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Localized Quantum States

François Ziegler

À la mémoire de Jean-Marie Souriau

Abstract Let $X$ be a symplectic manifold and $\text{Aut}(L)$ the automorphism group of a Kostant-Souriau line bundle on $X$. Quantum states for $X$, as defined by J.-M. Souriau in the 1990s, are certain positive-definite functions on $\text{Aut}(L)$ or, less ambitiously, on any “large enough” subgroup $G \subset \text{Aut}(L)$. This definition has two major drawbacks: when $G = \text{Aut}(L)$ there are no known examples; and when $G$ is a Lie subgroup the notion is, as we shall see, far from selective enough. In this paper we introduce the concept of a quantum state localized at $Y$, where $Y$ is a coadjoint orbit of a subgroup $H$ of $G$. We show that such states exist, and tend to be unique when $Y$ has lagrangian preimage in $X$. This solves, in a number of cases, A. Weinstein’s “fundamental quantization problem” of attaching state vectors to lagrangian submanifolds.

1 Introduction: The quantization problem

Quantum mechanics is a unitary representation of the symmetry group of classical mechanics—or a large subgroup thereof. This prescription, which infinitesimally

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goes back to Dirac [D30, §21], first became precise in 1965 when Kostant and Souriau constructed the symmetry group in question: namely, it is the automorphism group of a Kostant-Souriau line (or circle) bundle, \( L \), over the symplectic manifold \( X \) which models the classical mechanical system under consideration.

(1.1) Example (the plane). Let \( X \) be \( \mathbb{R}^2 \) with points \( x = (p, q) \) and 2-form \( \omega = dp \wedge dq \). Then \( L = X \times C \) with points \( x = (x, z) \), projection \( x \mapsto x \), connection 1-form \( \sigma = pdq + dz/iz \), and hermitian structure \( |x| = |z| \). An automorphism, \( g \in \text{Aut}(L) \), is a diffeomorphism of the form

\[
g(x, z) = (s(x), z e^{jS(x)})
\]

where \( s \) is a symplectomorphism of \( X \) and the function \( S \) is determined up to an additive constant by the condition that \( pdq - s'(pdq) = dS \). The Lie algebra \( \text{aut}(L) \) of infinitesimal automorphisms of \( L \) is isomorphic to the Poisson bracket algebra \( C^\infty(X) \); to any \( (\sigma, \cdot, \cdot) \)-preserving vector field \( Z \) we can attach the function \( H(x) = \sigma(Z(\xi)) \) called its \text{hamiltonian}, and conversely any \( H \in C^\infty(X) \) gives rise to the infinitesimal automorphism

\[
Z(x, z) = (\eta(x), iz\ell(x))
\]

where \( \eta = (-\partial H/\partial q, \partial H/\partial p) \) is the symplectic gradient of \( H \), and \( \ell = H - p\partial H/\partial p \). (This isomorphism is established in greater generality in [K70; S70]; in the case at hand it was already known to Lie and Van Hove [L90, p. 270; V51, §5].)

Given a symplectic manifold \( X \) and a Kostant-Souriau line bundle \( L \) over it, one would now of course like to know which representation(s) of \( \text{Aut}(L) \)---or of subgroups thereof---furnish the quantum theory. As \( \text{Aut}(L) \)-invariant “polarizations” are not available, Souriau was led to propose instead the following axiomatic, polarization-independent definition.

(1.4) Definition ([S88; S90a; S92]). A quantum representation (of \( \text{Aut}(L) \), for \( X \)) is a unitary \( \text{Aut}(L) \)-module \( \mathcal{H} \) such that, for every unit vector \( \varphi \in \mathcal{H} \), the matrix coefficient \( m(g) = \langle \varphi, g\varphi \rangle \) satisfies

\[
\left| \sum_{j=1}^n c_j m(\exp(Z_j)) \right| \leq \sup_{x \in X} \left| \sum_{j=1}^n c_j e^{iH_j(x)} \right|
\]

for all choices of an integer \( n \), complex numbers \( c_1, \ldots, c_n \) and complete, commuting vector fields \( Z_1, \ldots, Z_n \in \text{aut}(L) \) with respective hamiltonians \( H_1, \ldots, H_n \). (Here \( \exp(Z_j) \in \text{Aut}(L) \) denotes the time 1 flow of the complete vector field \( Z_j \in \text{aut}(L) \).) As we shall see in §2, (1.5) can be reformulated (after [Z96]) as requiring that

\[
\text{the quantum spectrum of ‘commuting observables’ is concentrated on their classical range, suitably compactified.}
\]

The problem of geometric quantization, in the words of [S84, p. 74], is now to find a quantum representation of \( \text{Aut}(L) \); or equivalently---see (3.3)---to find a state \( m \) of
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This is a tall order, which we will not address here beyond observing that 1°) the “obstruction theorem” of [V51] does not prove its impossibility, yet 2°) the solution is not the so-called prequantization representation (also introduced in [V51]; see §2). Instead we shall study, as the start of this introduction suggests, states and representations of Lie subgroups \( G \subset \text{Aut}(L) \) that satisfy the inequalities induced by (1.5). The main points of our investigation are as follows:

- In §3 we show that Souriau’s resulting notions of quantum state and representation (of a Lie group \( G \), for one of its coadjoint orbits \( X \)) are by themselves not selective enough, because the compactification in (1.6) can fail utterly to distinguish between coadjoint orbits.
- In [Z96] this was remedied by suppressing this compactification. Here in contrast we take it seriously, because we find that it (and only it) makes room for interesting, localized states—defined in §4 by the property that their further restriction to a Lie subgroup \( H \subset G \) is quantum for a coadjoint orbit \( Y \) of \( H \).
- In §5 we prove existence and uniqueness, whenever \( G \) is a nilpotent Lie group and \( \mathfrak{h} \) is what Kirillov called a maximal subordinate subalgebra to \( x \in \mathfrak{g}^* \), of a quantum state for \( X = G(x) \) localized at \( Y = \{ x_\lambda \} \). This vastly generalizes states of the Heisenberg group discussed in [B74; A03].
- In §6 we prove existence and uniqueness, whenever \( G \) is a compact Lie group, \( T \) a maximal torus and \( x \) an integral, \( T \)-fixed point in \( \mathfrak{g}^* \), of a quantum state for \( X = G(x) \) localized at \( Y = \{ x_\lambda \} \). The resulting Gel’fand-Na ˘ımark-Segal representation is the irreducible one with highest weight \( \lambda = x_\lambda \).
- In §7 we prove existence and sometimes uniqueness of several quantum states of Euclid’s group for the coadjoint orbit \( X \) relevant in geometrical optics, localized at orbits \( Y \) having lagrangian preimages in \( X \). These states provide legitimate hilbertian models of the physicists’ plane, spherical and cylindrical waves.

Finally the Appendix collects a number of known facts on positive-definite functions, states, and unitary representations of groups used throughout the paper.

2 Prequantization is not quantum

We start by giving the promised geometric recasting (1.6) of inequalities (1.5). To this end, let us agree to call perspective on \( X \) any finite-dimensional, commutative subalgebra \( \mathfrak{a} \) of \( \text{aut}(L) \) consisting of complete vector fields. Given such an \( \mathfrak{a} \) and \( x \in X \), write \( x_{\mathfrak{a}} \) for the character \( \mathbb{Z} \mapsto e^{iH(x)} \) of \( \mathfrak{a} \), where \( H \) is the hamiltonian of \( Z \); and regard \( x \mapsto x_{\mathfrak{a}} \) as a map of \( X \) to the (compact) Pontryagin dual \( \hat{\mathfrak{a}} \) of the discretized additive group \( \mathfrak{a} \). Then we have:

(2.1) Theorem. A unitary \( \text{Aut}(L) \)-module \( \mathcal{H} \) is a quantum representation for \( X \) if and only if for each unit vector \( \varphi \in \mathcal{H} \) and each perspective \( \mathfrak{a} \) on \( X \), the state \( (\varphi, \exp_\mathfrak{a}(\cdot \varphi)) \) of \( \mathfrak{a} \) has its spectral measure concentrated on the closure of \( X_{\mathfrak{a}} \) in \( \hat{\mathfrak{a}} \).

(We refer to the Appendix for the notions of state (A.1) and spectral measure (A.20). The closure of \( X_{\mathfrak{a}} \) in \( \hat{\mathfrak{a}} \) is the compactification mentioned in (1.6), and can
be viewed as an abstract device allowing us to treat the inequalities (1.5) all at once; the group \( \hat{a} \) itself is known as the Bohr compactification \( ba^* \) of the ordinary dual \( a^* \) of \( a \); see [H63, 26.11].

**Proof.** Suppose that \( \mathcal{X} \) satisfies (1.5), and let \( a \) be a perspective on \( X \). Then the function \((\varphi, \exp_{|a|}(\cdot)\varphi) = m \circ \exp_a \) is the pull-back of a state by a group homomorphism, hence is a state as one readily verifies. By Bochner’s theorem (A.20) this state has a spectral measure \( \nu \) so that \((m \circ \exp_a)(Z) = \int_{\mathcal{X}} \chi(Z) d\nu(\chi) \). Now (1.5) says that we have \(|\nu(f)| \leq \sup_{x \in \mathcal{X}} |f(x)| \), or in other words

\[
|\nu(f)| \leq \sup_{x \in \mathcal{X}} |f(\chi)|,
\]

for every trigonometric polynomial \( f(\chi) = \sum_j c_j \chi(Z_j) \) with \( c_j \in \mathbb{C}, Z_j \in a \). By Stone-Weierstrass, these are uniformly dense in the continuous functions on \( \hat{a} \), so therefore (2.2) still holds for all continuous \( f \). In particular if \( f \) vanishes on the closure \( bX_{|a|} \) of \( X_{|a|} \) in \( \hat{a} \) then \( \nu(f) = 0 \), which is to say that

\[
\text{supp}(\nu) \subset bX_{|a|},
\]

or in other words, that \( \nu \) is concentrated on \( bX_{|a|} \) [B67, n° V.5.7].

Conversely let \( c_j \) and \( Z_j \) be given as in Definition (1.4). Then the \( Z_j \) span a perspective \( a \) on \( X \), and \( f(\chi) = \sum_j c_j \chi(Z_j) \) defines a continuous function on \( \hat{a} \). Assuming (2.3) for \( a \), the mean value inequality gives us (2.2) and hence (1.5).

**(2.4) Example (continued).** The space of \( L^2 \) sections of the line bundle \( L \) of (1.1) is naturally a unitary \( \text{Aut}(L) \)-module, often called the prequantization representation. Identifying sections \( \sigma \) with functions \( \varphi \in L^2(X) \) by writing \( \sigma(x) = (x, \varphi(x)) \), the action of an automorphism (1.2) reads

\[
(g\varphi)(x) = e^{iS(r^{-1}(x))}\varphi(s^{-1}(x)).
\]

We claim:

**(2.6) Proposition.** The prequantization representation (2.5) of \( \text{Aut}(L) \) in \( L^2(X) \) is not quantum for \( X \).

**Proof.** We consider the hamiltonian \( H(p, q) = \sin p \). It gives rise to an infinitesimal automorphism (1.3) whose flow writes \( e^{itZ}(p, q, z) = (p, q + t \cos p, ze^{it\sin p - p \cos p}) \). The resulting action (2.5) on sections is

\[
(e^{itZ}\varphi)(p, q) = e^{it\sin p - p \cos p}\varphi(p, q - t \cos p).
\]

In order to compute its spectral measure, we introduce the partial Fourier transform \( \hat{\varphi}(p, k) = \sqrt{2\pi}^{-1} \int e^{i\varphi(p, q)} dq \) on which the transported action becomes

\[
(e^{itZ}\hat{\varphi})(p, k) = e^{it\sin p + (k - p) \cos p}\hat{\varphi}(p, k).
\]
This demonstrates that the spectral measure (A.20) of \( i\mathbb{Z} \mapsto (\varphi, e^{ik}\varphi) \) is the image of 
\( |\hat{\varphi}(p, k)|^2 dp \, dk \) by the map \((p, k) \mapsto \sin p + (k-p) \cos p\). Now if (2.5) were quantum for \( X \), then by Theorem (2.1) this image measure would be always concentrated on the range \([-1, 1]\) of \( H \) (in \( \mathbb{R} \), which we have identified with the dual \( \hat{a}^* \subset \hat{a} \) of the perspective \( a = RZ \)); but this is clearly not the case.

