Theory and Applications of Graphs

Volume 7 | Issue 2 Article 2

July 2020

Decomposition of Certain Complete Graphs and Complete Multipartite Graphs into Almost-bipartite Graphs and Bipartite Graphs

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Recommended Citation
Sethuraman, G. and Sujasree, M. (2020) "Decomposition of Certain Complete Graphs and Complete Multipartite Graphs into Almost-bipartite Graphs and Bipartite Graphs," *Theory and Applications of Graphs*: Vol. 7 : Iss. 2 , Article 2.
DOI: 10.20429/tag.2020.070202
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol7/iss2/2

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Cover Page Footnote

We would like to thank the editor for processing and accepting our research article. Also we would like to thank the reviewer for the valuable suggestions and recommending our paper for publication.

Abstract

In his classical paper [14], Rosa introduced a hierarchical series of labelings called ρ, σ, β and α labeling as a tool to settle Ringel's Conjecture [13] which states that if T is any tree with m edges then the complete graph K_{2m+1} can be decomposed into $2m+1$ copies of T. Inspired by the result of Rosa [14] many researchers significantly contributed to the theory of graph decompositions using graph labelings. In this direction, in 2004, Blinco et al. [6] introduced γ-labeling as a stronger version of ρ-labeling. A function g defined on the vertex set of a graph G with n edges is called a γ-labeling if

(i) g is a ρ-labeling of G,

(ii) G is a tripartite graph with vertex tripartition (A, B, C) with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c,

(iii) $g(a) < g(v)$ for every edge $\{a, v\} \in E(G)$ where $a \in A$,

(iv) $g(c) - g(\bar{b}) = n$.

Further, Blinco et al. [6] proved a significant result that the complete graph K_{2cn+1} can be cyclically decomposed into $c(2cn+1)$ copies of any γ-labeled graph with n edges, where c is any positive integer. Recently, in 2013, Anita Pasotti [4] introduced a generalisation of graceful labeling called d-divisible graceful labeling as a tool to obtain cyclic G-decompositions in complete multipartite graphs. Let G be a graph of size $e = d\ .\ m$. A d-divisible graceful labeling of the graph G is an injective function $g : V(G) \to \{0, 1, 2, \dots, d(m+1)-1\}$ such that $\{|g(u) - g(v)|/\{u, v\} \in E(G)\} = \{1, 2, \dots, d(m+1)-1\} \backslash \{m+1, 2(m+1), \dots, (d-1)(m+1)\}$. A d-divisible graceful labeling of a bipartite graph G is called as a d-divisible α-labeling of G if the maximum value of one of the two bipartite sets is less than the minimum value of the other one. Further, Anita Pasotti [4] proved a significant result that the complete multipartite graph $K_{(\frac{e}{d}+1)\times 2dc}$ can be cyclically decomposed into copies of d-divisible α-labeled graph G, where e is the size of the graph G and c is any positive integer ($K_{(\frac{e}{d}+1)\times 2dc}$ contains $\frac{e}{d}+1$ parts each of size $2dc$). Motivated by the results of Blinco et al. [6] and Anita Pasotti [4], in this paper we prove the following results.

i) For $t \geq 2$, disjoint union of t copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3, n \geq 4$ plus an edge admits γ-labeling.

ii) For $t \geq 2$, t-levels shadow graph of the path P_{dn+1} admits d-divisible α-labeling for any admissible d and $n \geq 1$.

Further, we discuss related open problems.

1 Introduction

Terms which are not defined here can be found in [15]. In an attempt to settle the Ringel's conjecture [13] which states that if T is any tree with m edges then the complete graph K_{2m+1} can be decomposed into $2m+1$ copies of T, in his classical paper [14], Rosa introduced a series of labelings called $\alpha, \beta, \sigma, \rho$-labeling.

Let G be a graph with n edges. A one-to-one function g from $V(G)$ to $\{0, 1, 2, \dots, n\}$ is called a β-labeling of G if $\{|g(u) - g(v)|/\{u, v\} \in E(G)\} = \{1, 2, \dots, n\}$. A β-labeling g of a graph G with n edges is called an α-labeling if there exists an integer k such that for every edge $\{u, v\} \in E(G)$ either $g(u) \leq k < g(v)$ or $g(v) \leq k < g(u)$. Given two vertices u and v by uv we denote the edge $\{u, v\}$.

