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On Δ^k -colouring of Powers of Paths and Cycles

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On Δ^k -colouring of Powers of Paths and Cycles

Cover Page Footnote

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Abstract

In an improper vertex colouring of a graph, adjacent vertices are permitted to receive same colours. An edge of an improperly coloured graph is said to be a bad edge if its end vertices have the same colour. A near-proper colouring of a graph is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements. The $\delta^{(k)}$ -colouring is a near-proper colouring of G consisting of k given colours, where $1 \leq k \leq \chi(G) - 1$, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it. In this paper, we discuss the number of bad edges of powers of paths and cycles.

Keywords: Improper colouring, near-proper colouring, $\delta^{(k)}$ -colouring, bad edges.

MSC 2020: 05C15, 05C38.

1 Introduction

Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected. The r -th power of an undirected graph G is another graph G^r that has the same set of vertices and two vertices are adjacent when their distance in G is at most r (see [6]). The power graph G^r is a complete graph when $r = d$ where d is the diameter of the graph G .

A *vertex colouring* of a graph G is a mapping $c : V(G) \rightarrow \mathcal{C}$, a set of colours. When no two adjacent vertices are assigned the same colour, it is called a *proper colouring* of G (see [2, 1]). The minimum number of colours required to colour a graph properly is called the *chromatic number* of G and is denoted by $\chi(G)$. A set $\{c_i \in \mathcal{C} : c(v) = c_i \text{ for at least one } v \in V(G)\}$ is called a *colour class* of the colouring of G .

Since its inception, the graph colouring methods have been widely used to model many practical problems. In many problems, availability of colours may be restricted. This led to the study on certain colouring techniques in which the colouring is not proper. In these types of colouring, some pairs of adjacent vertices may have the same colour. Such edges are called *bad edges*.

A *near-proper colouring* is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements [4]. From the terminology of near-proper colouring, the notion of $\delta^{(k)}$ -colouring has been introduced in [4] as follows:

Definition 1.1. [4] For $1 \leq k < \chi(G)$, the $\delta^{(k)}$ -colouring of a graph G is a near-proper colouring of G consisting of k given colours, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it.

The number of bad edges in a graph G obtained from a $\delta^{(k)}$ -colouring of G is denoted by $b_k(G)$.

Unless mentioned otherwise, the colour class C_1 will be the relaxed colour class (that is, the colour class which will have adjacency between the vertices in it), throughout the discussion. In this paper, the bad edges of the graphs are illustrated by dotted lines.

Different types of improper colouring of graphs in general and $\delta^{(k)}$ -colouring in particular have their own significance in real-life situations or practical problems. Since graph colouring methods are widely used to model many of such problems, the constraints, limitations and/or restrictions applicable to those problems will lead to different types of improper colouring of the corresponding graphical models. The $\delta^{(k)}$ -colouring and variants play a vital role, when we seek for the most economical or affordable solutions to those problems. Some interesting studies in this direction can be seen in [4, 5, 3].

Motivated by the above-mentioned studies, we investigate the $\delta^{(k)}$ -colouring of powers of paths and cycles in this paper.

2 $\delta^{(k)}$ -Colouring of Some Graph Powers

Among different classes of connected graphs, paths and cycles are the most elementary graph classes. In this section, we discuss the $\delta^{(k)}$ -colouring and the optimum number of bad edges corresponding to this colouring of the powers of paths and cycles. Because of their structural characteristics, the powers of paths and cycles possess much importance in our study on $\delta^{(k)}$ -colouring and the corresponding bad edges.

2.1 $\delta^{(k)}$ -Colouring of Powers of Paths

Theorem 2.1. *For $n \equiv m \pmod{r+1}$, the number of bad edges obtained from a $\delta^{(k)}$ -colouring in P_n^r , where $2 \leq r < n-1$ and $k \leq r$ is,*

$$b_k(P_n^r) = \begin{cases} \frac{(r-k+2)(r-k+1)}{2} (2\lfloor \frac{n}{r+1} \rfloor - 1), & \text{if } 0 \leq m < k, \\ \frac{(r-k+2)(r-k+1)}{2} (2\lfloor \frac{n}{r+1} \rfloor - 1) + \frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2} (2r - m - k + 2), & \text{if } m \geq k. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of a path and let k where $k \leq r$, be the number of available colours. We consider the colour class C_1 to have adjacency between the vertices in it. In all the possible cases of powers of paths, we colour the vertices in the following pattern:

Assign the colour c_k to the vertex v_1 . The vertices $v_2, v_3, \dots, v_{r+1}$ cannot be given the colour c_k as they are adjacent to the vertex v_1 . Hence we assign the colour c_{k-1} to the vertex v_2 . Continuing this colouring pattern to all the vertices, the vertex v_{k-1} is assigned the colour c_2 (note that, the colours are assigned in the descending order for the ease of explanation). Now, all the vertices $v_1, v_2, \dots, v_{k-1}$ are at least adjacent to the vertex v_{r+1} and hence the vertices from v_k to v_{r+1} are assigned the colour c_1 to satisfy the requirements of $\delta^{(k)}$ -colouring. Thus, using the $k-1$ colours, excluding the colour c_1 , the vertices $v_1, v_2, \dots, v_{k-1}$ are properly coloured and the remaining $r+k-2$ vertices namely, $v_k, v_{k+1}, \dots, v_{r+1}$ are assigned the colour c_1 . Since the vertex v_1 is not adjacent to the vertex v_{r+2} and since the vertices $v_2, v_3, \dots, v_{r+1}$ are at least adjacent to the vertex v_{r+2} , the vertex v_{r+2} can be given no colours other than c_k or c_1 . Here, we assign the vertex the colour c_k . In this manner, we assign the vertices $v_{r+3}, v_{r+4}, \dots, v_{r+k}$ the colours $c_{k-1}, c_{k-2}, \dots, c_2$ respectively. The remaining $r-k+2$ vertices namely, $v_{r+k+2}, v_{r+k+3}, \dots, v_{2(r+1)}$ are again assigned the

colour c_1 to maintain the requirements of $\delta^{(k)}$ -colouring. We continue this colouring pattern to the rest of the vertices and determine the minimum number of bad edges resulting from this $\delta^{(k)}$ -colouring.

Now, consider $n \equiv m \pmod{r+1}$, where $m \leq r$. Assigning the k available colours to the n vertices using the above mentioned colouring pattern, we observe that there are $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs and a vertex disjoint complete subgraph of order m where $m \leq r$. And each vertex disjoint complete subgraph lead to $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. Also, it can be noted that there are $(\lfloor \frac{n}{r+1} \rfloor - 1) \frac{(r-k+1)(r-k+2)}{2}$ bad edges between consecutive vertex disjoint complete subgraphs.