(2.9) Remarks. It is comforting to see Definition (1.4) eliminate the prequantization representation (2.5), which physicists since Van Hove [V51] have rejected as “too big”. But let us emphasize that it does so for different reasons.

For Van Hove, the trouble with (2.5) is that restricting it to the automorphisms 
\( g(p, q; z) = (p + b, q + c, ze^{-i(a+bq)}) \)
by which the Heisenberg group

\[
G = \left\{ g = \begin{pmatrix} 1 & b & a \\ 1 & c & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}
\]

acts on \( L \), produces a representation

\[
(g\varphi)(p, q) = e^{-ia}e^{-ib(q-c)}\varphi(p-b, q-c)
\]

of \( G \) which is reducible and thus not equivalent to the Schrödinger representation. (Van Hove went on to demand that any acceptable representation of \( \text{Aut}(L) \) be irreducible on \( G \), and then to prove his famous “obstruction theorem” that no such representation could possibly exist.)

Definition (1.4), in contrast, imposes no such irreducibility condition (we fully expect that a representation satisfying it will not be irreducible on \( G \)); and the sense in which it declares (2.5) “too big” is purely spectral: this representation assigns too large a spectrum to the bounded quantity \( \sin p \). Another advantage is that Definition (1.4) excludes more undesired representations—such as the following, once proposed by Gotay and rejected by Velhinho (see [V98; G00]).

(2.12) Example (the 2-torus). Consider the pair \( L \to X \) of (1.1) and three numbers \( A, B, C \) with \( A = BC = 2\pi \). Then a particular Kostant-Souriau line bundle over the torus \( X = \mathbb{R}^2/(B\mathbb{Z} \times C\mathbb{Z}) \) is the quotient \( L = L/\Gamma \) of \( L \) by the action of the subgroup \((a, b, c) \in A\mathbb{Z} \times B\mathbb{Z} \times C\mathbb{Z} \) of (2.10). Its \( L^2 \) sections can be identified with functions on \( X \) that satisfy

\[
\varphi(p + b, q + c) = e^{-ibq}\varphi(p, q)
\]

for all \((b, c) \in B\mathbb{Z} \times C\mathbb{Z} \), and are square integrable over any rectangle of size \( B \times C \). Specializing to \( C = 1 \), the flow with hamiltonian \( \sin p \) on \( L \) commutes with \( \Gamma \) and so descends to act on \( L \) and on its sections (2.13) by the same formula (2.7) as before. Arguing much as in (2.6) (with a Fourier series replacing the Fourier transform), one readily obtains:

(2.14) Proposition. The prequantization representation of \( \text{Aut}(L) \) in \( L^2 \) sections of \( L \to X \) is not quantum for the 2-torus \( X \).
3 Quantum states for coadjoint orbits

It is unknown whether any representation satisfying Definition (1.4) exists beyond the simple case where $X$ is a single point. So, heeding the advice at the start §1, we shall look instead for representations of Lie subgroups of Aut($L$), where $L \to X$ is a Kostant-Souriau line bundle; or equivalently (see (A.3)), for states of Lie groups $G$ having a smooth action $G \to$ Aut($L$).

Such an action has a canonical moment map $\Phi : X \to g^*$, where $\langle \Phi(\cdot), Z \rangle$ is the hamiltonian of the image of $Z \in \mathfrak{g}$ in aut($L$). We will regard $G$ as “large enough” if these hamiltonians separate points of $X$; then the moment map is one-to-one, and we may as well assume that $X$ is a coadjoint orbit of $G$. Thus we come to:

(3.1) Definition ([S88; S90a; S92]). Let $X$ be a coadjoint orbit of the Lie group $G$. A quantum state (of $G$, for $X$) is a state $m$ of $G$ such that

$$\left| \sum_{j=1}^{n} c_j m(\exp(Z_j)) \right| \leq \sup_{x \in X} \left| \sum_{j=1}^{n} c_j e^{i\langle x, Z_j \rangle} \right|$$

for all choices of an integer $n$, complex numbers $c_j$, and commuting $Z_j$ in the Lie algebra $\mathfrak{g}$ of $G$. A quantum representation (of $G$, for $X$) is a unitary $G$-module $\mathcal{H}$ such that, for every unit $\varphi \in \mathcal{H}$, the function $m(g) = (\varphi, g\varphi)$ is a quantum state.

(3.3) Theorem ([S88, 5.2b]). A state $m$ of $G$ is quantum for $X$ if and only if the resulting Gel’fand-Na˘ımark-Segal representation, $\text{GNS}_m$ (A.3), is quantum for $X$.

Diffeologists can regard Definition (1.4) as a special case of (3.1), for they know that the base of a Kostant-Souriau line bundle $L \to X$ is always a coadjoint orbit of Aut($L$) in the diffeological sense [S88, 4.3b]. Repeating the proof of (2.1) we can again recast the definition in more geometrical fashion, as follows.

(3.4) Theorem ([Z96]; Fig. 1). A state $m$ of $G$ is quantum for $X$ if and only if for each abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, the state $m \circ \exp_\mathfrak{a}$ of $\mathfrak{a}$ has its spectral measure concentrated on the closure $bX_\mathfrak{a}$ of $X_\mathfrak{a}$ in $\hat{\mathfrak{a}}$.

Here $|\mathfrak{a}$ means restriction to $\mathfrak{a}$, and as before $\hat{\mathfrak{a}}$ denotes the (compact) character group of the discrete additive group $\mathfrak{a}$. This densely contains the group of all continuous characters, which we may and will identify with $\mathfrak{a}^\ast$ by letting $u \in \mathfrak{a}^\ast$ stand for

Fig. 1 Projection of a coadjoint orbit $X$ of $G$ to the dual of an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$.
the character $e^{i\omega \cdot s}$ of $\mathfrak{a}$. Likewise we define $\mathfrak{g}$ and regard $\mathfrak{g}^*$ as a dense subgroup; in doing so we must be careful to distinguish between usual closure in $\mathfrak{g}^*$ and closure in $\mathfrak{g}$, which we denote by $X \to bX$ for Bohr closure. Finally we remark that the notation $bX_{\mathfrak{a}}$ is unambiguous, i.e. we have $(bX)_{\mathfrak{a}} = b(X_{\mathfrak{a}})$: the projection of $bX$ lies in the closure of $X_{\mathfrak{a}}$ by continuity; moreover it is compact and so contains this closure.

Now the point of (3.4) is that the effect of Bohr closure is quite drastic:

(3.5) **Theorem ([H114]).**
(a) If $G$ is noncompact simple, any nonzero coadjoint orbit is Bohr dense in $\mathfrak{g}$.
(b) If $G$ is connected nilpotent, any coadjoint orbit has the same Bohr closure as its affine hull.

(3.6) **Corollary.**
(a) If $G$ is noncompact simple, every unitary representation is quantum for every nonzero coadjoint orbit.
(b) If $G$ is simply connected nilpotent, any unitary representation is quantum for an orbit $X$ if and only if its restriction to $C_X = \text{exp}([Z \in \mathfrak{a} : \langle \cdot, Z \rangle \text{ is constant on } X])$ is the character $\exp(Z) \mapsto e^{i\langle X,Z \rangle}$ times the identity.

(Here of course $(X,Z)$ denotes the common value of $\langle x,Z \rangle$ for all $x \in X$.)

**Proof.** (a) is immediate from Theorems (3.4) and (3.5a). To prove (b), let $\mathcal{H}$ be a unitary $G$-module, pick a unit vector $\varphi \in \mathcal{H}$ and write $m(g) = \langle \varphi, g\varphi \rangle$.

Suppose $\mathcal{H}$ is quantum for $X$. If $\mathfrak{a}$ is any 1-dimensional subalgebra of $\mathfrak{c}_X$, then $X_{\mathfrak{a}}$ consists of the single point $Z \mapsto (X,Z)$. So (3.4) says that $m(\exp(Z)) = e^{i\langle X,Z \rangle}$ for all $Z \in \mathfrak{a}$ and hence for all $Z \in \mathfrak{c}_X$. Since $\|g\varphi - m(g)\varphi\|^2 = 1 - |m(g)|^2$ this implies that $C_X$ acts by $\exp(Z)\varphi = e^{i\langle X,Z \rangle}\varphi$, as claimed.

Conversely, suppose that $C_X$ acts by this character. Let $\mathfrak{a}$ be any abelian subalgebra of $\mathfrak{g}$, and write $\iota : \mathfrak{a} \cap \mathfrak{c}_X \to \mathfrak{a}$ for the natural injection and $\iota' : (\mathfrak{a} \cap \mathfrak{c}_X)^* \to \mathfrak{a}^*$ and $\hat{\iota} : \hat{\mathfrak{a}} \to (\mathfrak{a} \cap \mathfrak{c}_X)^*$ for the dual projections. The relation $m \circ \exp_{|\mathfrak{a}^*|_{\mathfrak{c}_X}} = m \circ \exp_{\mathfrak{a}^*} \circ \iota'$ shows that the spectral measure of $m \circ \exp_{|\mathfrak{a}^*|_{\mathfrak{c}_X}}$ is the image by $\hat{\iota}$ of that of $m \circ \exp_{\mathfrak{a}^*}$.

As the former is concentrated on the point $X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}}$ by hypothesis, it follows that the latter is concentrated on the preimage $\hat{\iota}^{-1}(X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}})$ of this point [B67, n° V.6.2, Cor. 2]. There remains to see that this preimage is precisely $bX_{\mathfrak{a}}$. This follows from the calculation

\begin{equation}
\hat{\iota}^{-1}(X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}}) = b\iota^{-1}(X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}}) = b\text{Aff}(X_{\mathfrak{a}}) = b\text{Aff}(X)_{\mathfrak{a}} = bX_{\mathfrak{a}}
\end{equation}

where ‘Aff’ stands for affine hull. Here the first equality is because both $\hat{\iota}^{-1}(X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}})$ and $\iota^{-1}(X_{|\mathfrak{a}^*|_{\mathfrak{c}_X}})$ are preimages of points, hence translates of closed subgroups. The second equality is because the affine hull of $X_{\mathfrak{a}}$ is the intersection of all hyperplanes containing it. The third is because the affine hull of a projected set is the projection of its affine hull. And finally the fourth equality is Theorem (3.5b). \qed

(3.8) **Remarks.** The results (3.6) were certainly unexpected by the author of Definition (3.1). They are in sharp contrast with our findings in §2: while it was easy to find non-quantum representations of $\text{Aut}(L)$, but unknown if a quantum one even...
exists (a question whose difficulty is probably on par with that of making sense of the Feynman integral), scaling our ambitions back to finding representations of Lie subgroups has now produced the opposite situation, where quantum representations are in such rich supply that it may even be impossible (3.6a) to find a non-quantum one! Clearly this indicates that—whatever may be the case of Definition (1.4)—Definition (3.1) still needs to be refined.

One way to do so is to keep our hopes up high in (1.4) and bet that asking for states that extend to \( \text{Aut}(L) \) will provide the much-needed selection. (Note that extending a state is a very different proposition from extending the resulting representation \textit{in the same space}, as Van Hove was trying to do. The GNS module (A.3) of an extended state is usually much bigger than that of the state’s restriction to a subgroup.)

A second, more conservative way is to lay the blame for (3.6) on the Bohr closure in (3.4) as the obvious culprit, and just suppress this closure. (Here we note that compactifying \( X \) is really a change at the \textit{classical} level: our quantum states have probability measures on \( bX \) rather than \( X \) as their classical analogues. In fact Souriau’s papers \cite{S88, S90a, S92} contain also a theory of “statistical states” which boil down to just that, probability measures on \( bX \).) This path was explored in \cite{Z96} with mixed results: one does recover the “orbit methods” of Borel-Weil and Kirillov-Bernat as special cases, but only after adding one or two hypotheses which may seem \textit{ad hoc}.