It is clear that α-labeling is a stronger version of β-labeling. β-labeling was later called as graceful labeling by Golomb [12] and this term is most widely used now. ρ-labeling is weaker version of graceful labeling. The precise definition of ρ-labeling is given below. Let G be a graph with n edges. A one-to-one function g from $V(G)$ to $\{0, 1, 2, \ldots, 2n\}$ is called a ρ-labeling of G if $\{\min\{|g(u) - g(v)|, 2n + 1 - |g(u) - g(v)|\}/\{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$.

Further, Rosa [14] proved the following two significant theorems.

Theorem 1.1. *Let G be a graph with n edges. Then there exists a cyclic G-decomposition of the complete graph K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 1.2. *If G is a graph with n edges that has an α-labeling, then the complete graph K_{2cn+1} can be cyclically decomposed into subgraphs isomorphic to G, where c is an arbitrary natural number.*

The interesting part of α-labeled graphs with n edges is that they not only decompose complete graphs K_{2cn+1} but also decompose the complete bipartite graphs $K_{an,bn}$. This interesting result proved by El-Zanati and Vanden Eynden [9] is precisely stated in the following theorem.

Theorem 1.3. *If a graph G with n edges has an α-labeling then there exists a cyclic decomposition of the complete bipartite graph $K_{an,bn}$ into subgraphs isomorphic to G, where a and b are arbitrary positive integers.*

These results attracted many researchers to significantly contribute in theory of graph decompositions using graph labelings. It is clear from the definition of α-labeling that if a graph G admits α-labeling then it must be necessarily bipartite. This restriction prompted Blinco et al. [6] to introduce γ-labeling in order to achieve cyclic G-decompositions in K_{2cn+1}, where G is a non-bipartite graph, c is any positive integer and n is the number of edges of the graph G. A function g defined on the vertex set of a graph G with n edges is called a γ-labeling if

(i) g is a ρ-labeling of G,

(ii) G is a tripartite graph with vertex tripartition (A, B, C) with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c,

(iii) $g(a) < g(v)$ for every edge $\{a, v\} \in E(G)$ where $a \in A$,

(iv) $g(c) - g(\bar{b}) = n$.

Further, in [6], Blinco et al. have proved the following significant theorem.

Theorem 1.4. *The complete graph K_{2cm+1} can be cyclically decomposed into copies of the γ-labeled graph G, where m is the number of edges of the graph G and c is any positive integer.*

Motivated by the above result of Blinco et al. [6], the almost-bipartite graphs $P_n + e$, $n \geq 4$, $K_{m,n}+e$, $m \geq 2$, $n > 2$, C_{2k+1}, $k \geq 2$, $C_{2m}+e$, $m > 2$, $C_3 \cup C_{4m}$, $m > 1$, $C_{2k+1} \cup C_{4n+2}$, $k \geq 1$, $n \geq 1$ are found to have γ-labeling (refer [5], [6], [7], [8], [10]). (A graph is said to be almost-bipartite if the removal of a particular edge makes the graph bipartite). For survey on γ-labeling refer the survey on graph labelings by Gallian [11]. Motivated by the results of Blinco et al. [6], in this paper we prove that for $t \geq 2$, disjoint union of t copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3, n \geq 4$ plus an edge admits γ-labeling.

Recently, in 2013, Anita Pasotti [4] introduced a generalisation of graceful labeling called d-divisible graceful labeling as a tool to obtain cyclic G-decomposition in complete multipartite graphs. Let G be a graph of size $e = d\ .\ m$. An injective function $g : V(G) \to \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that $\{|g(u) - g(v)|/\{u, v\} \in E(G)\} = \{1, 2, \ldots, d(m + 1) - 1\}\backslash\{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}$ is called as a d-divisible graceful labeling of the graph G. A d-divisible graceful labeling of a graph G can exist only if d is a divisor of the size e of G, hence, for this reason, any divisor d of e is said to be admissible for the existence of a d-divisible graceful labeling of G. A d-divisible graceful labeling of a bipartite graph G is called as a d-divisible α-labeling of G if the maximum value of one of the two bipartite sets is less than the minimum value of the other one.

Further, Anita Pasotti [4] has proved the following significant theorems.

