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Bohr density of simple linear group orbits

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Abstract. We show that any non-zero orbit under a non-compact, simple, irreducible linear
group is dense in the Bohr compactification of the ambient space.

1. Introduction
Let \( V \) be a locally compact abelian group, \( V^* \) its Pontryagin dual and \( bV \) its Bohr
compactification, that is, \( bV \) is the dual of the discretized group \( V^* \). On identifying \( V \)
with its double dual we have a dense embedding \( V \hookrightarrow bV \), namely,

\[
\{\text{continuous characters of } V^*\} \hookrightarrow \{\text{all characters of } V^*\}.
\]

The relative topology of \( V \) in \( bV \) is known as the Bohr topology of \( V \). Among its many
intriguing properties (surveyed in [G07]) is the observation due to Katznelson [K73a] (see
also [G79, §7.6]) that very ‘thin’ subsets of \( V \) can be Bohr dense in very large ones.

While Katznelson was concerned with the case \( V = \mathbb{Z} \) (the integers), we shall illustrate
this phenomenon in the setting where \( V \) is the additive group of a real vector space, and
the subsets of interest are the orbits of a Lie group acting linearly on \( V \). Indeed our aim is
to establish the following result, which was conjectured in [Z96, p. 45].

**THEOREM 1.** Let \( G \) be a non-compact, simple real Lie group and \( V \) a non-trivial,
irreducible, finite-dimensional real \( G \)-module. Then every non-zero \( G \)-orbit in \( V \) is dense
in \( bV \).

We prove this in §3 on the basis of four lemmas found in §2. Before that, let us
record a similar property of nilpotent groups. In that case, orbits typically lie in proper
affine subspaces, so we cannot hope for Bohr density in the whole space; but we have the
following theorem.

**THEOREM 2.** Let \( G \) be a connected nilpotent Lie group and \( V \) a finite-dimensional
\( G \)-module of unipotent type. Then every \( G \)-orbit in \( V \) is Bohr dense in its affine hull.
Proof. Recall that unipotent type means that the Lie algebra \( g \) of \( G \) acts by nilpotent operators. So \( Z \mapsto \exp(Z)v \) is a polynomial map of \( g \) onto the orbit of \( v \in V \), and the claim follows immediately from [Z93, Theorem]. \( \square \)

2. Four lemmas

Our first lemma gives several characterizations of Bohr density—each of which can also be regarded as providing a corollary of Theorem 1.

**Lemma 1.** Let \( \mathcal{O} \) be a subset of the locally compact abelian group \( V \). Then the following are equivalent:

1. \( \mathcal{O} \) is dense in \( bV \);
2. \( \alpha(\mathcal{O}) \) is dense in \( \alpha(V) \) whenever \( \alpha \) is a continuous morphism from \( V \) to a compact topological group;
3. every almost periodic function on \( V \) is determined by its restriction to \( \mathcal{O} \);
4. Haar measure \( \eta \) on \( bV \) is the weak* limit of probability measures \( \mu_T \) concentrated on \( \mathcal{O} \).

Proof. (1) \( \iff \) (2): Clearly (2) implies (1) as the special case where \( \alpha \) is the natural inclusion \( \iota : V \hookrightarrow bV \). Conversely, suppose (1) holds and \( \alpha : V \to X \) is a continuous morphism to a compact group. By the universal property of \( bV \) [D82, Theorem 16.1.1], \( \alpha = \beta \circ \iota \) for a continuous morphism \( \beta : bV \to X \). Now continuity of \( \beta \) implies \( \beta(\iota(\mathcal{O})) \subset \beta(\iota(\mathcal{O})) \), which is to say that \( \beta(bV) \subset \overline{\alpha(\mathcal{O})} \) and hence \( \alpha(V) \subset \overline{\alpha(\mathcal{O})} \), as claimed.

(1) \( \iff \) (3): Recall that a function on \( V \) is almost periodic if and only if it is the pull-back of a continuous \( f : bV \to C \) by the inclusion \( V \hookrightarrow bV \). If two such functions coincide on \( \mathcal{O} \) and \( \mathcal{O} \) is dense in \( bV \), then clearly they coincide everywhere. Conversely, suppose that \( \mathcal{O} \) is not dense in \( bV \). Then by complete regularity [H63, Theorem 8.4] there is a non-zero continuous \( f : bV \to C \) which is zero on the closure of \( \mathcal{O} \) in \( bV \). Now clearly this \( f \) is not determined by its restriction to \( \mathcal{O} \).