\section{Localized states}

In this paper we want to explore a third way—one that doesn’t suppress the compactification of \( X \) implicit in Definition (3.1), but instead takes it seriously. Our investigation is motivated by the discovery, among quantum states, of objects that solve in some cases (albeit in a rather unexpected way), what A. Weinstein \cite{W82} has called the \textit{fundamental quantization problem}: to attach (possibly distributional) “wave functions” to lagrangian submanifolds of \( X \). It will turn out that these states not only exist, but can be uniquely characterized quite simply:

\begin{definition} \label{def:localized}
Let \( X \) be a coadjoint orbit of the Lie group \( G \), and \( Y \) a coadjoint orbit of a closed subgroup \( H \subset G \), contained in \( X_\mathfrak{h}_0 \). We say that a quantum state \( m \) for \( X \) is \textbf{localized at} \( Y \subset \mathfrak{h}^* \) if the restriction \( m|_H \) is a quantum state for \( Y \).
\end{definition}

We also think of this as meaning that the state is \textbf{localized on} \( \pi^{-1}(Y) \), where \( \pi \) is the projection \( X \rightarrow \mathfrak{h}^* \). We recall from \cite[Prop. 1.1]{K78} that this set is generically a \textit{coisotropic} submanifold of \( X \)—hence at least half-dimensional, and suitable for constraining a system to.

We shall almost exclusively apply Definition \eqref{def:localized} to cases where \( H \) is connected and \( Y = \{y\} \) is a point-orbit. To be a quantum state for \( \{y\} \) then means the following.

\begin{proposition, definition \textit{(Integral point-orbits).} \label{prop:integral}
Let \( H \) be a connected Lie group and \( \{y\} \) a point-orbit of \( H \) in \( \mathfrak{h}^* \). A quantum state \( n \) of \( H \) for \( \{y\} \) exists if and
only if \( y \) is \textbf{integral} in the sense that \( H \) admits a character \( \chi \) with differential \( iy \). It is then unique and given by that character, i.e. \( n(\exp(Z)) \) equals

\begin{equation}
\chi(\exp(Z)) = e^{iyZ}.
\end{equation}

\textbf{Proof.} Since \( y \) is an \( H \)-invariant point in \( \mathfrak{h}^* \), we have \( \langle y, [Z, Z'] \rangle = 0 \) for all \( Z, Z' \in \mathfrak{h} \). Thus \( iy \) defines a Lie algebra homomorphism from \( \mathfrak{h} \) to the abelian Lie algebra \( \mathfrak{u}(1) = i\mathbb{R} \). This integrates into a character \( \tilde{\chi} : \tilde{H} \to U(1) \) of the simply connected covering group \( \tilde{H} \) of \( H \), which descends to \( H \) if and only if \( y \) is integral.

Suppose that \( n \) is a quantum state for \( \{y\} \). For each line \( a = RZ \) in \( \mathfrak{h} \), Theorem (3.4) says that \( n \circ \exp|_a \) has its spectral measure concentrated on the point \( \{y\}_a \), hence must be given by \( (n \circ \exp|_a)(Z) = e^{i\langle y, Z \rangle} \). Therefore \( n \) must coincide with \( \chi \).

\hfill \( \Box \)

\textbf{(4.4) Corollary.} Suppose \( H \) is a closed connected subgroup of the Lie group \( G \), and \( \{y\} \) an integral point-orbit of \( H \) in \( \mathfrak{h}^* \) with resulting character \( \chi \). Then a quantum state \( m \) of \( G \) is localized at \( \{y\} \subset \mathfrak{h}^* \) if and only if the cyclic vector \( (A.7) \) of the resulting GNS module is an eigenvector of type \( \chi \) under \( H \).

\textbf{Proof.} Suppose \( m \) is localized at \( \{y\} \), i.e. \( m_H \) is a quantum state for \( \{y\} \). Then we have \( m_H = \chi \) by the previous Proposition, and \( (A.13) \) implies \( m(hg) = \chi(h)m(g) \) for all \( (g, h) \in G \times H \). Therefore the cyclic vector \( \varphi = \overline{m} \) satisfies

\begin{equation}
(h\overline{m})(g) = \overline{m}(h^{-1}g) = \overline{m}(h^{-1})\overline{m}(g) = \chi(h)\overline{m}(g),
\end{equation}

i.e. \( h\varphi = \chi(h)\varphi \), as claimed. Conversely, suppose that this last relation holds. Then we have \( m(h) = (\varphi, h\varphi) = (\varphi, \chi(h)\varphi) = \chi(h) \). So \( m_H = \chi \), which is to say that \( m \) is localized at \( \{y\} \).

\hfill \( \Box \)

Definition (4.1) will allow us to extract interesting objects from the generally unclassifiable maze (3.6) of all quantum representations. This is somewhat reminiscent of the representation theory of Lie algebras, where one can’t in general describe the class of simple modules [B90a], but where imposing the presence of eigenvectors produces a manageable classification problem [B90b].

\section{5 Nilpotent groups}

In this section we assume that \( G \) is a connected, simply connected nilpotent Lie group. Then \( \exp : \mathfrak{g} \to G \) is a diffeomorphism whose inverse we denote \( \log : G \to \mathfrak{g} \). Moreover we fix a coadjoint orbit \( X \subset \mathfrak{g}^* \) and a point \( x \in X \), and we recall that a connected subgroup \( H = \exp(\mathfrak{h}) \) of \( G \) is called \textbf{subordinate to} \( x \) if, equivalently,

- (a) \( e^{ix \circ \log_H} \) is character of \( H \);
- (b) \( x_\mathfrak{h} \) is a point-orbit of \( H \) in \( \mathfrak{h}^* \);
- (c) \( \langle x, [\mathfrak{h}, \mathfrak{h}] \rangle = 0 \).
Any subordinate subgroup has \( \dim(G/H) \geq \frac{1}{2} \dim(X) \); if this bound is attained then one calls \( H \) a polarization at \( x \). Polarizations are maximal subordinate subgroups, but some maximal subordinate subgroups are not polarizations.

**Theorem.** Let \( H \) be maximal subordinate to \( x \in X \). Then there is a unique quantum state for \( X \) localized at \( \{x_b\} \subset \mathfrak{h}^* \), namely

\[
(5.2) \quad m(g) = \begin{cases} 
  e^{ix \circ \log}(g) & \text{if } g \in H, \\
  0 & \text{otherwise.}
\end{cases}
\]

The associated GNS representation (A.3) is \( \text{ind}_H^G e^{ix \circ \log} \), where induction is in the sense of discrete groups.\(^1\)

**Proof.** The fact that \( m \) must coincide with \( e^{ix \circ \log} \) in \( H \) is just (4.2). To see that it must vanish outside \( H \), we consider the sequence \( H = G_0 \subset G_1 \subset G_2 \ldots \) where \( G_{i+1} \) is the normalizer of \( G_i \) in \( G \). Since \( G \) is nilpotent, the \( G_i \) are connected and all equal to \( G \) after finitely many steps [B72b, Prop. III.9.16]; so it is enough to show inductively that \( m \) vanishes in \( G_{i+1} \setminus G_i \) for all \( i \).

Case \( i = 0 \). Let \( g \in G_1 \setminus H \). Applying Weil’s inequality (A.13) twice, we get

\[
(5.3) \quad e^{ix \circ \log}(h)m(g) = m(hg) = m(gg^{-1}hg) = m(g)e^{ix \circ \log}(g^{-1}hg)
\]

for all \( h \in H \). Thus, if \( m(g) \) was nonzero, \( g \) would both normalize \( H \) and stabilize its character \( e^{ix \circ \log} \). Since the normalizer and stabilizer in question are connected [B72a; B72b] it would follow that \( Z = \log(g) \) normalizes \( \mathfrak{h} \) and stabilizes \( x_b \). Putting \( \mathfrak{f} = \mathfrak{h} \oplus \mathbb{R} \mathfrak{Z} \), we would conclude that \( \langle x, [\mathfrak{f}, \mathfrak{f}] \rangle \) is zero. But then \( K = \exp(\mathfrak{f}) \) would be subordinate to \( x \), and so \( H \) would not be maximal subordinate to \( x \). This contradiction shows that \( m(g) = 0 \).

Case \( i > 0 \). Let \( g \in G_{i+1} \setminus G_i \). Then \( g \) normalizes \( G_i \) but not \( H \), so we can fix an \( h \in H \) such that \( g^{-1}hg \in G_i \setminus H \). Putting \( g_n = h^ng \) it follows that \( g^{-1}_pgq \in G_i \setminus H \) whenever \( p \neq q \). The induction hypothesis then shows that \( m(g^{-1}_pgq) = 0 \), which is to say that the \( \delta^{g\circ} (= 1 \text{ at } g_n, \text{ and } 0 \text{ elsewhere}) \) make an orthonormal set relative to the sesquilinear form (A.2). Therefore Bessel’s inequality gives

\[
(5.4) \quad \sum_n |m(g_n)|^2 = \sum_n \langle (\delta^g, \delta^{g_n}) \rangle_m^2 \leq \langle (\delta^g, \delta^g) \rangle_m = 1.
\]

Now this forces \( m(g) = 0 \), because we have \( |m(g_n)| = |e^{ix \circ \log}(h^n)m(g)| = |m(g)| \) for all \( n \). Finally the last assertion of the Theorem is a special case of (A.17), and the fact that the state (5.2) is indeed quantum for \( X \) will result from (5.6c) below, because maximal subordinate subgroups always contain \( C \) (3.6c). \( \square \)

The representations

\[
(5.5) \quad i(x, H) = \text{ind}_H^G e^{ix \circ \log} \]

\(^1\) Here and elsewhere we reserve the lower case ‘ind’ for discrete induction, as opposed to the usual ‘Ind’ when \( G \) already has another locally compact topology.
found in (5.1) make sense whenever \( H \) is subordinate to \( x \), and are closely analogous to the representations \( \Gamma(x, H) = \text{Ind}_H^G e^{ix \cdot \log H} \) fundamental in Kirillov’s theory \( [K62a] \). These enjoy, we recall, the following key properties:

(a) \( \Gamma(x, H) \) is irreducible if and only if \( H \) is a polarization at \( x \).
(b) \( \Gamma(x, H) \) and \( \Gamma(x, K) \) are equivalent if \( H \) and \( K \) are any two polarizations at \( x \).

In sharp contrast to this, we shall prove:

(5.6) Theorem.

(a) \( \Gamma(x, H) \) is irreducible if and only if \( H \) is maximal subordinate to \( x \).
(b) \( \Gamma(x, H) \) and \( \Gamma(x, K) \) are inequivalent whenever \( H \) and \( K \) are any two different polarizations at \( x \).
(c) \( \Gamma(x, H) \) is quantum for \( X \) if and only if \( H \) contains \( C_x \) (3.6).

Proof. (a): Suppose that \( H \) is subordinate to \( x \) but not maximally so, i.e., \( H \) is strictly contained in another subordinate subgroup \( K \). Since \( K \) is nilpotent, the normalizer \( N \) of \( H \) in \( K \) contains \( H \) strictly \( [B72b, \text{Prop. III.9.16}] \). Now, given \( s \in N \setminus H \), one verifies readily that \( (J f)(g) = f(gs) \) defines a unitary intertwining operator \( J : \Gamma(x, H) \to \Gamma(x, H) \) which is not scalar since \( (m_e, Jm_e) = 0 \) (A.9, A.16). So \( \Gamma(x, H) \) is reducible.

Conversely, suppose \( \Gamma(x, H) \) is reducible. Then some double coset \( D = HgK \), other than \( H \), must satisfy the Mackey-Shoda conditions (A.19) with \( \chi = \eta = e^{ix \cdot \log H} \). But then \( g \) must normalize \( H \); indeed, if some \( h \in H \) were outside \( gHg^{-1} \), so would be \( h^n \) for all \( n \neq 0 \); so we would have \( h^n gH \neq h^n gH \) whenever \( p \neq q \), and so \( D/H \) would be infinite, contradicting (A.19b). Thus \( g \) normalizes \( H \) and stabilizes \( e^{ix \cdot \log H} \) (A.19), and we conclude just as in the proof of (5.1) (case \( i = 0 \)) that \( H \) is not maximal subordinate to \( x \).

(b): Let \( H \) and \( K \) be polarizations at \( x \), and suppose there is a double coset \( D = HgK \) satisfying the conditions of (A.19) with \( \chi = e^{ix \cdot \log H}, \eta = e^{ix \cdot \log K} \). As above, it follows that \( H = gKg^{-1} \) and \( \chi(h) = \eta(g^{-1}hg) \) for all \( h \in H \). Thus we have \( e^{ix \cdot g(x)h} = 1 \), or in other words, \( g(x) \in x + h^\perp = H(x) \) \([B72a, \text{pp. 69–70}] \). Since \( H \) contains the stabilizer \( G_x \), these forces \( g \in H \) and hence \( K = H = D \). Thus, (A.19) says that \( \text{Hom}(\Gamma(x, H), \Gamma(x, K)) \) has dimension 1 if \( H = K \), and 0 otherwise.