Theorem 1.5. *(Anita Pasotti [4]) The complete multipartite graph $K_{(\frac{e}{d}+1)\times 2d}$ can be cyclically decomposed into copies of the d-divisible graceful labeled graph G, where e is the size of the graph G.*

Theorem 1.6. *(Anita Pasotti [4]) The complete multipartite graph $K_{(\frac{e}{d}+1)\times 2dc}$ can be cyclically decomposed into copies of the d-divisible α-labeled graph G, where e is the size of the graph G and c is any positive integer.*

In the literature survey [11], one can observe that a very few families of graphs are identified to have d-divisible α-labeling. Anita Pasotti [4] has proved that path and star admit d-divisible α-labeling for any admissible d. She [3] also proved that for any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits $(2m-1)$-divisible α-labeling. In [1] and [2], Anna Benini and Anita Pasotti proved the following results. A hairy cycle of size e admits an e-divisible α-labeling if and only if it is bipartite. The hairy cycle $H(2t, \lambda)$ admits d-divisible α-labeling for any admissible d. The ladder L_{2k} has 2-divisible α-labeling if and only if k is even.

Inspired by the decomposition theorems proved by Anita Pasotti, in this paper we prove that for $t \geq 2$, t-levels shadow graph of the path P_{dn+1} admits d-divisible α-labeling for any admissible d and $n \geq 1$. t-levels shadow graph of a graph is defined as follows. t-levels shadow graph of a graph G, denoted $S_t(G)$ is obtained by taking $t \geq 2$ copies $G_1, G_2, \ldots, G_t$ of G and joining each vertex v_{ij} in G_i to the copies of its adjacent vertices in G_{i+1}, for $1 \leq j \leq n$ and $1 \leq i \leq t-1$, where $n = |V(G)|$.

2 γ-labeling of disjoint union of complete bipartite graphs plus an edge

In this section we prove that disjoint union of t copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3$ and $n \geq 4$ plus an edge admits γ-labeling.

Theorem 2.1. *For $t \geq 2$, disjoint union of t copies of a complete bipartite graph with one part containing at least three vertices and another part containing at least four vertices, plus an edge admits γ-labeling.*

Proof. Consider the complete bipartite graph $K_{m,n}$, where $m \geq 3, n \geq 4$.
Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the two parts of $K_{m,n}$.
For any $i = 1, 2, \ldots, t$, let $U_i = \{u_{i1}, u_{i2}, \ldots, u_{im}\}$ and $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ be the two parts of the i-th copy $K^i_{m,n}$ of the complete bipartite graph $K_{m,n}$.

Set $U = \bigcup_{i=1}^{t} U_i$ and $V = \bigcup_{i=1}^{t} V_i$.
Clearly, U and V are the two parts of the disjoint union of the t copies of $K_{m,n}$, denoted by $\bigcup_{i=1}^{t} K^i_{m,n}$.
Join the vertices v_{11} and v_{12} by an edge $\hat{e}$.

Denote the new graph thus obtained by $(\bigcup_{i=1}^{t} K^i_{m,n}) + \hat{e}$.

Observe that $|V((\bigcup_{i=1}^{t} K^i_{m,n}) + \hat{e})| = t(m+n)$ and $|E((\bigcup_{i=1}^{t} K^i_{m,n}) + \hat{e})| = tmn + 1$.

Define $g : V((\bigcup_{i=1}^{t} K^i_{m,n}) + \hat{e}) \to \{0, 1, 2, \ldots, 2N\}$, where $N = tmn + 1$ in the following way.
First we define the labels of the vertices in the set U in the following way.
For $1 \leq j \leq m$, define $g(u_{1j}) = 2(j-1)$ and $g(u_{2j}) = 2j + 1$.
For each i, $3 \leq i \leq t$, define
$g(u_{i1}) = g(u_{(i-1)m}) + m$,
$g(u_{ij}) = g(u_{i(j-1)}) + 1$, for each j, $2 \leq j \leq m$.
Now we define the labels of the vertices in the set V in the following manner.
Define $g(v_{11}) = 2N - 1$, $g(v_{12}) = N - 1$, $g(v_{13}) = 2N$, $g(v_{14}) = N - 2$.
For $5 \leq k \leq n$, define

$$g(v_{1k}) = \begin{cases} g(v_{1(k-1)}) - 2m + 1, \text{if } k \text{ is odd} \\ g(v_{1(k-1)}) - 1, \text{if } k \text{ is even.} \end{cases}$$

$$\text{Define } g(v_{21}) = \begin{cases} g(v_{1n}) - 4(r-1), \text{if } m = 2r, r \geq 2 \text{ and } n \text{ is even} \\ g(v_{1n}) - (4r-2), \text{if } m = 2r+1, r \geq 1 \text{ and } n \text{ is even} \\ g(v_{1n}) + 2, \text{if } n \text{ is odd.} \end{cases}$$

We define the labels of the vertices v_{2k}, for $k, 2 \leq k \leq n$ in two cases depending on n is even or odd.