Now, when $0 \leq m < k$, there are no bad edges in the vertex disjoint subgraph of order m . Thus, the total number of bad edges obtained from this $\delta^{(k)}$ -colouring in P_n^r when $n \equiv m \pmod{r+1}$ where $0 \leq m < k$ is $\lfloor \frac{n}{r+1} \rfloor \frac{(r-k+2)(r-k+1)}{2} + (\lfloor \frac{n}{r+1} \rfloor - 1) \frac{(r-k+1)(r-k+2)}{2} = \lfloor \frac{n}{r+1} \rfloor (r-k+2)(r-k+1) - \frac{(r-k+1)(r-k+2)}{2}$ (see the Figure 1 for illustration).

Figure 1 $\delta^{(2)}$ -colouring of P_{10}^2

However, when $n \equiv m \pmod{r+1}$ where $m \geq k$, the vertex disjoint complete subgraph of order m will have $m-k+1$ vertices coloured only with the colour c_1 which are adjacent to few vertices among the $r-k+2$ vertices assigned with the colour c_1 in the previous section. The last vertex of this section is adjacent to $r+1-m$ vertices. Similarly, the second last, third last and so on the first vertex of this section is adjacent to $r+2-m, r+3-m, \dots, r+(m-(k-1))-m = r+1-k$ vertices of the previous section respectively. Therefore, $S_{m-k+1} = \frac{m-k+1}{2}(r+1-m+r+1-k) = \frac{m-k+1}{2}(2r-m-k+2)$ (note that, S_{m-k+1} is the sum of the arithmetic progression $r+2-m, r+3-m, \dots, r+1-k$ with $m-k+1$ terms, first and last terms to be $r+2-m$ and $r+1-k$ respectively and the common difference to be 1). Thus, the number of bad edges between these two sections of vertices having coloured with only the colour c_1 is $\frac{m-k+1}{2}(2r-m-k+2)$. Hence, the total number of bad edges obtained from $\delta^{(k)}$ -colouring in P_n^r when $n \equiv m \pmod{r+1}$ and $m \geq k$ is, $b_k(P_n^r) = \frac{(r-k+2)(r-k+1)}{2}(2\lfloor \frac{n}{r+1} \rfloor - 1) + \frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2}(2r-m-k+2)$ (see the Figure 2 for illustration).

Figure 2 $\delta^{(3)}$ -colouring of P_{11}^3

We call the above mentioned $\delta^{(k)}$ -colouring to be the optimum colouring and the bad edges obtained from this $\delta^{(k)}$ -colouring to be the minimum. The explanation for the same is given below.

Since, in P_n^r every vertex is adjacent to the vertices that are of at most distance r , the graph P_n^r induces complete subgraphs of order $r + 1$. As each vertex is in a complete subgraph, we consider the vertex disjoint complete subgraphs each of order $r + 1$. Now, there are $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs in P_n^r which can be properly coloured using $r + 1$ colours and hence we consider the k to be less than or equal to r . Now, the minimum number of bad edges in a complete graph is already proved in ([4]) as $b_k(K_n) = \frac{x(x+1)}{2}$, where $x = 1, 2, 3, \dots, n - 2$ and $k = n - x$. Substituting the values of n as $r + 1$ and $x = n - k = r + 1 - k$, we have $\frac{(r-k+1)(r-k+2)}{2}$ number of bad edges in each vertex disjoint complete subgraph of order $r + 1$. Thus, there are a total of $\lfloor \frac{n}{r+1} \rfloor \frac{(r-k+1)(r-k+2)}{2}$ number of bad edges in the vertex disjoint complete subgraphs. Since, the underlined graph is a path graph, there will always be adjacency between two consecutive vertex disjoint complete subgraphs only. So, to determine the minimum number of bad edges between any two vertex disjoint complete subgraphs, it is enough to find the minimum number of bad edges between any two consecutive vertex disjoint complete subgraphs. Since, $r - k + 2 < r + 1$, every vertex among the $r - k + 2$ vertices is adjacent to at most $r + 1$ vertices of the other vertex disjoint complete subgraph. To minimise the adjacency between the vertices assigned the colour c_1 in a complete subgraph to that of the vertices coloured with c_1 in the consecutive complete subgraph, there can be three ways as explained below.

Case 1: Consider the first two vertex disjoint subgraphs. Assign the first $r - k + 2$ vertices of both the complete subgraphs the colour c_1 and the rest of the $k - 1$ vertices of the two complete subgraphs the colours other than c_1 . Among the $r - k + 2$ vertices that are assigned the colour c_1 , the vertex v_1 is not adjacent to any vertex of second complete subgraph and hence v_1 will not contribute to any bad edge between the complete subgraphs. Now, the vertex v_2 , assigned the colour c_1 , is adjacent to the vertex v_{r+2} of the second complete subgraph which is assigned the colour c_1 . This leads to one bad edge. The third vertex v_3 , assigned the colour c_1 is at most adjacent to the vertices v_{r+2} and v_{r+3} of the second complete subgraph which are assigned the colour c_1 . This leads to two bad edges between the two vertex disjoint complete subgraphs. In this way, the vertex v_{r-k+2} of the first complete subgraph is adjacent to the $r - k + 1$ -th and the previous vertices (i.e the vertices $v_{r+2}, v_{r+3}, \dots, v_{2r-k+2}$) that are assigned the colour c_1 of the second complete subgraph and thus this lead to $r - k + 1$ bad edges. Thus, the total number of bad edges between each consecutive vertex disjoint complete subgraph is $\frac{(r-k+1)(r-k+2)}{2}$.

Case 2: Assign the colours $c_k, c_{k-1}, \dots, c_2$ properly to the first $k - 1$ vertices of both the vertex disjoint complete subgraphs and the rest $r - k + 2$ vertices the colour c_1 (as mentioned in our $\delta^{(k)}$ -colouring). Now, the vertex v_k , assigned the colour c_1 , is at most adjacent to the vertex v_{r+k} which is assigned the colour c_2 . Hence, this vertex lead to no bad edges between the two vertex disjoint complete subgraphs. Now as explained in *Case a*, the remaining $r - k + 1$ vertices lead to bad edges between the complete subgraphs. Thus, the total number of bad edges between each consecutive vertex disjoint complete subgraph is $\frac{(r-k+1)(r-k+2)}{2}$.