(c): We know from (A.17) that \( \Gamma(x, H) = \text{GNS}_m \) where \( m \) is the state (5.2). By (3.3), this module is quantum for \( X \) if and only if \( m \) is. By (3.6), that is true if and only if (5.2) coincides with \( e^{ix \cdot \log} \) on \( C_x \), which is to say that \( C_x \) lies in \( H \). \( \square \)

(5.7) Example (Heisenberg’s orbit). The results (5.1, 5.6) are already instructive in the simplest case of the group (2.10) with Lie algebra

(5.8)\[
\mathfrak{g} = \left\{ Z = \begin{pmatrix} 0 & \beta & \alpha \\ 0 & \gamma & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}.
\]

We consider the coadjoint orbit \( X \) of the linear form \( Z \mapsto -\alpha \). It is isomorphic to \( (\mathbb{R}^3, dp \wedge dq) \) under the map \( \Phi \) given by \( \langle \Phi(p, q), Z \rangle = \beta \gamma - \alpha \). By (3.6), a state \( m \) of \( G \) is quantum for \( X \) if and only if it restricts to the character \( e^{-\alpha} \) of the center.
(5.11d). Its statistical interpretation then gives (among others) the variables \(p\) and \(q\) probability distributions \(\mu\) and \(\nu\) defined by

\[
(5.9) \int_\mathbb{R} e^{i\mu(p,\gamma)} \, d\mu(p) = m \circ \exp(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \int_\mathbb{R} e^{-i\nu(q,\beta)} \, d\nu(q) = m \circ \exp(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Here we write elements of the Bohr compactification \(b\mathbb{R}\) as (possibly discontinuous) homomorphisms \((p, \cdot)\) and \((q, \cdot) : \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}\), and \(b, c\) are the one-dimensional subalgebras of matrices of the indicated form. Choosing \(x = \Phi(k, \ell)\) say, we have

\[
(5.10) e^{ix \cdot \log(g)} = e^{-iu \gamma + ic/2} e^{ik(c - f(b)}
\]

and the maximal subordinate subgroups to \(x\) are the polarizations \(H_t (t \in \mathbb{R} \cup \infty)\) listed in Table 1.

<table>
<thead>
<tr>
<th>Representation</th>
<th>acts on (\ell^2) functions</th>
<th>by</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (i(x, H_\infty) = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix})</td>
<td>(\phi(p) = f \left( \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} \right))</td>
<td>((g\phi)(p) = e^{-iu \gamma} \phi(p - b))</td>
</tr>
<tr>
<td>(b) (i(x, H_0) = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix})</td>
<td>(\psi(q) = f \left( \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} \right))</td>
<td>((g\psi)(q) = e^{-iu \gamma} \phi(q - c))</td>
</tr>
<tr>
<td>(c) (i(x, H_t) = \begin{pmatrix} 1 &amp; t \quad -tb \ 1 - t^2 &amp; 1 \end{pmatrix})</td>
<td>(\psi(r) = f \left( \begin{pmatrix} 1 &amp; t \ 1 - t^2 &amp; 1 \end{pmatrix} \right))</td>
<td>((g\psi)(r) = e^{-iu \gamma} e^{ib(t - c)} \phi(r - c - bt))</td>
</tr>
<tr>
<td>(d) (i(x, C_X) = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix})</td>
<td>(\varphi(p, q) = f \left( \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} \right))</td>
<td>((g\varphi)(p, q) = e^{-iu \gamma} e^{-ib(q - c)} \phi(p - b, q - c))</td>
</tr>
</tbody>
</table>

**Table 1** The representations \(i(x, H)\) attached to the subordinate subgroups \(H_t (t \in \mathbb{R} \cup \infty)\) and \(C_X\). While each acts nominally in sections of \(G \times_H C \rightarrow G/H\), i.e. on equivariant functions \(f : G \rightarrow C\) (A.6, A.17a), the middle column trivializes this bundle to realize the representation in \(\ell^2(G/H)\).

(a): A state localized at \(\{x_{b,\infty}\} \subset b^*_\infty\) is one in which \(p\) is certainly \(k\). Theorem (5.1) asserts that there is a unique such state, which is discontinuous: \(m(g) = e^{-iu \delta^b_\infty e^{ikc}}\) (Kronecker’s delta). Its statistical interpretation (5.9) reads

\[
(5.12) \int_\mathbb{R} e^{i\mu(p,\gamma)} \, d\mu(p) = e^{ik\gamma}, \quad \int_\mathbb{R} e^{-i\nu(q,\beta)} \, d\nu(q) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } \beta = 0, \quad \text{otherwise},
\]

i.e. while \(\mu\) is Dirac measure at \(k\) (as desired), \(\nu\) is Haar measure on \(b\mathbb{R}\) (A.23). So Theorem (5.1) entails a version of Heisenberg’s principle: \(p\) may be certain, but then \(q\) is necessarily equidistributed on the whole line.

The GNS representation \(i(x, H_\infty)\) obtained from (5.1) was apparently first considered (as representation of a certain \(C^*\)-algebra) in the papers [B74; E81]. It acts in \(\ell^2(\mathbb{R})\) by the very same action (5.11a) by which the Schrödinger representation \(I(x, H_\infty)\) acts in \(L^2(\mathbb{R})\). We know from (5.6a) that it is irreducible, and from (5.1) that its cyclic vector \(\phi(p) = \delta^b_\infty\) (obtained by taking \(f = \mathcal{m}\) in (5.11a); cf. (A.7)) is an
eigenvector of the “translation” subgroup exp(Γ)—benefiting the fact that the resulting measure ν is translation-invariant.

(b): A state localized at \{x_{0n}\} ⊂ ℏ \mathbb{R} is one in which q is certainly ℓ. Again (5.1) provides the unique such state: \( m(g) = e^{-i(\ell b)\delta_0} \), with statistical interpretation

\[
(5.13) \quad \int_{\mathbb{R}} e^{ip\gamma} \, d\mu(p) = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \int_{\mathbb{R}} e^{-i(q, b)} \, d\nu(q) = e^{-i(q, b)},
\]

i.e. \( \mu \) is Haar measure on \( b\mathbb{R} \) while \( \nu \) is Dirac measure at \( \ell \). The resulting representation (5.11b) is sometimes called the polymer representation, after [A03, §III.B]. Although related to (5.11a) by an automorphism of \( G \), it is inequivalent to it (5.6b).

(c): More generally, a state localized at \( \{x_{0n}\} \subset \mathbb{R} \) is one in which \( q + pt \) is certainly \( \ell + kt \). To further illustrate why the resulting modules (5.11c) are inequivalent for different values of \( t \), we map their space \( L^2(\mathbb{R}) \) to \( L^2(b\mathbb{R}) \) by the Fourier transform \( \tilde{\psi}(p) = \sum e^{ip\gamma} \psi(r) \) and compute the transported actions, obtaining

\[
(5.14) \quad (g \tilde{\psi})(p) = e^{-ib\ell \delta_0} e^{i(p, b) - i(b, \ell)} \tilde{\psi}(p - b).
\]

In \( L^2(\mathbb{R}) \), these actions are all unitarily equivalent to each other (and to (5.11a)), because the factor \( e^{i(b, \ell)} \) is the coboundary, \( u(p - b)/u(p) \), of \( u(p) = e^{-ip/2} \).

But in \( L^2(b\mathbb{R}) \) that is no longer the case, because \( u \) is not almost periodic.

(d): Finally (and unrelated to localization), (5.6c) lets us induce from the center \( C_X \) itself, using \( m(g) = e^{-i\ell \delta_0} \delta_0 \). The resulting module (5.11d) is simply an \( L^2 \) version of the prequantization representation (2.11). Like the latter, it is reducible (5.6a) (and in fact finite type II [K62b, Thm 11]); as such it would have been rejected by Van Hove, but Definition (3.1) welcomes it.

**(5.15) Remark.** Another extant argument to discard (5.11d) (or (2.11)) is that it “would violate the uncertainty principle since square integrable sections of \( L \) can have arbitrarily small support” [S80, p. 7]. This however is based on a misinterpretation of \( \varphi(p, q) \), whose square modulus should not be regarded as a probability density in phase space. For example if \( \varphi \) is the characteristic function of the origin, then the state \( (\varphi, g\varphi) \) is our \( m(g) = e^{-i\ell \delta_0} \delta_0 \), whose statistical interpretation (5.9) reads

\[
(5.16) \quad \int_{\mathbb{R}} e^{ip\gamma} \, d\mu(p) = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \int_{\mathbb{R}} e^{-i(q, b)} \, d\nu(q) = \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{otherwise} \end{cases}.
\]

So both \( \mu \) and \( \nu \) are Haar measure on \( b\mathbb{R} \), and far from being concentrated at 0, \( p \) and \( q \) are both equidistributed on the whole line.\(^2\)

**(5.17) Example (Bargmann’s orbit).** The effects of Bohr closure in the previous example were still rather mild, insofar as \( X \) was equal to its affine hull (cf. (3.5b)).

\(^2\) One can also reason purely in the \( L^2 \) version: although the function \( \varphi_\epsilon(p, q) = \psi(2\pi \epsilon + x) e^{-i(\ell p + q)/2\epsilon} \) “shrinks to the origin” as \( \epsilon \to 0 \), one computes without trouble that the resulting state \( (\varphi_\epsilon, g\varphi_\epsilon) \) (which incidentally, tends pointwise to \( m \)) assigns to \( p \) and \( q \) probability distributions whose product of variances, \( \Delta p \Delta q = \frac{1}{2} \sqrt{1 + \frac{\ell^2}{2\epsilon}} \), tends not to zero but to infinity.
So we move on to the next simplest example, where $G$ (resp. $g$) consists of all real matrices of the form

$$
g = \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & a \\ 1 & b & c & \frac{1}{2}c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{resp.} \quad Z = \begin{pmatrix} 0 & \beta & 0 & \alpha \\ 0 & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Forgetting the first row and column yields the Galilei group of space-time transformations

$$
g \begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} 1 & b & c & \frac{1}{2}c^2 \\ 1 & b & t & c \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} r + bt + c \\ t + e \end{pmatrix}
$$

of which $G$ is a central extension. We denote elements of $g^*$ as 4-tuples $(M, p, q, E)$, paired to $g$ by $(x, Z) = \begin{pmatrix} p \gamma \\ \rho \end{pmatrix} - E\rho - Ma$, and we consider the orbit of $(1, 0, 0, 0)$. It is isomorphic to $(\mathbb{R}^2, dp \wedge dq)$ under the map $\Phi$:

$$
\Phi(p, q) = (1, p, q, \frac{1}{2}p^2).
$$

Theorem (3.6b) says that a state $m$ of $G$ is quantum for $X$ if and only if it restricts to the character $e^{-ia}$ of the center $C_X = \{g : b = c = e = 0\}$. Its statistical interpretation then assigns to the variables $(p, E)$ and $r := q + pt (t \in \mathbb{R}$ a fixed parameter) probability distributions $\mu$ and $\nu$, defined by

$$
\int_{b\mathbb{R}^2} e^{i\langle p, \gamma \rangle - \langle E, x \rangle} d\mu(p, E) = m \circ \exp_{b\mathbb{R}^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\int_{b\mathbb{R}} e^{-i\langle r, \beta \rangle} d\nu(r) = m \circ \exp_{b\mathbb{R}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

where $c$ and $b_1$ are the abelian subalgebras of matrices of the indicated form. Adding the center to $c$ and $b_1$, and exponentiating produces (abelian) subgroups $H_\infty$ and $H_1$ which turn out to be exactly all maximal subordinate subgroups to any $x = \Phi(k, \ell)$. Of these only $H_\infty$ is a polarization; the others are all conjugate under the stabilizer of $x$ in $G$, so it will suffice to specialize our results to $H_\infty$ and $H_0$ (Fig. 2).