Case 1. n is even
For $2 \leq k \leq n$, define

$$g(v_{2k}) = \begin{cases} g(v_{2(k-1)}) - 1, \text{if } k \text{ is even} \\ g(v_{2(k-1)}) - 2m + 1, \text{if } k \text{ is odd.} \end{cases} .$$

DOI: 10.20429/tag.2020.070202

Case 2. n is odd
For $2 \leq k \leq n$, define
$g(v_{2k}) = \begin{cases} g(v_{2(k-1)}) - 2m + 1, \text{if } k \text{ is even} \\ g(v_{2(k-1)}) - 1, \text{if } k \text{ is odd.} \end{cases}$.
For each $i, 3 \leq i \leq t$, define the labels of the vertices v_{ik}, for each $k, 2 \leq k \leq n$ in the following way.
For each $i, 3 \leq i \leq t$, define
$g(v_{i1}) = g(v_{(i-1)n}) + m - 1$,
$g(v_{ik}) = g(v_{i(k-1)}) - m$, for each $k, 2 \leq k \leq n$.

Observation 1. Vertex labels of $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are distinct.

We prove that the vertex labels of the graph $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are distinct depending on n is even or odd.

Case 1. n is even

If the labels of the vertices of the graph $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are arranged as,
$g(u_{11})$, $g(u_{12})$, $g(u_{21})$, $g(u_{13})$, $g(u_{22})$, $g(u_{14})$, $\ldots$, $g(u_{1m})$, $g(u_{2(m-1)})$, $g(u_{2m})$, $(g(u_{ij}))_{i=3,j=1}^{i=t,j=m}$, $g(v_{tn})$, $g(v_{t(n-1)})$, $g(v_{t(n-2)})$, $\ldots$, $g(v_{t2})$, $g(v_{(t-1)n})$, $g(v_{t1})$, $g(v_{(t-1)(n-1)})$, $g(v_{(t-1)(n-2)})$, $\ldots$, $g(v_{(t-1)2})$, $g(v_{(t-2)n})$, $g(v_{(t-2)(n-1)})$, $g(v_{(t-1)1})$, $g(v_{(t-2)(n-2)})$, $\ldots$, $g(v_{(t-2)2})$, $g(v_{(t-2)1})$, $g(v_{(t-3)n})$, $\ldots$, $g(v_{3n})$, $g(v_{41})$, $g(v_{3(n-1)})$, $g(v_{3(n-2)})$, $\ldots$, $g(v_{33})$, $g(v_{32})$, $g(v_{2n})$, $g(v_{2(n-1)})$, $g(v_{31})$, $g(v_{2(n-2)})$, $g(v_{2(n-3)})$, $\ldots$, $g(v_{22})$, $g(v_{21})$, $g(v_{1n})$, $g(v_{1(n-1)})$, $g(v_{1(n-2)})$, $g(v_{1(n-3)})$, $\ldots$, $g(v_{14})$, $g(v_{12})$, $g(v_{11})$, $g(v_{13})$,
then it forms a monotonically increasing sequence.

