12)}{2}$$

$$= \frac{12n - 6}{2}$$

$$= 6n - 3$$

$$= 3(2n - 1)$$

And since $2n - 1 = k(2p - 1)$,

$$3(2n - 1) = 3k(2p - 1)$$

And since $3k(2p - 1)$ is a multiple of $k$, the points $A_{n+2}$ and $A_{n-4}$ are on the same column thereby defining a parabolic shape.  

**Infinite Parabolas**

It may be natural to wonder if this pattern could somehow be constructed without the limit of columns. We hypothesized that the parabolic shapes we had been finding would continue indefinitely when the limit of columns was taken away. We came up with a few
ideas of how to show this continuation and settled on wrapping the sequence around a cylinder. Instead of cutting the sequence off at the k\textsuperscript{th} block, and restarting it on the next line as a new row of a two dimensional lattice, we wrapped the sequence around the three dimensional cylinder so that the row is continuous as it wraps down the side of the cylinder.

Theoretically, there can be an infinite number of columns on the first row, but to see what was happening, we examined a smaller k so that the parabolic shapes would wrap around the cylinder. In order to visualize what was happening as these parabolic shapes continued down the cylinder, we flattened out the cylinder. The easiest way to show the continuation was to place these graphs next to each other, one line above the previous graph so that each new row was a continuation of the previous row. Then, to make things even clearer, we colored the spaces in between every two points. Because there is a new row with every repetition of the pattern, we colored one row at a time for the first parabolic shape, two rows at a time for the second parabolic shape, three rows at a time for the third and so on. The green follows the first parabolic shape, purple the second, and orange the third. As you can see, the parabolic shapes continue indefinitely and overlap; each of these parabolic shapes could be expanded to cover the entire space.
Music

These graphs, when flipped sideways, could function as music staffs. Each column equates to one of the lines or spaces in between lines found on a music staff. We translated some of the different examples into music to see if other patterns might be easier to see or hear when listening. We also wanted to see if anything could be produced from these patterns that would be considered musically pleasing. [4]

You can see where the main parabolic shapes are in the music. We kept the key in C major, because C major has no sharps or flats and we did not want to add any extra variables.
By listening to these patterns, you can hear more clearly than you can see that these patterns are being repeated exactly. If you copy the sections of the pattern from where the pattern repeats and play the sections simultaneously, you will hear that the exact same notes are being played. Because we did not take into account the extra lines in between points and just focused on the columns being used, these sections are exactly the same.

So the next thing we wanted to try was to see what would happen if we played multiple graphs simultaneously. We tried two different options. First we tried adjacent values. Musically when you have two notes that are right next to each other, you get what
is called dissonance. This is an unpleasant sound. It turns out that playing graphs of adjacent numbers gives you a lot of dissonant chords and does not sound very nice.

Alternately, we found that the music sounded nicer when using multiples of the same base number.
Another interesting point that is visible in this form is that all graphs use the same notes (or the parabolic shape uses the same format regardless of the k). The only difference is that you see more of the parabolic shape the larger the k is. This goes back and re-establishes in a different format that these parabolic shapes continue forever when unrestricted by columns.
Concluding Remarks

We only really scratched the surface of what can be learned from a sequence on a lattice. We learned how to construct this pattern for the triangular numbers and for the central polygonal numbers, but much can still be learned from examining other sequences or how sequences would react when combined. We looked at how this sequence would react when functioning in a three-dimensional form and found that the sequence would continue without the column limit. We examined the musical properties of this sequence and what could be learned mathematically by listening to the sequence, but this could also be pushed by examining other sequences or other combinations of this sequence.
References