(a): A state localized at $\{x_{b_1}\} \subset b_1\mathbb{R}$ is one in which $(p, E)$ is certainly $(k, k^2)$. Theorem (5.1) says that the unique such state is $m(g) = e^{-ia}\delta_0 e^{ik(r - \frac{1}{2}p^2 \epsilon)}$. Computing as in (5.11a), we find that the resulting representation $i(x, H_\infty)$ acts in $l^2(\mathbb{R})$ by

$$
(g\phi)(p) = e^{-ia} e^{ik(r - \frac{1}{2}p^2 \epsilon)} \phi(p - b),
$$

with statistical interpretation as follows: in the state $(\phi, g\phi)$, the variable $p$ is distributed according to $|\phi(p)|^2$ times counting measure on $\mathbb{R}$, the pair $(p, E)$ according to the image of that measure under $p \mapsto (p, \frac{1}{2}p^2)$, and the variable $r (5.22)$ according to $\nu = |\phi(r)|^2$ times Haar measure on $b\mathbb{R}$, where $\phi(r) = \sum_{p} e^{-i\langle r, \beta \rangle - \frac{1}{2}p^2 \epsilon} \phi(p)$. We
Localized Quantum States

\[ g \ast h \ast \infty = \{(0 0 0 \alpha 0 0 \gamma 0 \epsilon 0)\} \ast (p, E) \]

\[ h \ast 0 = \{(0 0 0 \beta 0 0 \alpha 0 \beta 0 \gamma 0 \epsilon 0)\} \ast q \]

Note that the action of $G$ transported to these latter functions writes

\[ (5.24) \quad (g \psi)(r) = e^{-i\alpha}e^{-i(b(t-e) - \frac{1}{2}b^2(t-e))} \psi(r - c - b(t-e)), \]

and that their restrictions to $r \in \mathbb{R} \subset \mathbb{b} \mathbb{R}$ constitute a non-standard Hilbert space of (almost-periodic) solutions of the Schrödinger equation $i\partial_t \psi = \frac{1}{2} \partial^2_r \psi$ (a “plane wave” [D30, §30]). For comparison, the standard solution space consists of transforms $\sqrt{2\pi}^{-1} \int e^{-i(p \cdot \gamma - \frac{1}{2}p^2t)\phi(p)} dp$ where $\phi \in I(x, H_0) = L^2(\mathbb{R})$ with action (5.23) [B54, §6g]. In either case it takes, of course, the Schrödinger equation to extract an irreducible subspace from the space of all functions of $(r, t)$.

(b): A state localized at $\{x_{\mathbb{b} \mathbb{R}}\} \subset \mathbb{b}_0$ is one in which $q$ is certainly $\ell$. Again (5.1) provides the unique such state: $m(g) = e^{-i\alpha}e^{-i\beta}e^{-i\gamma}e^{-i\epsilon}$. This turns out to be interesting. Indeed, computing as in (5.11) exhibits the resulting GNS module $i(x, H_0)$ as $L^2(\mathbb{R}^2)$ in which $G$ acts by the very formula (5.24). By (5.6a) this is irreducible even though $I(x, H_0)$ is not. The need for Schrödinger’s equation has evaporated!

The statistical interpretation sheds some light on this: inserting $m$ into (5.21), (5.22), we find

\[ (5.25) \quad \int_{\mathbb{R}^2} e^{i(p, r)(E, c)} d\mu(p, E) = \delta_0^\alpha \delta_0^\beta \delta_0^\gamma \delta_0^\epsilon, \quad \int_{\mathbb{R}} e^{-i(r, p)} d\nu_t(r) = \begin{cases} e^{-i\beta} & \text{if } t = 0 \\ \delta_0^\beta & \text{else} \end{cases}, \]

i.e. while $\nu_0$ is Dirac measure at $\ell$, both $\mu$ and $\nu_t (t \neq 0)$ are Haar measure. Thus we see that Theorem (5.1) gives Heisenberg’s principle the form: position $q$ at any instant may be certain, but then momentum-energy $(p, E)$ is necessarily equidistributed in the whole plane, irrespective of the relation $E = \frac{1}{2}p^2$ in (5.20); and position $q + pt$ at any other instant is also equidistributed.

Fig. 2 Projection of Bargmann’s orbit (5.20) to the duals of abelian subalgebras $\mathbb{b}_0$ and $\mathbb{h}_0$. 
This blurring of the relation \( E = \frac{1}{2} p^2 \) “explains”, at the symbol level, the disappearance of Schrödinger’s equation. It is only under consideration here because, first, we do allow spectral measures concentrated on \( bX_a \) and not just \( X_a \) (3.4), and secondly, the paraboloid (5.20) is Bohr dense in its affine hull (3.5b). This may legitimate, in our opinion, the use of Bohr closure implicit in Definition (3.1).

6 Compact groups

In this section \( G \) is a compact connected Lie group. We fix a maximal torus \( T \subset G \), and we write \( W \) for the resulting Weyl group, \( W = \text{Normalizer}(T)/T \). It is finite and acts on \( t \) and \( t^* \) by conjugation. We also fix a \( W \)-invariant inner product on \( t \) and use it to identify \( t \) and \( t^* \). We have a canonical inclusion \( t^* \hookrightarrow g^* \) as follows: being maximal abelian, \( t \) coincides with the space of all \( T \)-fixed points in \( g \); whence a canonical projection, \( \int_T \text{Ad}(t) \, dt : g \to t \), whose transpose identifies \( t^* \) with the \( T \)-fixed points in \( g^* \). We let:

\[(6.1) \quad R \text{ consist of the nonzero weights of } g_C \text{ (adjoint action), a.k.a. roots;}\]
\[(6.2) \quad \overline{T} \text{ be the closure of a chosen connected component of } t \times \bigcup_{\alpha \in R} \ker(\alpha);\]
\[(6.3) \quad \mu \text{ be defined on } \overline{T} \text{ by: } \lambda \leq \mu \iff \lambda \text{ is in the convex hull of } W(\mu) \text{ [B85, p. 250].}\]

One knows:

\[(6.4) \quad \overline{T} \text{ is a fundamental domain for the } W \text{-action on } t = t^* \text{ [B85, p. 202];}\]
\[(6.5) \quad \text{each coadjoint orbit intersects } t^* \text{ in a } W \text{-orbit, hence } \overline{T} \text{ in a point [B79, p. 74];}\]
\[(6.6) \quad \text{each irreducible continuous representation of } G \text{ has a } \leq \text{-highest weight in } \overline{T} \text{ which characterizes it [B85, p. 252].}\]

(6.7) **Theorem.** Every quantum representation of \( G \) is continuous. The irreducible representation with highest weight \( \lambda \in \overline{T} \) is quantum for the coadjoint orbit through \( \mu \in \overline{T} \) if and only if \( \lambda \leq \mu \) (6.3).

**Proof.** A unitary representation is continuous if and only if the state \( m(g) = (\varphi, g\varphi) \) is continuous for each unit vector in it [H63, 22.20a]. So it is enough to show that every quantum state (for \( X \) say) is continuous. Now since \( X \) is compact we have \( bX = X \), so for each abelian \( \alpha \subset g \) (3.4) says that \( m \circ \exp_{\alpha} \) has its spectral measure concentrated on \( X_{\alpha} \) (in \( \hat{\alpha} \)). By [B59, Korollar p. 421] it is equivalent to say that it is the image under \( \alpha^* \hookrightarrow \hat{\alpha} \) of a measure \( \nu \) concentrated on \( X_{\hat{\alpha}} \) (in \( \alpha^* \)). So we have
\[
\langle m \circ \exp_{\alpha}(Z) \rangle = \int_{\hat{\alpha}} e^{\langle a_{\alpha}, Z \rangle} \, dv(u),
\]
which shows that \( m \circ \exp_{\alpha} \) is continuous (A.20). Continuity of \( m \) at \( g \in G \) now follows by writing \( g \) as a direct sum of lines \( a_1, \ldots, a_n \) and using the chart \((Z_1, \ldots, Z_n) \leftrightarrow g \exp(Z_1) \cdots \exp(Z_n) \), together with the inequality
\[
(6.8) \quad |m(g_{g_1} \cdots g_n) - m(g)| \leq \sqrt{2} \text{Re}(1 - m(g_1)) + \cdots + \sqrt{2} \text{Re}(1 - m(g_n))
\]
which is obtained from (A.12) by induction on \( n \).

Suppose \( \lambda \neq \mu \). Let \( V \) be the module with highest weight \( \lambda \), and \( X \) the orbit of \( \mu \). If \( \varphi \in V \) is a highest weight vector and \( m(g) = (\varphi, g\varphi) \), then
\[
(m \circ \exp_{\alpha})(Z) = e^{i\langle \alpha, Z \rangle} \]

has its spectral measure concentrated at $\lambda \notin \text{Conv}(W(\mu))$. But this convex hull is precisely $X_\text{ct}$ by Kostant’s theorem (see e.g. [Z92]), so Theorem (3.4) says that $m$ and hence $V$ are not quantum for $X$.

Conversely, suppose $\lambda \leq \mu$. Pick a unit vector $\varphi \in V$, write $m(g) = (\varphi, g\varphi)$ and let $E_\nu$ be the eigenprojector onto the subspace of weight $\nu$ vectors in $V$. Then $Z \in t$ acts on $V$ by $\sum_{\nu: \text{weight } \nu} i(\nu, Z)E_\nu$, so we have $(m \circ \exp_h)(Z) = \sum_{\nu: \text{weight } \nu} e^{i\langle \nu, Z \rangle} ||E_\nu \varphi||^2$. Thus the spectral measure of $m \circ \exp_h$ is concentrated on the set of weights of $V$. Since these all lie in $\text{Conv}(W(\lambda)) \subset \text{Conv}(W(\mu)) = X_\text{ct}$ by definition of $\leq$, we see that $m$ satisfies the condition of Theorem (3.4) for $a = t$, for every unit $\varphi \in V$.

But every maximal abelian subalgebra of $g$ is a conjugate $a = g^{-1}tg$ of this one (e.g. [B79, pp. 73–74]). In that case, the obvious relation

\begin{equation}
(\varphi, \exp_{ia}(\cdot)\varphi) = (g\varphi, \exp_{\mu a}(\cdot)g\varphi) \circ \text{Ad}(g)_{ia}
\end{equation}

shows that the spectral measure of $(\varphi, \exp_{ia}(\cdot)\varphi)$ is, dually, the image of the spectral measure of $(g\varphi, \exp_{\mu a}(\cdot)g\varphi)$ by the map $j : t^* \rightarrow a^*$ transpose to $\text{Ad}(g)_{ia} : a \rightarrow t$.

Since the latter measure is concentrated on $X_\text{ct}$ for every $g\varphi$ (by the previous case), it follows that the former is concentrated on $j(X_\text{ct}) = X_\text{ct}$ for every $\varphi$, and we conclude by Theorem (3.4) that $V$ is quantum for $X$.

Theorem (6.7) shows that even in the compact case Definition (3.1) fails to recover the whole substance of the orbit method, which is (usually) understood to impose $\lambda = \mu$, i.e. attach each representation to the orbit through its highest weight. While [Z96] discusses various reasonable conditions one can add to regain this condition (e.g. it suffices to restrict attention to modules weakly contained in sections of the Kostant-Souriau line bundle over the orbit [Z96, Thm 5.23]), we concentrate here on studying the representations obtained from states localized at an orbit $Y$ of a subgroup.