Case 3: Assign the first vertex of a vertex disjoint complete subgraph the colour c_1 , then the second with c_2 , the third with c_1 , the fourth with c_3 and so on. Now, if $r - k + 1 \geq k - 1$, there is a possibility of getting the last few vertices coloured only with c_1 . Now, the first vertex is not adjacent to the consecutive vertex disjoint complete subgraphs and this will not lead to any bad edges. Thus, as explained in the above two cases the remaining $r - k + 1$

vertices in each complete subgraphs will lead to bad edges between any two vertex disjoint consecutive complete subgraphs, or if the colour c_1 is assigned to a vertex other than the first vertex, that particular vertex will always be adjacent to a vertex assigned the colour other than c_1 . Thus, in either of the cases only $r - k + 1$ vertices of any two consecutive vertex disjoint complete subgraphs leads to bad edges between any two consecutive vertex disjoint complete subgraphs and the total number is $\frac{(r-k+1)(r-k+2)}{2}$.

Thus, from all of the above three cases it can be observed that the minimum number of bad edges between any two consecutive vertex disjoint complete subgraphs in P_n^r is $\frac{(r-k+1)(r-k+2)}{2}$. Since there are $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs of order $r + 1$, the minimum number of bad edges between the subgraphs is $(\lfloor \frac{n}{r+1} \rfloor - 1) \frac{(r-k+1)(r-k+2)}{2}$. Now, among the n vertices there are m vertices, where $m \leq r$, which form a vertex disjoint subgraph of order m . These m vertices and the previous vertex disjoint complete subgraph of order $r + 1$ can have lesser number of bad edges between them. Thus, taking this into consideration the following two cases are addressed.

Case 4: Let $n \equiv m \pmod{r+1}$ and $m < k$, then the remaining m vertices can be assigned the colours other than c_1 . This type of colouring where the last m vertices are given distinct colours other than c_1 can be obtained if and only if each m vertices are not adjacent to the same colour as that of itself and this is only possible if all the first $k - 1$ vertices of each vertex disjoint complete subgraph of order $r + 1$ is coloured with $k - 1$ distinct colours and the remaining $r - k + 2$ with the colour c_1 . This will lead to have no bad edges among the last m vertices which induces a complete subgraph on m vertices and also no bad edges between the $\lfloor \frac{n}{r+1} \rfloor$ -th complete subgraph and the m vertices. Thus, the total number of bad edges resulting from a $\delta^{(k)}$ -colouring when $n \equiv m \pmod{r+1}$ and $m < k$ is $\lfloor \frac{n}{r+1} \rfloor \frac{(r-k+2)(r-k+1)}{2} + (\lfloor \frac{n}{r+1} \rfloor - 1) \frac{(r-k+1)(r-k+2)}{2} = \lfloor \frac{n}{r+1} \rfloor (r - k + 2)(r - k + 1) - \frac{(r-k+1)(r-k+2)}{2}$.

Case 5: Let $n \equiv m \pmod{r+1}$ and $m \geq k$. Since $m \geq k$, not every m vertices can be coloured with colours other than c_1 . But we try to maximise the use of the colours other than c_1 to colour the last m vertices to minimise the number of bad edges between the last vertex disjoint complete subgraph and the complete subgraph induced by the m vertices. As explained in the above case, this can be obtained only if every first $k - 1$ vertices of each complete subgraph of order $r + 1$ is assigned the $k - 1$ colours distinctly and the remaining $r - k + 2$ vertices the colour c_1 . This colouring will lead in properly assigning the $k - 1$ vertices among the m vertices the colours $c_k, c_{k-1}, \dots, c_2$ and the remaining $m - k + 1$ vertices the colour c_1 , to attain the requirements of $\delta^{(k)}$ -colouring. Thus, this lead to $\frac{(m-k+1)(m-k)}{2}$ bad edges in the vertex disjoint complete subgraph on m vertices. Now, between the last vertex disjoint complete subgraph and the complete subgraph induced by the m vertices the number of bad edges as explained above is $\frac{m-k+1}{2}(2r - m - k + 2)$. Hence, the total number of bad edges when $n \equiv m \pmod{r+1}$ and $m \geq k$ is $\frac{(r-k+2)(r-k+1)}{2}(2\lfloor \frac{n}{r+1} \rfloor - 1) + \frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2}(2r - m - k + 2)$. $\square$

The path graph P_n is the subgraph of the cycle graph C_n and so it is obvious that few points discussed about the powers of paths in the above theorem will be similar when considered the powers of cycles. Now, as discussed in the 2.1, the cycle graph when taken the power also induces $\lfloor \frac{n}{r+1} \rfloor$ complete subgraphs of order $r + 1$ and the number of bad edges in each of the vertex disjoint complete subgraphs is same as that of the bad edges in each of

the vertex disjoint complete subgraphs of powers of paths which is equal to $\frac{(r-k+2)(r-k+1)}{2}$. However, it can be noted that, unlike in the powers of paths, the first and the last vertex disjoint complete subgraphs ($\lfloor \frac{n}{r+1} \rfloor$ th) of order $r+1$, have edges between them and so there are a total of $\lfloor \frac{n}{r+1} \rfloor \frac{(r-k+2)(r-k+1)}{2}$ bad edges between the vertex disjoint complete subgraphs in powers of cycle. For different values of n , r and k , the bad edges between the first and last vertex disjoint complete subgraphs vary. The $\delta^{(k)}$ -colouring of powers of cycles will be discussed in the following Theorems.

2.2 $\delta^{(k)}$ -Colouring of Powers of Cycles

First, recall that the diameter of a cycle graph of order n is $\lfloor \frac{n}{2} \rfloor$. The cases when $r = 1$ and $r = \lfloor \frac{n}{2} \rfloor$ have already been discussed in [4]. Hence, we discuss the $\delta^{(k)}$ -colouring of the r -th power of a cycle C_n for $1 < r < \lfloor \frac{n}{2} \rfloor$ in the following theorems.