Case 2. n is odd

If the labels of the vertices of the graph $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are arranged as,
$g(u_{11})$, $g(u_{12})$, $g(u_{21})$, $g(u_{13})$, $g(u_{22})$, $g(u_{14})$, $\ldots$, $g(u_{1m})$, $g(u_{2(m-1)})$, $g(u_{2m})$, $(g(u_{ij}))_{i=3,j=1}^{i=t,j=m}$, $g(v_{tn})$, $g(v_{t(n-1)})$, $g(v_{t(n-2)})$, $\ldots$, $g(v_{t2})$, $g(v_{(t-1)n})$, $g(v_{t1})$, $g(v_{(t-1)(n-1)})$, $g(v_{(t-1)(n-2)})$, $\ldots$, $g(v_{(t-1)2})$, $g(v_{(t-2)n})$, $g(v_{(t-2)(n-1)})$, $g(v_{(t-1)1})$, $g(v_{(t-2)(n-2)})$, $\ldots$, $g(v_{(t-2)2})$, $g(v_{(t-2)1})$, $g(v_{(t-3)n})$, $\ldots$, $g(v_{3n})$, $g(v_{41})$, $g(v_{3(n-1)})$, $g(v_{3(n-2)})$, $\ldots$, $g(v_{33})$, $g(v_{32})$, $g(v_{2n})$, $g(v_{2(n-1)})$, $g(v_{31})$, $g(v_{2(n-2)})$, $g(v_{2(n-3)})$, $\ldots$, $g(v_{22})$, $g(v_{1n})$, $g(v_{21})$, $g(v_{1(n-1)})$, $g(v_{1(n-2)})$, $g(v_{1(n-3)})$, $\ldots$, $g(v_{14})$, $g(v_{12})$, $g(v_{11})$, $g(v_{13})$,
then it forms a monotonically increasing sequence.

Hence the vertex labels of the graph $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are distinct.

Observation 2. Edge labels of $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ are distinct.

The edge $v_{11}v_{12}$ has the label N.

We prove that the edge labels of $\bigcup_{i=1}^{t} K_{m,n}^{i}$ are distinct in two cases depending on n is even or odd.

Case i. n is even

The labels of the edges in the first copy $K_{m,n}^{1}$ can be arranged as a sequence,
$S_{11} : ((N-1, N-2, N-3, \ldots, N+2m+1-mn, N+2m-mn), (2m, 2m-1, \ldots, 2, 1))$.
For each $i, 2 \leq i \leq t$, the labels of the edges in the i^{th} copy $K_{m,n}^{i}$ can be arranged as a sequence,
$S_{1i} : (N+2m-(i-1)mn-1, N+2m-(i-1)mn-2, \ldots, N+2m-imn+2,$
$N+2m-imn+1, N+2m-imn)$.
The labels of the edges in the above sequences together with the label of the edge $v_{11}v_{12}$, $|g(v_{11}) - g(v_{12})| = N$ can be rearranged as a monotonic decreasing sequence
$S : (N, N-1, N-2, \ldots, 3, 2, 1)$.
Thus the edge labels are distinct when n is even.

Case ii. n is odd

The labels of the edges in the first copy $K_{m,n}^{1}$ can be arranged as a sequence,
$S_{21} : ((N-1, N-2, N-3, \ldots, N+3m-mn+2, N+3m-mn+1, N+3m-mn),$
$(N+3m-mn-1, N+3m-mn-3, N+3m-mn-5, \ldots, N+m-mn+3,$
$N+m-mn+1), (2m, 2m-1, \ldots, 2, 1))$.
The labels of the edges in the second copy $K_{m,n}^{2}$ can be arranged as a sequence,
$S_{22} : (N+3m-mn-2, N+3m-mn-4, N+3m-mn-6, \ldots, N+m-mn+2,$
$N+m-mn, N+m-mn-1, N+m-mn-2, N+m-mn-3, \ldots,$
$N+2m-2mn+2, N+2m-2mn+1, N+2m-2mn)$.
For each $i, 3 \leq i \leq t$, the labels of the edges in the i^{th} copy $K_{m,n}^{i}$ can be arranged as a sequence,
$S_{2i} : (N+2m-(i-1)mn-1, N+2m-(i-1)mn-2, \ldots, N+2m-imn+2,$
$N+2m-imn+1, N+2m-imn)$.
The labels of the edges in the above sequences together with the label of the edge $v_{11}v_{12}$, $|g(v_{11}) - g(v_{12})| = N$ can be rearranged as a monotonic decreasing sequence
$S : (N, N-1, N-2, \ldots, 3, 2, 1)$.
Thus the edge labels are distinct when n is odd.

Hence the edge labels of the graph $(\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e}$ are distinct.

DOI: 10.20429/tag.2020.070202

Observation 3. g is a γ-labeling.