(1) \( \iff \) (4) [K73a]: Suppose that \( \eta \) is the weak* limit of probability measures \( \mu_T \) concentrated on \( \mathcal{O} \). So we have \( \mu_T(f) \to \eta(f) \) for every continuous \( f \), and the complement of \( \mathcal{O} \) in \( bV \) is \( \mu_T \)-null [B04, Definition V.5.7.4 and Proposition IV.5.2.5]. If \( f \) vanishes on the closure of \( \mathcal{O} \) in \( bV \) then so do all \( \mu_T(|f|) \) and hence also \( \eta(|f|) \), which forces \( f \) to vanish everywhere. So \( \mathcal{O} \) is dense in \( bV \). Conversely, suppose that \( \mathcal{O} \) is dense in \( bV \). We have to show that given continuous functions \( f_1, \ldots, f_n \) on \( bV \) and \( \varepsilon > 0 \), there is a probability measure \( \mu \) concentrated on \( \mathcal{O} \) such that \( |\eta(f_j) - \mu(f_j)| < \varepsilon \) for all \( j \). Writing

\[
F = (f_1, \ldots, f_n) \quad \text{and} \quad \eta(F) = (\eta(f_1), \ldots, \eta(f_n)),
\]

we see that this amounts to \( \|\eta(F) - \mu(F)\| < \varepsilon \), where the norm is the sup norm in \( C^n \). Now by [B04, Corollary V.6.1] \( \eta(F) \) lies in the convex hull of \( F(bV) \) (which is compact by Carathéodory’s theorem [B87, Corollary 11.1.8.7]). So \( \eta(F) \) is a convex combination \( \sum_{i=1}^N \lambda_i F(w_i) \) of elements of \( F(bV) \). But \( F(\mathcal{O}) \) is dense in \( F(bV) \), so we can find \( w_i \in \mathcal{O} \) such that \( \|F(w_i) - F(w_i)\| < \varepsilon \). Putting \( \mu = \sum_{i=1}^N \lambda_i \delta_{w_i} \), where \( \delta_{w_i} \) is Dirac measure at \( w_i \), we obtain the desired probability measure \( \mu \). \( \square \)
Remark 1. One might wonder if condition (2) is equivalent to the following a priori weaker but already interesting property:

(2') $O$ has dense image in any compact quotient group of $V$.

Here is an example showing that (2') does not imply (2). Let $V = \mathbb{R}$ and $O = \mathbb{Z} \cup 2\pi \mathbb{Z}$. Then clearly $O$ has dense image in every compact quotient $\mathbb{R}/a\mathbb{Z}$. On the other hand, considering the irrational winding $\alpha : \mathbb{R} \to T^2$ defined by $\alpha(v) = (e^{iv}, e^{2\pi iv})$, one can check without difficulty that $\overline{\alpha(O)} = T \times \{1\} \cup \{1\} \times T$, which is strictly smaller than $\overline{\alpha(V)} = T^2$.

Remark 2. A net of probability measures $\mu_T$ converging to Haar measure on $bV$ as in (4) has been called a generalized summing sequence by Blum and Eisenberg [B74]. They observed, among others, the following characterization.

**Lemma 2.** The following conditions are equivalent:

1. $\mu_T$ is a generalized summing sequence;
2. the Fourier transforms $\hat{\mu}_T(u) = \int_{bV} \omega(u) d\mu_T(\omega)$ converge pointwise to the characteristic function of $\{0\} \subset V^*$.

**Proof.** This characteristic function is the Fourier transform of Haar measure $\eta$ on $bV$. Thus, condition (2) says that $\mu_T(f) \to \eta(f)$ for every continuous character $f(\omega) = \omega(u)$ of $bV$, whereas condition (1) says that $\mu_T(f) \to \eta(f)$ holds for every continuous function $f$ on $bV$. Since linear combinations of continuous characters are uniformly dense in the continuous functions on $bV$ (Stone–Weierstrass), the two conditions imply each other. \qed

For our third lemma, let $G$ be a group, $V$ a finite-dimensional $G$-module, and write $V^*$ for the dual module wherein $G$ acts contragrediently: $\langle gu, v \rangle = \langle u, g^{-1}v \rangle$.