Theorem 2.2. For C_n^r , where $2 \leq r < \lfloor \frac{n}{2} \rfloor$ and $2 \leq k < \chi(G)$ and $k \leq r$, the number of bad edges obtained from $\delta^{(k)}$ -colouring is given by,

$$b_k(C_n^r) = \begin{cases} (\frac{n}{r+1})(r-k+2)(r-k+1), & \text{if } n \equiv 0 \pmod{r+1}, \\ \frac{(r-k+1)(r-k+2)}{2}(2\lfloor \frac{n}{r+1} \rfloor - 1) + \frac{(r-k+3)(r-k+2)}{2}, & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of C_n^r taken in order. Let $c_1, c_2, \dots, c_k$ be the k available colours and $C_1, C_2, \dots, C_k$ be their respective colour classes. By Definition 1.1, only the vertices of one colour class are permitted to have adjacency among themselves. Hence, we assume that the vertices in the colour class C_1 to have adjacency among themselves. As explained above, the number of bad edges in each vertex disjoint complete subgraph is $\frac{(r-k+2)(r-k+1)}{2}$. Now, between any two consecutive vertex disjoint subgraphs there are a minimum of $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. Thus, the total number of bad edges between any two consecutive vertex disjoint complete subgraphs is $\lfloor \frac{n}{r+1} \rfloor \frac{(r-k+1)(r-k+2)}{2}$. Now, we need to address the following cases.

Case 1: Let $n \equiv 0 \pmod{r+1}$. Since $n \equiv 0 \pmod{r+1}$, there are $\frac{n}{r+1}$ vertex disjoint complete subgraphs. Thus, the total number of bad edges between any two consecutive vertex disjoint complete subgraphs is $(\frac{n}{r+1}) \frac{(r-k+1)(r-k+2)}{2}$. The total number of bad edges in each vertex disjoint complete graph is $\frac{(r-k+2)(r-k+1)}{2}$. Thus, the total number of minimum bad edges from a $\delta^{(k)}$ -colouring is $(\frac{n}{r+1})(r-k+2)(r-k+1)$.

Below given is a $\delta^{(k)}$ -colouring explaining the proof. Let $v_1, v_2, \dots, v_n$ be the vertices of C_n^r taken in order (for the ease of explanation we colour the graph in clockwise direction). We follow the following colouring pattern to assign colours to the first $r+1$ vertices. Colour the vertices $v_1, v_2, \dots, v_k$ with the colours $c_1, c_2, \dots, c_k$ respectively and the vertices $v_{k+1}, v_{k+2}, \dots, v_{r+1}$ with the colour c_1 to maintain the requirements of $\delta^{(k)}$ -colouring. Since, the vertex v_2 and the vertices $v_3, \dots, v_{r+1}$ is at least adjacent to vertex v_{r+2} , this vertex cannot be assigned any colour other than c_1 . Now, since vertex v_3 and the vertices $v_4, v_5, \dots, v_{r+1}$ are at least adjacent to v_{r+3} , this vertex can be assigned the colour c_1 or c_2 . To minimise the number of bad edges we assign the vertex the colour c_2 . Similarly, vertex v_{r+4} is assigned the colour c_3 , v_{r+5} the colour c_4 and so on, the vertex v_{r+k+1} the colour c_k . Now, the remaining

$r - k + 1$ vertices from the v_{r+k+1} -th vertex are given the colour c_1 . This colouring pattern is followed on each vertex disjoint complete subgraph of order $r + 1$. Thus, each vertex disjoint complete subgraph will have $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. Now, between any two consecutive vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ number of bad edges (see the Theorem 2.1). Thus, the total number of bad edges is $(\frac{n}{r+1})(r - k + 2)(r - k + 1)$ (see the Figure 3 for illustration).

Figure 3 The $\delta^{(2)}$ -colouring of C_6^2

Case 2: Let $n \equiv 1 \pmod{r+1}$. Since, paths are the subgraphs of cycles and as proved in the Theorem 2.1, there will always be $\frac{(r-k+2)(r-k+1)}{2}$ bad edges in each of the vertex disjoint complete subgraphs and between any two consecutive complete subgraphs there will always be a minimum of $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. Now, since $n \equiv 1 \pmod{r+1}$, there is a vertex say v_n which does not belong to any of the vertex disjoint complete subgraphs. Now, this vertex can be assigned the colour c_1 or any colour other than c_1 . The two possibilities are discussed below.

Subcase 2.1: If $c(v_n) = c_1$. In this case, any $\delta^{(k)}$ -colouring that colours each vertex disjoint complete subgraph with k colours will lead to $\frac{(r-k+2)(r-k+1)}{2}$ bad edges in each of the vertex disjoint complete subgraphs and $\frac{(r-k+2)(r-k+1)}{2}$ bad edges between any two vertex disjoint complete subgraphs. But, due to the vertex v_n , the number of bad edges between the last and the first vertex disjoint complete subgraphs (the one to the left of the vertex v_n (the last) and the one to right of the vertex v_n (the first)) as proved in *Case 1*, can be decreased by $r - k + 1$. Hence, the minimum number of bad edges between the last and the first vertex disjoint complete subgraphs is $\frac{(r-k)(r-k+1)}{2}$ and the vertex v_n is at most adjacent to v_{n-r} -th and v_r -th vertices of the last and first vertex disjoint complete subgraphs. Thus, the minimum number of bad edges arising from the vertex v_n are $2(r - k + 1)$. A $\delta^{(k)}$ -colouring that explains this case is discussed below.

We follow the same colouring pattern as that of the *Case 1* of the Theorem 2.4 and see that there are $\frac{(r-k+2)(r-k+1)}{2}$ bad edges in each of the vertex disjoint complete subgraphs and between each consecutive vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. The vertex v_n is assigned the colour c_1 . Now, the vertices v_{r+1} and $v_{n-(r+1)}$ of the first and last vertex disjoint complete subgraphs respectively are assigned the colour c_1 . Thus, the