In order to prove that g is a γ-labeling, we partition the vertex set $V((\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e})$ as (A, B, C), where $A = U, B = V \setminus \{v_{11}\}$ and $C = \{v_{11}\}$. Then, by the above labeling we have $g(u_{ij}) < g(v_{ik})$ for any $u_{ij} \in A$ and for any $v_{ik} \in B \bigcup C$. The label of the edge $v_{11}v_{12} = N = (2N - 1 - (N - 1))$. Hence, the graph $(\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e}$ admits γ-labeling. $\square$

Illustration

γ-labeling that is defined as in the proof of Theorem 2.1 for the disjoint union of three copies of the complete bipartite graph $K_{4,5}$ plus an edge, $(\bigcup_{i=1}^{3} K_{4,5}^{i}) + \hat{e}$ and the disjoint union of two copies of the complete bipartite graph $K_{3,4}$ plus an edge, $(\bigcup_{i=1}^{2} K_{3,4}^{i}) + \hat{e}$ are given in Figure 1 and Figure 2 respectively.

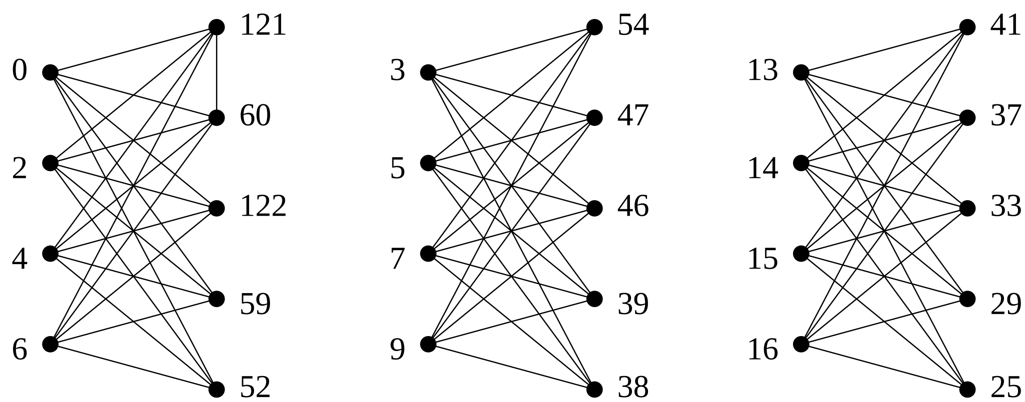

Figure 1: γ-labeling of $(\bigcup_{i=1}^{3} K_{4,5}^{i}) + \hat{e}$

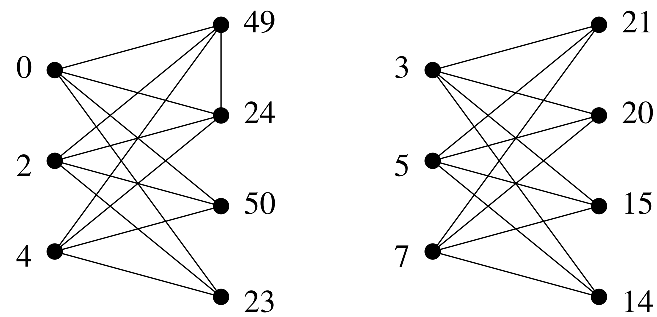

Figure 2: γ-labeling of $(\bigcup_{i=1}^{2} K_{3,4}^{i}) + \hat{e}$

The following corollary is an immediate implication of Blinco et al.'s theorem, Theorem 1.4.

Corollary 2.2. *The complete graph K_{2cr+1} can be cyclically decomposed into copies of the graph $(\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e}$, where c is any positive integer, $m \geq 3, n \geq 4, t \geq 2$ and $r = |E((\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e})|$.*

3 d-divisible α-labeling of t-levels shadow graph of path

In this section we prove that for $t \geq 2$, t-levels shadow graph of the path P_{dn+1}, $S_t(P_{dn+1})$ with $d \geq 1$, $n \geq 1$ admits d-divisible α-labeling for all $d \geq 1$.

Theorem 3.1. *For $t \geq 2$, the t-levels shadow graph of the path P_{dn+1}, $S_t(P_{dn+1})$ with $d \geq 1$ and $n \geq 1$ admits d-divisible α-labeling for all $d \geq 1$.*

Proof. Consider the path P_{dn+1}, where $d \geq 1$, $n \geq 1$.
For the convenience, we let $P_{dn+1} : v_1, v_2, \ldots, v_{dn}, v_{dn+1}$, $n \geq 1$, $d \geq 1$.
Suppose $G_1, G_2, \ldots, G_t$ are the t copies of P_{dn+1}.
Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i(dn+1)}\}$ be the vertex set of the i^{th} copy G_i of P_{dn+1}.