**Lemma 3.** Suppose that $V$ is irreducible and $\phi(g) = \langle u, gv \rangle$ is a non-zero matrix coefficient of $V$. Then every other matrix coefficient $\psi(g) = \langle x, gy \rangle$ is a linear combination of left and right translates of $\phi$.

**Proof.** Irreducibility of $V$ and (therefore) $V^*$ ensures that $u$ and $v$ are cyclic, that is, their $G$-orbits span $V^*$ and $V$. So we can write $x = \sum_i \alpha_i g_i u$ and $y = \sum_j \beta_j g_j v$, whence $\psi(g) = \sum_{i,j} \alpha_i \beta_j \phi(g_i^{-1} g g_j)$. \qed

Our fourth and final preliminary result is the following famous lemma.

**Lemma 4.** (Van der Corput) Suppose that $F : [a, b] \to \mathbb{R}$ is differentiable, its derivative $F'$ is monotone, and $|F'| \geq 1$ on $(a, b)$. Then $\int_a^b e^{F(t)} \ dt \leq 3$.

**Proof.** See [S93, p. 332], or [R05, Lemma 3] which actually gives the sharp bound 2. \qed

3. **Proof of Theorem 1**

By Lemma 1, it is enough to show that Haar measure on $bV$ is the weak* limit of probability measures $\mu_T$ concentrated on the orbit under consideration; or equivalently (Lemma 2), that the Fourier transforms of the $\mu_T$ tend pointwise to the characteristic function of $\{0\} \subset V^*$. (Here we identify the Pontryagin dual with the dual vector space or module.)
In particular, the matrix coefficient
\[ \langle \text{decomposition} \rangle \]
\[ \text{everywhere} \]
is not identically zero on \( K \). Hence contains weights \( \nu \) for which we know that \( \langle \nu, H \rangle \) is positive. Therefore, maximizing \( \langle \nu, H \rangle \) over \( N \) produces a positive number \( \langle \nu_0, H \rangle \), in terms of which our exponent and its derivatives can be written
\[ \frac{d^n}{dt^n} F_{kk'}(t) = e^{\langle \nu_0, H \rangle t} \sum_{\nu \in \alpha^*} f_{\nu}(k, k') \langle \nu, H \rangle^n e^{\langle \nu - \nu_0, H \rangle t}, \]
where \( \langle \nu - \nu_0, H \rangle < 0 \) in all non-zero terms except the one indexed by \( \nu_0 \). (Here we assume, as we may, that \( H \) was initially chosen outside the kernels of all pairwise...
differences of weights of $V$.) From this it is clear that for almost all $(k, k')$ there is a $T_0$ beyond which the first two derivatives of $F_{kk'}$ are greater than 1 in absolute value. So Lemma 4 applies and gives

$$\left| \int_{T_0}^{T} e^{i F_{kk'}(t)} \, dt \right| \leq 3 \quad \text{for all } T.$$ 

Therefore, $\lim_{T \to \infty} (1/T) \int_{0}^{T} e^{i F_{kk'}(t)} \, dt = 0$ for almost all $(k, k')$, whence the conclusion (*) by dominated convergence. This completes the proof.

4. **Outlook**

Theorem 1 says that the $G$-action on $V \setminus \{0\}$ is minimal [P83] in the Bohr topology. It would be interesting to determine if it is still minimal, and/or uniquely ergodic, on $bV \setminus \{0\}$.

It is also natural to speculate whether our theorems have a common extension to more general group representations. Here we shall content ourselves with noting two obstructions. First, Theorem 1 clearly fails for semisimple groups with compact factors. Secondly, Theorem 2 fails for $V$ not of unipotent type, as one sees by observing that the orbits of $\mathbb{R}$ acting on $\mathbb{R}^2$ by $\exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ (i.e., hyperbolas) already have non-dense images in $\mathbb{R}^2/\mathbb{Z}^2$.

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