vertex v_n will be adjacent to $r-k+1$ vertices of both the complete subgraphs that are assigned the colour c_1 , leading to $2(r-k+1)$ bad edges all together, between the vertex v_n and the vertex disjoint complete subgraphs. Now, we discuss the number of bad edges between the last and first vertex disjoint complete subgraphs. The vertices from v_{n-1} to $v_{n-(r-k+1)}$ that are assigned the colour c_1 are the only vertices that lead to bad edges between the last and the first vertex disjoint complete subgraphs and hence we consider only these vertices. Since, there is a vertex v_n , unlike in the *Case 1*, the vertex v_{n-1} is at most adjacent to the vertex v_{r-1} . Thus, among the $r+1$ vertices of the first vertex disjoint complete subgraph, the vertex v_{n-1} is not adjacent to the remaining two vertices that are assigned the colour c_1 . Thus, among the $r-k+2$ vertices that are assigned the colour c_1 , v_{n-1} is adjacent to $r-k$ vertices coloured with c_1 leading to $r-k$ bad edges. Similarly, the vertices $v_{n-(r-k)}$ and $v_{n-(r-k+1)}$ are at most adjacent to v_k and v_{k-1} vertices respectively (i.e. $v_{r-(r-k)}$ and $v_{r-(r-k+1)}$) and hence both the vertices will be adjacent to only one vertex coloured with the colour c_1 i.e the vertex v_1 and this thereby lead to one bad edge each respectively. Thus, there are $r-k+r-k-1+r-k-2+\dots+1$ bad edges from vertex v_{n-1} to $v_{n-(r-k)}$ and a bad edges lead by the vertex $v_{n-(r-k+1)}$. Hence, the total number of bad edges between these two vertex disjoint complete subgraphs are $\frac{(r-k)(r-k+1)}{2} + 1$. Thus, the total number of bad edges resulting from this $\delta^{(k)}$ -colouring is $\lfloor \frac{n}{r+1} \rfloor (r-k+2)(r-k+1) - \frac{(r-k+1)(r-k+2)}{2} + 2(r-k+1) + \frac{(r-k)(r-k+1)}{2} + 1 = \frac{(r-k+1)(r-k+2)}{2} (2\lfloor \frac{n}{r+1} \rfloor - 1) + 2(r-k+1) + \frac{(r-k)(r-k+1)}{2} + 1$.

Subcase 2.2: If $c(v_n) = c_k$, where $k = 2, 3, \dots, \chi(G) - 1$. Then, only the $r+1$ -th vertex of both the vertex disjoint complete subgraphs from v_n can be assigned the colour c_k . Thus, in this case the vertex v_n does not lead to any bad edges. Each vertex disjoint complete subgraph will have a minimum of $\frac{(r-k+2)(r-k+1)}{2}$ bad edges and between each complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. But in this case, as mentioned earlier the $r+1$ -th vertex of the first (i.e. v_{r+1} -th vertex) and last (i.e. $v_{n-(r+1)}$ -th vertex) vertex disjoint complete subgraphs are assigned the colour c_k and following this pattern from either of the directions, every $r+1$ -th vertex of every complete subgraph is assigned the colour c_k in order to maintain the requirements of $\delta^{(k)}$ -colouring. Thus, either of one vertex disjoint complete subgraph that is adjacent to first or the last complete subgraph cannot be given the colour c_k (this is because each colour other than c_1 can only be repeated $\lfloor \frac{n}{r+1} \rfloor$ times in the graph and the colour c_k is already been assigned to the vertex v_n and so there will be a vertex disjoint complete subgraph that cannot be assigned the colour c_k) and so this particular vertex disjoint complete subgraph will have $r-k+3$ vertices assigned the colour c_1 . Hence, this will lead to $\frac{(r-k+3)(r-k+2)}{2}$ bad edges in one vertex disjoint complete subgraph and the rest $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. Now, between the last and first vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges and this is because to determine the number of bad edges between these vertex disjoint complete subgraphs it is enough to consider the $r+1$ vertices from the vertex v_{n-r} to v_n (instead of the original vertex disjoint complete subgraph that has the vertices $v_{n-(r+1)}$ to v_{n-1}). This is possible as the graph is regular and the vertex v_n is assigned a colour other than c_1 . Thus, the number of bad edges between these two vertex disjoint complete subgraphs is $\frac{(r-k+1)(r-k+2)}{2}$ (See the Theorem 2.1). Thus, the total number of bad edges resulting from this case is given as $\lfloor \frac{n}{r+1} \rfloor (r-k+2)(r-k+1) - \frac{(r-k+1)(r-k+2)}{2} + \frac{(r-k+3)(r-k+2)}{2}$. A $\delta^{(k)}$ -colouring explaining the above discussed case is given as follows:

To maintain the minimum number of bad edges between any two consecutive complete subgraphs which is $\frac{(r-k+1)(r-k+2)}{2}$ (See the Theorem 2.1), we assign the first $k-2$ vertices (from the vertex v_n) of either of the complete subgraphs, say, the last one with $k-2$ distinct colours say $c_2, c_3, \dots, c_{k-2}$. Now since, the v_{n-1} -th vertex of the last complete subgraph is at most adjacent to the $r-1$ -th vertex of the first complete subgraph, the colour c_2 can be assigned to the vertex v_r . Similarly, the vertex v_{n-2} is at most adjacent to the vertex v_{r-2} , and hence the vertex v_{r-1} can be assigned the colour c_3 . The vertex $v_{n-(k-2)}$ is at most adjacent to the vertex v_{r-k+2} of the first complete subgraph and so the vertex v_{r-k+3} can be assigned the colour c_{k-2} . Thus, the remaining $r-k+2$ vertices of the last and the first complete subgraphs are assigned the colour c_1 respectively. Now, to find the number of bad edges between the last and first vertex disjoint complete subgraphs, it is enough to consider the vertices from v_{n-r-1} to v_{n-k+1} from the last vertex disjoint complete subgraph and the vertices from v_1 to v_{r-k+1} from the first vertex disjoint complete subgraph, as these are the vertices which lead to bad edges between the last and the first vertex disjoint complete subgraphs. The vertex v_{n-r-1} (assigned the colour c_1) is at most adjacent to the vertex v_1 , (assigned the colour c_1), leading to one bad edge. Similarly, v_{n-r} is at most adjacent to the vertex v_2 , leading to two bad edges. Continuing in this manner, the vertices v_{n-k} and v_{n-k+1} are at most adjacent to the vertices v_{r-k} and v_{r-k+1} , leading to $r-k$ and $r-k+1$ bad edges respectively. Thus, the minimum number of bad edges between the last and the first vertex disjoint complete subgraphs is $\frac{(r-k+1)(r-k+2)}{2}$.