Then the t-levels shadow graph of the path P_{dn+1}, $S_t(P_{dn+1})$ has the vertex set $W = \bigcup_{i=1}^{t} V_i$.

Therefore, $|V(S_t(P_{dn+1}))| = t|V(P_{dn+1})| = t(dn + 1)$.
By the definition of the t-levels shadow graph of the path P_{dn+1}, the graph $S_t(P_{dn+1})$ can be visualised as t copies of the path P_{dn+1} and a pair of $t-1$ copies of P_{dn+1} which connect the vertices of the copies G_i and G_{i+1} of the path P_{dn+1}, $1 \leq i \leq t-1$.
Therefore, $|E(S_t(P_{dn+1}))| = tdn + 2(t-1)dn = (3t-2)dn$.
Since the path P_{dn+1} is bipartite, the i^{th} copy of P_{dn+1}, G_i is also bipartite having the bipartition (V_{i1}, V_{i2}), where
$V_{i1} = \{v_{ij}/1 \leq j \leq dn+1 \text{ and } j \text{ odd}\}$ and

$V_{i2} = \{v_{ij}/1 \leq j \leq dn+1 \text{ and } j \text{ even}\}$, for $1 \leq i \leq t$.
Let $N = d((3t-2)n+1)-1$.
Define $g : V(S_t(P_{dn+1})) \to \{0, 1, 2, \ldots, N\}$ in the following way.
$g(v_{12}) = N$.
For $1 \leq i \leq t$, $g(v_{i1}) = i-1$.
For $2 \leq i \leq t$, $g(v_{i2}) = g(v_{(i-1)2}) - 2$.
For all the remaining vertices of $S_t(P_{dn+1})$ we define g depending on $d = 1$ and $d > 1$.

When $d = 1$ define g as follows.

For $1 \leq j \leq \ell$, $g(v_{1(2j+1)}) = g(v_{1(2j-1)}) + 3t - 2$, where

$$\ell = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

For $2 \leq j \leq k$, $g(v_{1(2j)}) = g(v_{1(2j-2)}) - (3t-2)$, where

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

For $3 \leq j \leq dn+1$, $g(v_{ij}) = \begin{cases} g(v_{(i-1)j}) + 1, & \text{for } j \text{ odd and } 2 \leq i \leq t, \\ g(v_{(i-1)j}) - 2, & \text{for } j \text{ even and } 2 \leq i \leq t. \end{cases}$

When $d > 1$ then define g in two cases depending on n is even or n is odd.

Case a. n is even

$g(v_{1(2j+1)}) = g(v_{1(2j-1)}) + 3t - 2$, $1 \leq j \leq \frac{dn}{2}$,

$$g(v_{1(2j)}) = \begin{cases} g(v_{1(2j-2)}) - (3t-2), & 2 \leq j \leq \frac{dn}{2} \text{ and} \\ & j \neq \frac{kn+2}{2}, k = 1, 2, \ldots, d-1 \\ g(v_{1(kn)}) - (3t-1), & \text{for } j = \frac{kn+2}{2}, k = 1, 2, \ldots, d-1. \end{cases}$$

Case b. n is odd

$$g(v_{1(2j+1)}) = \begin{cases} g(v_{1(2j-1)}) + 3t - 2, & 1 \leq j \leq \ell, \ell = \frac{dn}{2} \text{ if } d \text{ is even,} \\ & \ell = \frac{dn-1}{2} \text{ if } d \text{ is odd and} \\ & j \neq \frac{kn+1}{2}, 1 \leq k \leq d-1 \text{ and } k \text{ odd,} \\ g(v_{1(kn)}) + 3t - 1, & \text{for } j = \frac{kn+1}{2}, 1 \leq k \leq d-1 \text{ and } k \text{ odd} \end{cases}$$

$$g(v_{1(2j)}) = \begin{cases} g(v_{1(2j-2)}) - (3t-2), & 2 \leq j \leq \ell, \ell = \frac{dn+1}{2} \text{ if } d \text{ is odd,} \\ & \ell = \frac{dn}{2} \text{ if } d \text{ is even and} \\ & j \neq \frac{kn+2}{2}, 2 \leq k \leq d-1 \text{ and } k \text{ even,} \\ g(v_{1(kn)}) - (3t-1), & \text{for } j = \frac{kn+2}{2}, 2 \leq k \leq d-1 \text{ and } k \text{ even} \end{cases}$$