Although we mentioned after (4.1) that the preimage of $Y$ in $X$ is generically coisotropic, the useful case to consider below lies at the opposite end, where this preimage is a single point—as happens when we take $Y$ to be an extreme point (such as $X \cap \mathcal{C}$) of the convex polytope $X_\text{ct}$:

\begin{equation}
(6.10) \text{Theorem.} \text{ Let } X \text{ be the coadjoint orbit through } \lambda \in \mathcal{C}. \text{ If } \lambda \text{ is integral, then there is a unique quantum state for } X \text{ localized at } \{\lambda\} \subset t^*, \text{ namely } m(g) = (\varphi, g\varphi) \text{ where } \varphi \text{ is a highest weight vector in the irreducible } G\text{-module with highest weight } \lambda. \text{ Otherwise there is no such state.}
\end{equation}

\begin{proof}
Let $m$ be such a state, and write $\text{GNS}_m = \bigoplus_j V_{\lambda_j}$ for the (orthogonal) decomposition of the resulting GNS module (A.3) into irreducibles with highest weights $\lambda_j$. Since $\text{GNS}_m$ is quantum for $X$ (3.3), all $\lambda_j \leq \lambda$ (6.7). Moreover we know that its cyclic vector $m_c$ (A.7) is a weight vector of weight $\lambda$ (4.4). So $\lambda$ must be integral, and $m_c$ is orthogonal to all summands with highest weights $\lambda_j < \lambda$, which must therefore vanish since $m_c$ is cyclic. Also by the orthogonality of vectors with different weight, $m_c$ is orthogonal to all except the maximal weight space in each remaining summand. So its decomposition writes $m_c = \sum_j c_j \varphi_j$ where $\varphi_j$ is a unit highest weight vector in $V_{\lambda_j} \equiv V_{\lambda_j}$. Now the equivalence and orthogonality of
the summands implies \((\varphi_j, g\varphi_k) = \delta_{jk}(\varphi, g\varphi)\) where \(\varphi\) is as in the statement of the Theorem. So we have

\begin{equation}
m(g) = (m_e, gm_e) = \sum_{jk} \bar{c}_j c_k(\varphi_j, g\varphi_k) = (\varphi, g\varphi),
\end{equation}
as claimed. (Of course it follows \textit{a posteriori} that there was only one summand.) \(\square\)

\section{Euclid’s group and localization on normal congruences}

We consider here the manifold \(X\) of oriented straight lines in Euclidean space \(\mathbb{R}^3\), i.e. pairs \(x = (\ell, \mathbb{R})\) of a line \(\ell = r + \mathbb{R} u\) and the choice \(u\) of one of the two unit vectors parallel to it. We can regard it either as the quotient of \(\mathbb{R}^3 \times S^2\) by the equivalence \((r, u) \sim (r', u')\) if \(u = u'\) and \(r - r' \parallel u\), or as the subspace \(TS^2 = \{(\ell, r) : r \perp u\}\) which is a section of that quotient (Fig. 3). Either way, \(X\) is naturally acted upon by Euclid’s group \(G\) (resp. its Lie algebra \(g\)) consisting of all matrices of the form

\begin{align}
g &= \begin{pmatrix} A & e \\ 0 & 1 \end{pmatrix}, \quad \text{resp.} \quad Z = \begin{pmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{pmatrix},
\end{align}

where \(A \in SO(3)\), \(e, \alpha, \gamma \in \mathbb{R}^3\) and \(j(\alpha) = \alpha \times \cdot\) (“vector product by \(\alpha\”). Moreover one can show that the most general \(G\)-invariant symplectic structure on \(X\) writes

\begin{equation}
\omega(\delta x, \delta' x) = k \left[ \langle \delta u, \delta' r \rangle - \langle \delta' u, \delta r \rangle \right] + s(\delta u, \delta' u \times \delta u)
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Identification of the manifold \(X\) of oriented lines (or light rays) with the tangent bundle \(TS^2\), after Hudson [H02]. Euclid’s group acts on oriented lines via its natural action on \(\mathbb{R}^3\).}
\end{figure}
for some \( k > 0 \) and \( s \in \mathbb{R} \). (The term in \( k \) was discovered by Lagrange [L05] and the term in \( s \) by Cartan [C96].) Identifying \( g^* \) with \( \mathbb{R}^6 \) where \( w = (\frac{1}{p}) \) is paired to \( Z \in g \) by \( \langle w, Z \rangle = \langle L, \alpha \rangle + \langle P, \gamma \rangle \), the resulting equivariant moment map \( \Phi : X \to g^* \).

\[
(7.3) \quad \Phi(x) = \left( r \times ku + su \right)_{ku},
\]

identifies \((X, \omega)\) with the coadjoint orbit \( X^{k,s} \) of \( (\alpha, e) \) endowed with its Kirillov-Kostant-Souriau 2-form. When so endowed, we think of \( X \) as the manifold of light rays with color \( k \) and helicity \( s \), and as the arena of geometrical optics [S70, 15.88]. In what follows we exhibit three kinds of lagrangian submanifolds (known classically as normal congruences) on which light accepts to be concentrated:

(7.4) Example (Localization on a parallel beam). Let \( H \) be the subgroup of \( G \) in which the rotation \( A \) has axis \( \mathbb{R}e_3 \), i.e. \( H = \{(\alpha, e) : A = e^{i(\alpha e_3)} \text{ for some } \alpha \in \mathbb{R}\} \). Then \( \{(\alpha, e) \}_{\alpha} \) is a point-orbit of \( H \) in \( b^* \), whose preimage in \( X \) is the fiber \( T_0 S^2 \subset TS^2 \), i.e. the lagrangian congruence of all lines normal to the plane \( e_3^* \).

(7.5) Theorem. If \( s \) is an integer, there is a unique quantum state for \( X^{k,s} \) localized at \( \{(\alpha, e) \}_{\alpha} \subset b^* \), viz.

\[
(7.6) \quad m\begin{pmatrix} A & e \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\tau}e^{i(\alpha, e)} & \text{if } A = e^{i(\alpha e_3)}, \\ 0 & \text{otherwise}. \end{cases}
\]

The resulting GNS module (A.3) is \( \text{ind}_H^G \chi^{k,s} \) where \( \chi^{k,s} = m_H \) and induction is in the sense of discrete groups; it is irreducible. If \( s \) is not an integer, then there is no such state.

Proof. The fact that a localized state must coincide with (7.6) in \( H \), and in particular that \( s \) must be an integer, is just (4.2). To see that it must vanish outside \( H \), pick \( g = (\frac{1}{p}, 0) \in G \setminus H \) (thus \( Ae_3 \neq e_3 \)) and then \( h = (1, 0) \in H \) such that \( e^{i(Ae_3-e_3, k)} \neq 1 \). Computing as in (5.3), we get

\[
(7.7) \quad e^{i(e_3, k)}m(g) = m(hg) = m(gg^{-1}hg) = m(g)e^{i(Ae_3, k)}
\]

which shows that \( m(g) = 0 \). The identification of GNS\(_m\) as an induced representation is a special case of (A.17), and its irreducibility is a simple application of (A.19). In fact, taking \( \chi = \eta = m_H \) there, the assignment \( gH \mapsto Ae_3 \) identifies \( G/H \) with the
sphere $S^2$, on which the residual left action of $H$ is by rotations about $Re_3$. So the only finite orbits (or double coset projections) are the poles $\pm e_3$, and consequently the double cosets satisfying (A.19b) are all contained in $H^+ = \{(A_1 0) : Ae_3 = \pm e_3\}$ which is the normalizer of $H$. But if $g \in H^+$ projects to the south pole (so $Ae_3 = -e_3$) then we have just seen that $\chi(g^{-1}hg)$ could differ from $\chi(h)$. So the double cosets that also satisfy (A.19a) are all contained in $\{A_1 0\} : Ae_3 = e_3\}$, which is just $H$. Hence the number of double cosets in (A.19) is just one, which shows that $\text{ind}H^+ \chi^{k,s}$ is irreducible.

There remains to show that the state (7.6) is indeed quantum for $\chi^{k,s}$. To this end we observe that $g$ has exactly two conjugacy classes of maximal abelian subalgebras. The first one consists of the translation ideal $t = \{(\begin{smallmatrix} \theta & 0 \\ 0 & 0 \end{smallmatrix}) : \gamma \in R^3\}$ alone. Identifying its dual with $R^3$ in the obvious way, it is clear on (7.3) that $X^{k,s}$ is the sphere of radius $k$, and on (7.6) that $m \circ \exp_{\tilde{g}_t}$ has its spectral measure concentrated at its north pole $ke_3$. So the condition of Theorem (3.4) is satisfied. The other conjugacy class consists of the infinitesimal stabilizers

\begin{equation}
\tilde{g}_t = \{Z = \alpha \begin{pmatrix} j(u) & r \times u \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} : \alpha, \gamma \in R\}
\end{equation}

of all oriented lines $x = (r^Ru_u, 0) \in X$. Identifying elements of $\tilde{g}_t^c$ with pairs $(\ell, p)$ so that $(\ell, p), Z = \ell \alpha + py$ (so $\ell$ and $p$ are respectively the angular momentum around and the linear momentum along the oriented line $x$), one deduces readily from (7.3) that the projection $X^{k,s}_{\tilde{g}_t}$ is the strip $\{(\rho, \ell) : \ell \in R, -k < p < k\}$ with the two points $\pm(\frac{k}{2})$ added. On the other hand (7.6) gives

\begin{equation}
(m \circ \exp_{\tilde{g}_t})(Z) = m\begin{pmatrix} e^{i(ku)} & (1 - e^{i(ku)})r + \gamma u \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i(3\pi/k + k\gamma)} & \text{if } u = \pm e_3 \\ 1_{2\pi}(\alpha)e^{i\gamma(ku, u)} & \text{otherwise,} \end{cases}
\end{equation}

where $1_{2\pi}$ is the characteristic function of $2\pi Z$. In the first case we see that the spectral measure of $m \circ \exp_{\tilde{g}_t}$ is Dirac measure at $\pm(\frac{k}{2})$. In the second we see that it is Haar measure on $bZ \subset bR$ (A.23) times Dirac measure at $(ke_3, u)$; so again the condition of Theorem (3.4) is satisfied.

\textbf{(7.9) Remarks.} (a) Although instructive, it is not actually necessary to check the condition of Theorem (3.4) separately for $a = \tilde{g}_t$ as we have just done. Indeed, concentration of the spectral measure of $m \circ \exp_{\tilde{g}_t}$ on the sphere $X^{k,s}_{\tilde{g}_t}$ suffices to ensure concentration of the spectral measure of $m \circ \exp_{bZ_{\tilde{g}_t}}$ on the segment $[-k, k]$ which is its image under the projection $\tilde{g}_t \to \tilde{g}_t \cap t$; and by [B67, no V.6.2, Cor. 2] this implies concentration of the spectral measure of $m \circ \exp_{\tilde{g}_t}$ on the strip $bX^{k,s}_{\tilde{g}_t} = bR \times [-k, k]$ which is the preimage of $[-k, k]$ under the projection $\tilde{g}_t \to \tilde{g}_t \cap t$.

(b) The module $\text{GNS}_m = \text{ind}^{\omega}_{\hat{H}} \chi^{k,s}$ and its cyclic vector have various realizations familiar in physics. It consists of $\ell^2$ sections of the $s$th tensor power of the tangent (complex line) bundle $TS^2 \to S^2$, or in other words, functions $f : SO(3) \to C$ satisfying $f(e^{i(ku)}U) = e^{-i\pi}f(U)$ and $\|f\|^2 = \sum_{x \in S^2} |f(U)|^2 < \infty$, where $U = (u_1u_2u_3)$; the group $G$ acts on them by
For further discussion of the 2-sphere $X$ is quantum for the Bohr mean of $\psi$, cyclic vector the “circularly polarized plane wave” $\psi(r) = e^{-ikz} (\zeta = (e_3, r))$, and natural “scalar field” $G$-action:

\begin{equation}
(g\psi)(r) = \psi(A^{-1}(r - c)).
\end{equation}

**Case $s = 1$.** Here $f$ has the form $f(U) = 0$. Putting $\psi(r) = \sum_{n \in \mathbb{Z}} e^{-i(n, kr)} f(u_3)$ one gets a Hilbert space of almost-periodic solutions of the Helmholtz equation $\Delta \psi + \kappa^2 \psi = 0$, with norm $||\psi||^2$ the Bohr mean of $|\psi|^2$, cyclic vector the “plane wave” $\psi(r) = e^{-ikz} (\zeta = (e_3, r))$, and natural “vector field” $G$-action:

\begin{equation}
(g\psi)(r) = \psi(A^{-1}(r - c)).
\end{equation}

**Example (Localization on a convergent beam).** Assume $s = 0$ and let $K$ be the rotation subgroup of $G$, i.e. $K = \{ (A_{01}) : A \in \text{SO}(3) \}$. Then $|0\rangle$ is a point-orbit of $K$ in $t^*$, whose preimage in $X$ is the zero section $S^2 \subset TS^2$, i.e. the lagrangian congruence of all lines normal to a sphere centered at the origin.

**Theorem.** There is a unique quantum state for $X^{k,0}$ localized at $|0\rangle \subset t^*$, viz.

\begin{equation}
m\begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \frac{\sin ||ke||}{||ke||}.
\end{equation}

The resulting GNS module (A.3) is irreducible and is $\text{Ind}_H^G \chi^{k,0}$, where $H$ and $\chi^{k,0}$ are as in (7.5).