We use the same colouring pattern to colour the rest of the vertex disjoint complete subgraphs, say we begin colouring the vertex disjoint complete subgraph that is adjacent to the first vertex disjoint complete subgraph and follow the pattern likewise in clockwise direction. Since, the $r+1$ -th vertex of every vertex disjoint complete subgraph is coloured with the colour c_k , the $r+1$ -th vertex of the vertex disjoint complete subgraph that is adjacent to the last complete subgraph cannot be given the colour c_k and so this particular vertex disjoint complete subgraph will have $r-k+3$ vertices assigned the colour c_1 . Hence, this particular vertex disjoint complete subgraph will have $\frac{(r-k+3)(r-k+2)}{2}$ and the rest $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. This vertex disjoint complete subgraph is coloured in such a way that the minimum number of bad edges between any two consecutive vertex disjoint complete subgraphs (see the Theorem 2.1) is maintained. The first $r-k+3$ vertices among the $r+1$ vertices of this complete subgraph is assigned the colour c_1 and the remaining $k-2$ the colours $c_{k-2}, c_{k-3}, \dots, c_2$ in this order respectively, if not coloured in this order there would be adjacency between the elements of colour classes other than C_1 . This colouring pattern will maintain the minimum number of bad edges between the second last vertex disjoint complete subgraph and its consecutive vertex disjoint complete subgraphs i.e. the third last and the last. Since there are $r-k+2$ vertices assigned the colour c_1 in the third last vertex disjoint complete subgraph and since the first vertex of this vertex disjoint complete subgraph coloured with c_1 is not adjacent to the second last vertex disjoint complete subgraph, this vertex will not lead to any bad edges between the second last vertex disjoint complete subgraph and itself. Now, the second vertex assigned the colour c_1 , is at most adjacent to the first vertex of the second last vertex disjoint complete subgraph which is assigned the colour c_1 and this vertex will lead to one bad edge. Again, the third vertex of the third vertex disjoint complete subgraph (assigned the colour c_1) is at most adjacent to the second vertex of the second vertex disjoint complete subgraph coloured with c_1 , leading to two bad edges.

Similarly, the $r - k + 2$ -th vertex of the third vertex disjoint complete subgraph (assigned the colour c_1) is at most adjacent to the $r - k + 1$ -th vertex of the second vertex disjoint complete subgraph coloured with c_1 , leading to $r - k + 1$ bad edges. Thus, between these two vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. Now, between the second last and the last vertex disjoint complete subgraph there will also be $\frac{(r-k+1)(r-k+2)}{2}$ bad edges as explained below. The first vertex of second last vertex disjoint complete subgraph coloured with c_1 is not adjacent to the last vertex disjoint complete subgraph, this vertex thereby will not lead to any bad edges between the last complete subgraph and itself. Now, the second vertex assigned the colour c_1 , is at most adjacent to the vertex $v_{n-(r+1)}$ of the last vertex disjoint complete subgraph which is assigned the colour c_k and this leads to no bad edges. The remaining $r - k + 1$ vertices assigned the colour c_1 , will lead to $\frac{(r-k+1)(r-k+2)}{2}$ bad edges between the last vertex disjoint complete subgraph and the second last vertex disjoint complete subgraph. Thus, the minimum total number of bad edges obtained from this $\delta^{(k)}$ -colouring is $(\lfloor \frac{n}{r+1} \rfloor - 1) \frac{(r-k+2)(r-k+1)}{2} + \frac{(r-k+3)(r-k+2)}{2} + \lfloor \frac{n}{r+1} \rfloor \frac{(r-k+1)(r-k+2)}{2} = \frac{(r-k+1)(r-k+2)}{2} (2\lfloor \frac{n}{r+1} \rfloor - 1) + \frac{(r-k+3)(r-k+2)}{2}$.

Now, when both the $\delta^{(k)}$ -colourings are compared, the bad edges in both the cases are the same (See Figure 4a and Figure 4b for illustration). $\square$

Now, from the Theorem 2.4 it is clear that since C_n^r is a regular graph and since it induces $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs, any $\delta^{(k)}$ -colouring with k colours will give same number of bad edges if the colours other than the colour c_1 is assigned to the maximum number of vertices in the graph. Now, every colour other than c_1 will have only one possibility of colouring a single vertex of each vertex disjoint complete subgraph. As, there are $k - 1$ colours (excluding the colour c_1) to colour each vertex disjoint complete subgraph, there are $(k - 1)\lfloor \frac{n}{r+1} \rfloor$ possibilities of using the colours other than c_1 in the optimal $\delta^{(k)}$ -colouring of powers of cycles and the remaining vertices are assigned the colour c_1 to maintain the requirements of $\delta^{(k)}$ -colouring.

The below theorem discusses the number of bad edges obtained from the $\delta^{(k)}$ -colouring of C_n^r when $n \equiv m \pmod{r+1}$ where $m \geq 2$.

Theorem 2.3. *For the powers of cycle C_n^r , where $2 \leq r < \lfloor \frac{n}{2} \rfloor$, the number of bad edges obtained from $\delta^{(k)}$ -colouring when $n \equiv m \pmod{r+1}$ where $m \geq 2$ is given by,*

$$b_k(C_n^r) = ab + a(b-1) + c + d + e,$$

where,

$$\begin{aligned} a &= \frac{(r-k+2)(r-k+1)}{2} \\ b &= \left\lfloor \frac{n}{r+1} \right\rfloor \\ c &= \begin{cases} r-m + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r-m < r-k+1 \text{ when } k > m, \\ r-k+1 + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r-m \geq r-k+1 \text{ when } k \leq m. \end{cases} \\ d &= \begin{cases} \frac{(r-k)(r-k+1)+2m}{2}, & \text{if } m \geq r-k, \\ \frac{m}{2}(2(r-k)-m+3), & \text{if } m < r-k, \\ m-r+k, & \text{if } m \leq k \text{ and if } m=k=r. \end{cases} \\ e &= \begin{cases} m(r-k+1), & \text{if } m \leq k, \\ k(r-k+1) + \frac{m-k}{2}(2r-m-k+1), & \text{if } m > k. \end{cases} \end{aligned}$$

Proof. Let $c_1, c_2, \dots, c_k$ be the k available colours and let $C_1, C_2, \dots, C_k$ be their respective colour classes. Now, as explained above, since, C_n^r is a $2r$ regular graph every vertex is in complete subgraph of order $r+1$. Now, any vertex can be assigned any colour only once so we consider the vertex disjoint complete subgraphs of C_n^r each of order $r+1$. There are $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs of order $r+1$ and the remaining m vertices do not belong to any of the vertex disjoint complete subgraphs, also since, $2 \leq m \leq r$, these m vertices will not form a complete subgraph of order $r+1$. However, these m vertices form a vertex disjoint complete subgraph of order m . Now, every colour other than c_1 will have only one possibility of colouring a single vertex of each vertex disjoint complete subgraph. Since there are $k-1$ colours (excluding the colour c_1) to colour each vertex disjoint complete subgraph, there are $(k-1)\lfloor \frac{n}{r+1} \rfloor$ possibilities of using the colours other than c_1 and the remaining vertices are assigned the colour c_1 to maintain the requirements of $\delta^{(k)}$ -colouring. A $\delta^{(k)}$ -colouring that explains this is same as that of the colouring mentioned for the case where $n \equiv 1 \pmod{r+1}$ and $k \leq r$, explained in the Theorem 2.4. In the case where $n \equiv 1 \pmod{r+1}$ and $k \leq r$, since $m=1$, the vertex v_n receives the colour c_1 and in this particular case all the m vertices $m \geq 2$ will receive the colour c_1 . Thus, every vertex disjoint complete subgraph will have $\frac{(r-k+2)(r-k+1)}{2}$ bad edges and between any two vertex disjoint complete subgraphs of order $r+1$, there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges (see the Theorem 2.4 for detailed explanation). The m vertices, assigned the colour c_1 , to maintain the prerequisites of $\delta^{(k)}$ -colouring, will lead to $\frac{m(m-1)}{2}$ bad edges. Now, since $m \leq r$, there can be bad edges between the last and the

first vertex disjoint subgraphs each of order $r + 1$, but the number can be less than that of the number of bad edges between the other vertex disjoint complete subgraphs of order $r + 1$. Also, there can be bad edges between the m vertices and the first and the last vertex disjoint complete subgraphs which are as discussed below.

Case 1: First we discuss the bad edges between the first and the last vertex disjoint complete subgraphs. The number of bad edges between these two vertex disjoint complete subgraphs can vary with different values of r , k and m . There are two cases for the same as explained below.

Case 1a: Let $r - m < r - k + 1$ when $k > m$. The vertex v_1 of the first vertex disjoint complete subgraph is adjacent to r vertices of the last vertex disjoint complete subgraph among which m vertices are not taken into consideration. Now, there are $r - k + 2$ vertices in each of the vertex disjoint complete subgraphs that are assigned the colour c_1 and the vertex v_1 is adjacent to a maximum of $r - k + 1$ vertices whose colour is c_1 . since, $r - m < r - k + 1$, the vertex v_1 is adjacent to $r - m$ vertices coloured only with the colour c_1 , leading to $r - m$ bad edges. Further, the remaining $k - 1$ vertices $v_2, v_3, \dots, v_k$, assigned the colour $c_2, c_3, \dots, c_k$ respectively, are adjacent to $r - m - 1, r - m - 2, \dots, r - m - (k - 1)$ number of vertices respectively of the last vertex disjoint complete subgraph which lead to no bad edges. Now, the remaining vertices v_{k+1} to v_{r+1} , assigned the colour c_1 , adjacent to the last vertex disjoint complete subgraph (if any), lead in $r - m - k + r - m - (k + 1) + \dots + 1 = \frac{(r-m-k)(r-m-k+1)}{2}$ bad edges. Thus, the total number of bad edges resulting from this case is $r - m + \frac{(r-m-k)(r-m-k+1)}{2}$.

Case 1b: Let $r - m \geq r - k + 1$ when $k \leq m$. Then, the vertex v_1 is adjacent to $r - k + 1$ vertices that are solely assigned the colour c_1 . The remaining vertices that are assigned the colour c_1 in the first vertex disjoint complete subgraph will lead in $\frac{(r-m-k)(r-m-k+1)}{2}$ bad edges as explained in the *case 1a*. Thus, the total number of bad edges resulting from this case is $r - k + 1 + \frac{(r-m-k)(r-m-k+1)}{2}$.

Now, we discuss the number of bad edges with respect to the m vertices and the first and last vertex disjoint complete subgraphs. First we focus on the bad edges between the former and then explain the later. Each of the two cases have subcases which are as detailed below.

Case 2: In this case, we discuss the number of bad edges with respect to the m vertices and the first vertex disjoint complete subgraph.

Case 2a: Let $m \geq r - k$. Here, the vertex v_n among the m vertices is adjacent to the vertices up to the vertex v_r . Now, since there are $r - k + 2$ vertices that receive the colour c_1 in the vertex disjoint complete subgraphs and since v_n is not adjacent to the vertex v_{r+1} which is also assigned the colour c_1 , it is adjacent to $r - k + 1$ vertices that are assigned the colour c_1 . This will lead in $r - k + 1$ bad edges. The next vertex v_{n-1} is adjacent to the vertices up to v_{r-1} and this vertex will lead in $r - k$ bad edges. Continuing in this manner, the vertex $v_{n-r+k+1}$ is adjacent to the vertices up to the vertex $v_{r-(r-k-1)} = v_{k+1}$, among which only two vertices, v_{k+1} and v_1 are assigned the colour c_1 and this leads to two bad edges. Now, since $m \geq r - k$, there will be $m - r + k$ vertices among the m vertices that are adjacent to the vertices coloured with colours other than c_1 and a single vertex v_1 that is assigned the colour c_1 . Thus, all these vertices will lead in a total of $m - r + k$ bad edges. Hence, the total number of bad edges resulting from this case is $r - k + 1 + r - k + \dots + 2 + m - r + k = \frac{(r-k)(r-k+1)+2m}{2}$.

Case 2b: Let $m < r - k$. Similar to the *Case 2a*, the vertex v_n will lead in $r - k + 1$ bad edges, the vertex v_{n-1} will lead in $r - k$ bad edges and so on. Now, since $m < r - k$, the m -th vertex v_{n-m+1} is adjacent to the vertices up to the vertex v_{r-m+1} of the first vertex disjoint complete subgraph and this leads to $r - k + 1 - m + 1 = r - k - m + 2$ bad edges. Thus, there are a total of $r - k + 1 + r - k + \dots + r - k - m + 2 = \frac{m}{2}(2(r - k) - m + 3)$ bad edges resulting from this particular case.

Case 2c: Now, there are two special cases from the *case 2a* where the value of $r - k + 1 + r - k + \dots + 2 = 0$ and the bad edges resulting from the special cases will be $m - r + k$. The cases arise when $m < k$ and $k = r$ and when $m = r = k$. In both the cases, there are only two vertices in the first vertex disjoint complete subgraphs that receive the colour c_1 i.e., the vertex v_1 and v_{r+1} . Now, none of the m vertices are adjacent to the vertex v_{r+1} . Thus, all the m vertices will lead to $m - r + k$ or simply m bad edges in both the cases.

Case 3: Next, we discuss the number of bad edges between the m vertices and the last vertex disjoint complete subgraph of order $r + 1$. There are two cases for this which are explained below.