**Proof.** Localization at $|0\rangle \subset t^*$ implies by (4.2) that $m_K = 1$. So Weil’s formula (A.13) gives $m((\gamma \xi)(A_{01})) = m((A_{01})(\gamma \xi)(A_{10}^{-1} 0_1)) = m(1_{01})$, i.e.

\begin{equation}
m\begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = m\begin{pmatrix} AAc \\ 0 & 1 \end{pmatrix} = m\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.
\end{equation}

If further $m$ is quantum for $X^{k,0}$ and $t = (\gamma 0_1 : \gamma \in \mathbb{R}^3)$, then the compactness of the 2-sphere $X^{k,0}$ implies as in the proof of (6.7) that $m(\xi) = \int_{S^2} e^{i(x,ke)} dv(u)$ for a unique probability measure $\nu$ on $S^2$. Now the second equality in (7.17) shows

---

*Helmholtz’s equation $\Delta F + \kappa^2 F = 0$ follows, for on divergence-free vector fields the curl provides a square root (à la Dirac) of $-\Delta = \text{curl curl -grad div}$.***
that $\nu$ has the rotation invariance property $\int_{S^2} f(A^{-1}u)\,dv(u) = \int_{S^2} f(u)\,dv(u)$ for all $f = e^{i\langle \cdot, k\epsilon \rangle}$. Since these span a uniformly dense subspace of the continuous functions on $S^2$ (Stone-Weierstrass) it follows that $\nu$ is the unique invariant probability measure on $S^2$. Therefore we obtain, using spherical coordinates with pole at $c/||c||$,

$$
(7.18) \quad m\left(\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\infty e^{-||k||c} \cos \theta \,d\theta \,d\phi = \frac{1}{2} \int_{-1}^1 e^{-||k||c} |z^2| = \frac{\sin ||k||c}{||k||c} \int_{-1}^1 \end{array}
$$

[20, p.174]. Together with (7.17) this proves (7.16). Now consider the module $Ind_H^G \chi_{k,0} \simeq L^2(S^2)$ with $G$-action $(gf)(v) = e^{i\langle k, \epsilon \rangle} f(A^{-1}v)$. It is irreducible by Mackey theory [B65, Thm 1], and we clearly have $m(g) = (f, gf)$ where $f(v) \equiv 1$. So (A.3) shows that $m$ is a state and $Ind_H^G \chi_{k,0} \simeq \text{GNS}_m$, as claimed. Finally it is clear from (7.18) that $m \circ \exp$ has its spectral measure concentrated on the sphere $S^2$, and from (7.9a) that $m \circ \exp_{\text{GNS}}$ has its own concentrated on the strip $bX_{k,0}^{\text{GNS}}$. So we conclude by Theorem (3.4) that $m$ is quantum for $X_{k,0}$.

(7.19) Remarks. (a) For any integer $s$ one readily proves in the same manner that $Ind_H^G \chi_{k,s}$ is irreducible and quantum for the orbit $X_{k,s}$. But only in the case $s = 0$ do we have a characterization of this representation as arising from a localized state.

(b) Just as the $Ind_H^G \chi_{k,s}$ can be realized in solution spaces of wave equations on $\mathbb{R}^3$ (7.11–7.13), so can the $Ind_H^G \chi_{k,s}$: simply replace $\sum_{m \in S^2}$ there by $\int_{S^2} \ldots \,dv(u)$. (The resulting norms on solution spaces are computed in [S90b, Thm 5.5].) In particular the cyclic vector $f(v) \equiv 1$ of $Ind_H^G \chi_{k,0}$ becomes the “spherical wave” $\psi(r) = \frac{\sin ||ky||c}{||ky||c}.$

(7.20) Example (Localization on a neon beam). Let $G_{\alpha} = \exp\left(\frac{\beta(x,\gamma)}{2} \right): \alpha, \gamma \in \mathbb{R}$ be the stabilizer of the vertical axis $\alpha (\frac{x}{r}) \in X$. Then $\{0\}$ is a point-orbit of $G_{\alpha}$ in $g_{\alpha}^\star$, whose preimage in $X \simeq TS^2$ is the normal bundle to the equator $S^1 \subset S^2$, i.e. the lagrangian congruence of all lines normal to a cylinder with directrix $a$.

(7.21) Theorem. There are (at least) two pure quantum states for $X_{k,0}$ localized at $\{0\} \subset g_{\alpha}^\star$, viz.

$$
(7.22) \quad m_\epsilon\left(\begin{array}{c} A \\ 0 \\ 1 \end{array}\right) = \begin{cases} J_0(||ke||c) & \text{if } Ae_3 = e_3, \\ (-1)^\epsilon J_0(||ke||c) & \text{if } Ae_3 = -e_3, \\ 0 & \text{otherwise,} \end{cases} (\epsilon = 0, 1),
$$

where $J_0$ is the zeroth-order Bessel function and $e_\perp = \text{projection of } e \text{ in the plane } e_\perp \perp e_3$.

We have $\text{GNS}_m = Ind_H^G \chi_{\epsilon}$ where $\chi_{\epsilon}(A)^{\frac{A}{0}} \chi_{\epsilon}(A)^{\frac{A}{0}} = (\pm 1)^\epsilon e^{i\langle \epsilon, e \rangle} \text{ if } Ae_3 = \pm e_3$, and

$$
(7.23) \quad H^+ = \{ (A)^{\frac{A}{0}} : G : Ae_3 = \pm e_3 \}, \quad T^+ = \{ (A)^{\frac{A}{0}} : G : A \in \{1, e^{i\langle \epsilon, e \rangle} \} \}.
$$

Proof. Let $m$ be a quantum state for $X_{k,0}$. As in the proof of (7.15), we have a probability measure $\lambda$ on $S^2$ such that $m(\frac{A}{0}) = \int_{S^2} e^{i\langle \lambda, \epsilon \rangle} d\lambda(u)$. Localization at $\{0\} \subset g_{\alpha}^\star$ further implies that $m$ is trivial on $G_{\alpha}$ and in particular on $\exp_{\text{GNS}}(0, \text{Re}_\alpha)$. Writing $\pi$ for the projection $u \mapsto ku_3$, it follows that the image $\pi(\lambda)$ is Dirac measure at $0$, hence that $\lambda$ is concentrated on the equator $S^1 \subset S^2$ [B67, n^V.6.2, Cor. 4]. Next, the triviality of $m(\frac{A}{0})$, $A \in \text{SO}(2) := \{ e^{i\langle \alpha, e \rangle} : \alpha \in \mathbb{R} \}$, implies that the relations (7.17) hold.
for $A \in \text{SO}(2)$ with the same proof. Therefore $\lambda$ is the $\text{SO}(2)$-invariant measure on $S^1$ and we have, with $H = \{(A, c) \in G : A \in \text{SO}(2)\}$ as before,

$$m_H \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = m_H \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \int_{S^1} \rho_{\text{ind}}(\lambda u) = \int_0(\|kc\|)$$

[\text{W22}, \S2.2]. This shows that the restriction $m_{H \cup}$ must be given by the first row of (7.22).

We do not know whether the next two rows give the only extensions of the first row to pure states of $G$; but we can prove that they do provide such extensions. Indeed, consider the module $V_e = \text{Ind}_{H}^{G} \chi_{-} \approx L^2(S^1)$ with $H^{-1}$-action $(gf)(u) = (\pm 1)^{f} e^{i(kc)} f(A^{-1}u)$ whenever $A e_3 = \pm e_3$. It is irreducible by Mackey theory [\text{B65}, Thm 1] and we clearly have $m_{H^{-1}}(g) = (f, g f)$ where $f(u) \equiv 1$. So (A.3) shows that $m_{H^{-1}}$ is a state and $V_e = \text{GNS}_{m_{H^{-1}}}$. Now [\text{B62}, Cor. 1] proves that

$$\dim(\text{Hom}_{H^{-1}}(\text{ind}_{H}^{G} V_e, \text{ind}_{H}^{G} V_e)) = \sum_{H^{-1}gH^{-1} \in H^{-1}G/H^{-1}} \dim(\text{Hom}_{H^{-1}gH^{-1} \in H^{-1}G/H^{-1}}(V_e, \text{ind}_{H}^{G} V_e))$$

where $\text{ind}_{H}^{G} V_e$ denotes the $gH^{-1}g^{-1}$-module in which $k \in gH^{-1}g^{-1}$ acts as $g^{-1}kg$ acts on $V_e$. Now if $g \in H^{-1}$, then its double coset $H^{-1}gH^{-1} = H^{-1}$ clearly contributes 1 to the sum in (7.25). On the other hand if $g \notin H^{-1}$, then $H^{-1} \cap gH^{-1}g^{-1}$ contains the translation group $T$. But any $I \in \text{Hom}_{H^{-1}gH^{-1} \in H^{-1}G/H^{-1}}$ satisfies by definition $I \in C_{H^{-1}gH^{-1} \in H^{-1}G/H^{-1}} V_e$ or in other words (since the left-hand side here is just $I f$)

$$1 - e^{i(kc\text{A}e_3)}(I f)(u) = 0 \quad \forall c \in \mathbb{R}.$$ 

As $A e_3 \neq \pm e_3$, the first factor is only zero (for all $c$) at two points of the equator, and we conclude that $I = 0$. So the sum in (7.25) is 1 and $\text{ind}_{H}^{G} V_e$ is irreducible; hence $m_{c}$ is pure, as claimed. Finally it is clear from (7.24) that $m_{e} \circ \text{exp}_{H}$ has its spectral measure concentrated on the (equator of) the sphere $\chi_{0}$, and from (7.9a) that $m_{e} \circ \text{exp}_{H}$ has its own concentrated on the strip $\chi_{0}$. So we conclude by Theorem (3.4) that $m_{e}$ is quantum for $\chi_{0}$.

\textbf{(7.27) Remarks.} (a) As emphasized during the proof, we do not know if (7.22) gives the only pure quantum states for $\chi_{0}$ (or $\chi_{0}^\perp$) localized at $\{0\} \subset g_{0}'$. 

(b) Much as in (7.9b) and (7.19b), one can realize the representation $G_{m_{e}}$ in a Hilbert space of solutions of $\Delta \psi + k^2 \psi = 0$, with cyclic vector the “cylindrical wave” $\psi(r) = J_{0}(\|kr\|)$ and norm $||\psi||^2 = \lim_{R \to \infty} R^{-2} \int_{|r| \leq R} \psi(r)^{2} dr$ [\text{S90b}, Thm 5.5]. On the other hand, we have not managed to produce a similar realization of $G_{m_{e}}$.

(c) The modules $\text{ind}_{H}^{G} \chi_{0}$ (7.5) and $\text{ind}_{H}^{G} \chi_{0}$ (7.15) were given by the $G$-action $(gf)(u) = e^{i(kc)} f(A^{-1}u)$ in $L^{2}(\mu_{0})$ and $L^{2}(\mu_{2})$, where $\mu_{d}$ is $d$-dimensional Hausdorff measure on the sphere. It would be interesting to determine if the same action in $L^{2}(\mu_{1})$ is also irreducible, and in particular if $L^{2}(\mu_{1})$ is isomorphic to $G_{m_{e}}$. (7.21).
A Appendix: Positive-definite functions, states, representations

(A.1) Definitions. Let \( G \) be a group, with identity element \( e \). Recall that a complex-valued function \( m \) on \( G \) is called **positive-definite** if the sesquilinear form

\[
(c, d)_m := \sum_{g,h \in G} \overline{c}_g d_h m(g^{-1} h),
\]

defined on \( \mathbb{C}[G] = \{ \text{complex-valued functions with finite support on } G \} \), is positive: \((c, c)_m \geq 0\). If further \( m(e) = 1 \), then \( m \) is called a **state** of \( G \). A state of \( G \) is called **pure** (or **extreme**) if it is not a convex combination of two states other than itself.

We can identify each function \( m \) on \( G \) with the linear functional on \( \mathbb{C}[G] \) defined by \( m(\delta^z) = m(g) \), where \( \delta^z \) denotes the basis function which is one at \( g \) and zero elsewhere; then \((A.2)\) writes \((c, d)_m = m(c^* \cdot d)\), where we are using the *-algebra structure of \( \mathbb{C}[G] \): \( \delta^z \cdot \delta^h = \delta^{zh}, \delta^zh = \delta^{zh} \). So states are the same as normalized positive linear functionals on \( \mathbb{C}[G] \).