Case 3a: Let $k \geq m$. Now, none of the m vertices are adjacent to the vertex v_{n-m-r} which is assigned the colour c_1 , thus the bad edges obtained in this case will be from the $r - k + 1$ vertices assigned the colour c_1 in the last vertex disjoint complete subgraph. Now, the first vertex among the m vertices i.e., the vertex v_{n-m+1} is adjacent to the r vertices i.e., up to the vertex $v_{n-m-r+1}$. This will lead in $r - k + 1$ bad edges. The next vertex v_{n-m+2} is adjacent to the vertices up to the vertex $v_{n-m-r+2}$ which is assigned the colour c_2 , thus this vertex v_{n-m+2} will also lead in $r - k + 1$ bad edges. Similarly, the vertex v_{n-m+3} is adjacent to the vertices up to the vertex $v_{n-m-r+3}$ which is assigned the colour c_3 leading to again $r - k + 1$ bad edges. Similarly, the m -th vertex v_n is adjacent to vertices up to $v_{n-m-r+m} = v_{n-r}$ which is assigned the colour c_{m+1} leading to again $r - k + 1$ bad edges. Thus, the total number of bad edges arising from this case is $m(r - k + 1)$.

Case 3b: Let $k < m$. As explained in the *case 3a* and since $k < m$, the k vertices among the m vertices will lead in $r - k + 1$ bad edges. Now, there are $m - k$ vertices remaining. The first vertex among the $m - k$ vertices will lead in $r - k$ bad edges, the second $r - k - 1$ and so on the $m - k$ -th vertex will lead in $r - k - (m - k - 1) = r - m + 1$ bad edges. Thus, the total number of bad edges resulting from this case is $k(r - k + 1) + r - k + r - k - 1 + \dots + r - m + 1$. Hence, the total number of bad edges for C_n^r resulting from the $\delta^{(k)}$ -colouring when $n \equiv m \pmod{r + 1}$ where $m \leq r$ and $k \leq r$ is given as $ab + a(b - 1) + c + d + e$, where,

$$\begin{aligned} a &= \frac{(r - k + 2)(r - k + 1)}{2} \\ b &= \left\lfloor \frac{n}{r + 1} \right\rfloor \\ c &= \begin{cases} r - m + \frac{(r - m - k)(r - m - k + 1)}{2}, & \text{if } r - m < r - k + 1 \text{ when } k > m, \\ r - k + 1 + \frac{(r - m - k)(r - m - k + 1)}{2}, & \text{if } r - m \geq r - k + 1 \text{ when } k \leq m. \end{cases} \end{aligned}$$

$$d = \begin{cases} \frac{(r-k)(r-k+1)+2m}{2}, & \text{if } m \geq r-k, \\ \frac{m}{2}(2(r-k)-m+3), & \text{if } m < r-k, \\ m-r+k, & \text{if } m \leq k \text{ and if } m=k=r. \end{cases}$$

$$e = \begin{cases} m(r-k+1), & \text{if } m \leq k, \\ k(r-k+1) + \frac{m-k}{2}(2r-m-k+1), & \text{if } m > k. \end{cases}$$

This completes the proof. $\square$

The below theorem discusses the number of bad edges for two cases of powers of cycles resulting from $\delta^{(k)}$ -colouring when $k > r$.

Theorem 2.4. *For C_n^r , where $2 \leq r < \lfloor \frac{n}{2} \rfloor$ and $3 \leq k < \chi(G)$ and $k > r$, the number of bad edges obtained from $\delta^{(k)}$ -colouring is given by,*

$$b_k(C_n^r) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{r+1}, \\ 1, & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}$$

Proof. Case 1: When $n \equiv 0 \pmod{r+1}$, we know that there are exactly $\frac{n}{r+1}$ vertex disjoint complete subgraphs which includes all the n vertices. Since, there are complete subgraphs of order $r+1$ in C_n^r , $\chi(C_n^r) \geq r+1$. Now, since there are exactly $\frac{n}{r+1}$ vertex disjoint complete subgraphs, every vertex disjoint complete subgraph can be properly coloured using $k = r+1$ colours leading to no bad edges in the graph. Thus, there are no bad edges resulting from a $\delta^{(k)}$ -colouring in C_n^r when $k > r$.

Case 2: Let $n \equiv 1 \pmod{r+1}$. There will be $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs including all the $n-1$ vertices among the n vertices and hence as explained in the above case, each vertex disjoint complete subgraph can be properly coloured with $k = r+1$ colours $c_1, c_2, \dots, c_{r+1}$. The remaining one vertex can be assigned a colour say c_{r+2} . Thus, when $n \equiv 1 \pmod{r+1}$, the $\chi(C_n^r) = r+2$. Now, if $k = r+1$, then we colour the vertices in the following manner. Assign the vertices $v_1, v_2, \dots, v_{r+1}$ the colours $c_1, c_2, \dots, c_{r+1}$ respectively. Thus, each vertex disjoint complete subgraph is properly coloured with the $r+1$ colours leading to no bad edges in each of the vertex disjoint complete subgraphs and also no bad edges between any two vertex disjoint complete subgraphs. Now, after assigning colours to the $n-1$ vertices that belong to $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs, there is a vertex say v_n which has to be assigned the colour c_1 , to maintain the requirements of $\delta^{(k)}$ -colouring and this vertex is adjacent to the vertex v_1 which is assigned the colour c_1 leading to one bad edge in the graph. Thus, $b_k(C_n^r) = 1$, when $n \equiv 1 \pmod{r+1}$ and $k = r+1$. Note that, in this case, any $\delta^{(k)}$ -colouring other than the above mentioned will always give one bad edge when $k = r+1$. $\square$

3 Conclusion

In this paper, we have discussed the $\delta^{(k)}$ -colouring of the powers of paths and cycles. Every case for different parities of n and for different values of k and r are keenly studied and an

optimal $\delta^{(k)}$ -colouring and the minimum number of bad edges obtained from the same is determined.

The study can be extended to many graph classes, graph powers and graph products. In this discussion, we have restricted the number of colour classes, which can have relaxation in proper colouring requirements, to one. This restriction may also be relaxed ensuring the minimum possible number of bad edges in the given graph.

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References

- [1] G Chartrand and P Zhang. *Chromatic graph theory*. Chapman and Hall/CRC press, 2008.
- [2] F Harary. *Graph theory*. Narosa Publ. House, New Delhi., 2001.
- [3] J Kok and EG Mphako-Banda. Chromatic completion number. *J. Math. Comput. Sci.*, 10(6):2971–2983, 2020.
- [4] J Kok and S Naduvath. $\delta^{(k)}$ -colouring of cycle related graphs. *Adv. Stud. Contemp. Math.*, 2018.
- [5] E G Mphako-Banda. An introduction to the k -defect polynomials. *Quaest. Math.*, 42, 2019.
- [6] D B West. *Introduction to graph theory*. Prentice Hall, Upper Saddle River, 2001.