(A.3) Theorem (Gelfand-Na˘ımark-Segal, Schwartz [S64]). A function \( m \) on \( G \) is a state if and only if there is a unitary \( G \)-module \( \mathcal{H} \), and a unit vector \( \varphi \in \mathcal{H} \), such that

\[
m(g) = (\varphi, g\varphi).
\]

We may even assume that \( \varphi \) is cyclic, i.e. its \( G \)-orbit spans a dense subspace of \( \mathcal{H} \).

Then the pair \((\mathcal{H}, \varphi)\) is unique and canonically isomorphic to \((\text{GNS}_m, m_E)\), where

(A.5) \( \text{GNS}_m \subset \mathbb{C}[G] \) is the subspace with reproducing kernel \( K(g, h) = m(g^{-1} h) \);

(A.6) \( G \) acts on it by \((g f)(g') = f(g^{-1} g')\);

(A.7) the cyclic vector \( m_E \) is the complex conjugate \( \overline{m} = K(\cdot, e) \) of \( m \).

Finally \( m \) is pure if and only if \( \text{GNS}_m \) is irreducible.

Proof. If \((A.4)\) holds, we get \( m(e) = 1 \) and \( m(c^* \cdot c) = (c\varphi, c\varphi) \geq 0 \); so \( m \) is a state. Conversely if \( m \) is a state, one observes that the form \((A.2)\) on \( \mathbb{C}[G] \) is invariant under the regular action, \( gc = \delta^c \cdot c \); dividing out the null vectors \( \mathbb{C}[G]^1 \) and completing, one obtains a unitary \( G \)-module \( \mathbb{C}[G]/\mathbb{C}[G]^1 \) in which \((A.4)\) holds with \( \varphi \) the class of \( \delta^e \).

The clever way to complete here is to take the antidual [S64]: we let \( \text{GNS}_m \) be the (contragredient) \( G \)-module consisting of all antilinear functionals \( f \) on \( \mathbb{C}[G] \), such that the quantity

\[
||f||^2 := \sup_{c \in \mathbb{C}[G]} |f(c)|^2/(c, c)_m
\]

is finite.

(It is understood that the numerator must vanish when the denominator does, so that \( f \) factors through the null vectors.) Clearly each \( d \in \mathbb{C}[G] \) defines an element \( m_d := (\cdot, d)_m \) of \( \text{GNS}_m \), and one verifies without trouble that \( d \mapsto m_d \) induces a \( G \)-equivariant linear isometry of \( \mathbb{C}[G]/\mathbb{C}[G]^1 \) into \( \text{GNS}_m \); whence by extension an
isometry $\mathbb{C}[G]/\mathbb{C}[G]^+ \to \text{GNS}_m$ which is onto by the Riesz representation theorem. In particular we have $(c, d)_m = (m_c, m_d)$ and thus (first for $f = m_d$, then in general by density) the “reproducing” property

$$f(c) = (m_c, f) \quad \forall f \in \text{GNS}_m$$

of the kernel $K(\cdot, c) := m_c(\cdot)$. Now abbreviate $f(\delta^g)$ to $f(g)$ and $m_{\delta^g}$ to $m_g$; in this way $\text{GNS}_m$ becomes a unitary $G$-module of functions on $G$, with reproducing kernel $K(g, h) = m_h(g) = m(g^{-1}h)$ and cyclic vector $m_{\delta^g} = m_c$. Finally if $\varphi$ in (A.4) is cyclic, then the map $c\varphi \mapsto m_c$ extends to the required isomorphism $\mathcal{H} \to \text{GNS}_m$; and for the equivalence $m$ pure $\Leftrightarrow \text{GNS}_m$ irreducible we refer to [H63, 21.34].

Before further exemplifying this construction, we record an important inequality (A.13) of Weil [W40, p. 57] and some of its consequences:

(A.10) **Theorem.** Every state satisfies $m(g^{-1}) = \overline{m(g)}$ and

\begin{align}
(A.11) & \quad |m(g)| \leq 1, \\
(A.12) & \quad |m(g) - m(h)| \leq \sqrt{2 \text{Re}(1 - m(g^{-1}h))}, \\
(A.13) & \quad |m(gh) - m(g)m(h)| \leq \sqrt{1 - |m(g)|^2} \sqrt{1 - |m(h)|^2}.
\end{align}

**Proof.** The first statement is because $(\delta^g, \delta^h)_m = (\delta^h, \delta^g)_m$ since (A.2) is hermitian. As it is positive we also have a Cauchy-Schwarz inequality: $|(c, d)_m|^2 \leq (c, c)_m(d, d)_m$. This becomes (A.11) if we take the pair $c^*, d$ to be $\delta^g, \delta^h$; (A.12) if we take it to be $\delta^g, \delta^h - \delta^g$; and (A.13) if we take it to be $\delta^g - m(g)\delta^h, \delta^h - m(g)\delta^h$.

(A.14) **Corollary.** For any state $m$ of $G$, the equation $|m(g)| = 1$ defines a subgroup $H$ of $G$, $m$ restricts to a character $\chi$ of $H$, and we have

$$f(gh) = \overline{\chi(h)} f(g) \quad \forall (f, g, h) \in \text{GNS}_m \times G \times H.$$  

**Proof.** The initial statements are clear from (A.13). For (A.15), let $d = \delta^h - m(h)\delta^g$. Then $|m_{gd}|^2 = (d, d)_m = 0$, whence $f(gh) - \overline{\chi(h)} f(g) = f(gd) = 0$ by (A.9).

Property (A.15) means that $\text{GNS}_m$ is a certain space of sections of the line bundle, $G \times_H \mathbb{C}$, associated to $G \to G/H$ by the character $\chi$. Which space exactly, and with what norm, depend on how $m$ extends $\chi$ off $H$. For instance, we will show that we get all $L^2$ sections if we take the extension by zero, i.e. the state

$$m(g) = \chi^*(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

(A.17) **Theorem (Blattner [B63]).** For $m = \chi^*$ as above, we have $\text{GNS}_m = \text{ind}^G_H \chi$ where induction is in the sense of discrete groups. That is to say, the space (A.8) consists exactly of all $f : G \to \mathbb{C}$ such that

(a) $f(gh) = \overline{\chi(h)} f(g)$ for all $h \in H$;

(b) $f(g) = 0$ for all $g \in G \setminus H$.

(c) $f(g^{-1}h) = \overline{f(h)}$ for all $g, h \in G$.

(d) $f(g) = f(hg)$ for all $g, h \in G$.
(b) the quantity \( \|f\|_2^* := \sum_{g \in G/H} |f(g)|^2 \) is finite.

**Proof.** First we confirm that (A.16) is positive-definite: splitting the sum (A.2) over the cosets of \( H \) one readily obtains \( (c, c)_m = \sum_{g \in G/H} m_c(g) \geq 0 \), where \( m_c(g) = \sum_{h \in H} c_{gh}(h) \) is the function defined before (A.9).

Assume that \( f \) satisfies (A.8). Then (A.15) proves (a), and taking \( e = \sum_{g \in G} f(g) \delta_g \) where \( \Gamma \subset G \) is finite with at most one point per \( H \)-coset, one finds that the quotient in (A.8) equals \( \sum_{g \in G} |f(g)|^2 \). This shows that \( \|f\|_2^* \leq \|f\|_2^2 \), whence (b).

Conversely, assume that \( f \) satisfies (a, b). Splitting the sum \( f(c) = \sum_{g \in G} c_{g^{-1}} f(g) \) over the cosets of \( H \) gives \( f(c) = \sum_{g \in G/H} \overline{m_c(g)} f(g) \). Inserting this and the above value of \( (c, c)_m \) into (A.8), and using Cauchy-Schwarz, one obtains \( \|f\|_2^2 \leq \|f\|_2^2 \). \( \square \)

The realization (A.8) is especially well suited to discuss intertwining operators \( J : \text{GNS}_m \to \text{GNS}_n \), for each will be characterized by a single function, \( Jm_c \).

In more detail, writing \( \ast \) for the involution \( f \mapsto f^\ast := f(-1) \) of \( C^G \), we have:

**Theorem (A.18).** Let \( m, n \) be two states of \( G \). Then \( J \mapsto Jm_c \) defines an injection \( \text{Hom}_C(\text{GNS}_m, \text{GNS}_n) \to \text{GNS}_m \cap \text{GNS}_n \).

**Proof.** By hypothesis the function \( j = Jm_c \) is in \( \text{GNS}_n \) and satisfies \( gj = Jm_c \). Thus, by (A.9), the adjoint of \( J \) is given by \( (J^\ast f)(g) = (m_g, J^\ast f) = (g, f) \). In particular, putting \( f = n_c \) one finds \( J^\ast n_c = j^\ast \). Therefore \( j^\ast \) is in \( \text{GNS}_m \), and it determines \( J \) by the dual calculation: \( (Jf)(g) = (n_c, f) = (J^\ast n_c, f) = (g, f) \). \( \square \)

**Corollary (Mackey-Shoda [M51, II.2]).** Let \( \chi \) and \( \eta \) be characters of subgroups \( H \) and \( K \) of \( G \). Then \( \text{Hom}_C(\text{ind}_H^G \chi, \text{ind}_K^G \eta) \) has its dimension bounded above by the number of double cosets \( D = HgK \) such that

(a) \( \chi(h) = \eta(g^{-1}hg) \) for all \( h \in H \cap gKg^{-1} \);

(b) \( HgK \) projects onto finite sets in both \( G/K \) and \( H \backslash G \).

**Proof.** By (A.18) this dimension does not exceed that of \( (\text{ind}_H^G \chi)^\ast \cap (\text{ind}_K^G \eta) \), whose members \( j \) satisfy \( j(h^{-1}hg) = \overline{\eta(k)}(g)\chi(h) \) by virtue of (A.17a).

Such a function is determined by one value per double coset \( D = HgK \). This value must vanish when (a) fails, as one sees by putting \( k = g^{-1}hg \) in the relation above; also when (b) fails: for \( \|f\|_2^2 \) is constant in \( D \), and this constant occurs \#(\( D/K \)) times in the series (A.17b) for \( \|f\|_2^2 \), resp. \#(\( H \backslash D \)) times in the series for \( \|f\|_2^2 \). \( \square \)

We conclude this Appendix with Bochner’s description of continuous positive-definite functions on *locally compact abelian* groups [W40, pp. 120–122]. If \( G \) is such a group, write \( \hat{G} \) for its Pontryagin dual, i.e. the group of all continuous characters \( \chi : G \to \mathbb{U}(1) \) with the topology of uniform convergence on compact sets.

**Theorem, Definition (Bochner).** The Fourier transformation \( \mu \mapsto m \):

(A.21) \[ m(g) = \int_G \chi(g) d\mu(\chi) \]

defines a bijection between all continuous positive-definite functions \( m \) on \( G \), and all positive bounded Radon measures \( \mu \) on \( \hat{G} \). In particular, states of \( G \) correspond to probability measures on \( \hat{G} \). We refer to \( \mu \) as the spectral measure of \( m \).
**Example.** In the setting of (A.20), suppose that $H$ is an open subgroup of $G$. The characteristic function $1_H$ of $H$ in $G$ is a continuous state of $G$ (A.16), and we claim that its spectral measure is the image of Haar measure on the annihilator $H^\perp = \{ \chi \in \hat{G} : \chi(h) = 1 \text{ for all } h \in H \}$ under the inclusion $H^\perp \hookrightarrow \hat{G}$, i.e. we have

$$1_H(g) = \int_{H^\perp} \eta(g) \, d\eta. \tag{A.23}$$

To prove this, we first observe that $H$ is also closed (as complement of the union of its cosets in $G$); so $G/H$ is discrete and its dual $\hat{G}/H \cong H^\perp$ is compact [H63, 23.17, 23.25, 23.29]. So Haar measure $d\eta$ on $H^\perp$ is a probability measure, and the right-hand side $m(g)$ of (A.23) is clearly 1 when $g \in H$. On the other hand, the translation invariance of $d\eta$ gives $m(g) = \int_{H^\perp} (\zeta \eta)(g) \, d\eta = \zeta(g)m(g)$ for all $\zeta \in H^\perp$. If $g \not\in H$ this implies $m(g) = 0$, for we can find $\zeta \in H^\perp$ such that $\zeta(g) \neq 1$ [H63, 23.26].

**References**


REFERENCES


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