For both the cases, for $3 \leq j \leq dn+1$, define

$$g(v_{ij}) = \begin{cases} g(v_{(i-1)j}) + 1, & \text{for } j \text{ odd and } 2 \leq i \leq t, \\ g(v_{(i-1)j}) - 2, & \text{for } j \text{ even and } 2 \leq i \leq t \end{cases}.$$

From the definition of g if the labels of the vertices of $S_t(P_{dn+1})$ are arranged as,
$g(v_{11}),\ g(v_{21}),\ \ldots,\ g(v_{t1}),$
$g(v_{13}),\ g(v_{23}),\ \ldots,\ g(v_{t3}),$
$g(v_{15}),\ g(v_{25}),\ \ldots,\ g(v_{t5}),$
$\vdots$
$g(v_{1(s-2)}),\ g(v_{2(s-2)}),\ \ldots,\ g(v_{t(s-2)}),$
$g(v_{1s}),\ g(v_{2s}),\ \ldots,\ g(v_{ts}),$
$g(v_{tk}),\ g(v_{(t-1)k}),\ \ldots,\ g(v_{1k}),$
$g(v_{t(k-2)}),\ g(v_{(t-1)(k-2)}),\ \ldots,\ g(v_{1(k-2)}),$
$\vdots$
$g(v_{t4}),\ g(v_{(t-1)4}),\ \ldots,\ g(v_{14}),$
$g(v_{t2}),\ g(v_{(t-1)2}),\ \ldots,\ g(v_{12}),$

where $s = \begin{cases} dn, & \text{if } dn+1 \text{ is even} \\ dn+1, & \text{if } dn+1 \text{ is odd,} \end{cases}$

$$k = \begin{cases} dn+1, & \text{if } dn+1 \text{ is even} \\ dn, & \text{if } dn+1 \text{ is odd,} \end{cases}$$

then the above sequence forms a strictly increasing sequence. Hence the vertex labels of $S_t(P_{dn+1})$ are distinct.

From the above arrangement of vertex labels observe that
$max\{g(u)/u \in V_{i1}, 1 \leq i \leq t\} = g(v_{t(dn)})$
$< min\{g(v)/v \in V_{i2}, 1 \leq i \leq t\} = g(v_{t(dn+1)})$, when $dn+1$ is even;
while when $dn+1$ is odd, $max\{g(u)/u \in V_{i1}, 1 \leq i \leq t\} = g(v_{t(dn+1)})$
$< min\{g(v)/v \in V_{i2}, 1 \leq i \leq t\} = g(v_{t(dn)})$.

We prove that the edge labels of $S_t(P_{dn+1})$ are distinct depending on $d = 1$ and $d > 1$.

Case 1. $d = 1$

When n is even, the edges of the graph $S_t(P_{dn+1})$ can be arranged as the following sequence, $(v_{12}v_{11},\ v_{12}v_{21},\ v_{11}v_{22},\ v_{21}v_{22},\ v_{31}v_{22},\ \ldots,\ v_{(i-1)1}v_{i2},\ v_{i1}v_{i2},\ v_{(i+1)1}v_{i2},\ \ldots,\ v_{(t-1)(n+1)}v_{tn},\ v_{tn}v_{t(n+1)})$.

When n is odd, the edges of $S_t(P_{dn+1})$ can be arranged as the following sequence, $(v_{12}v_{11},\ v_{12}v_{21},\ v_{11}v_{22},\ v_{21}v_{22},\ \ldots,\ v_{(i-1)1}v_{i2},\ v_{i1}v_{i2},\ v_{(i+1)1}v_{i2},\ \ldots,\ v_{(t-1)n}v_{t(n+1)},\ v_{tn}v_{t(n+1)})$.

Then from the definition of g for both the cases we have the corresponding edge label sequence,
$(N, N-1, N-2, \ldots, 3, 2, 1)$.
Hence, it is clear that the edge labels are distinct.
Therefore, when $d = 1$, g is a 1-divisible α-labeling of $S_t(P_{dn+1})$. That is, g is an α-labeling of the graph $S_t(P_{dn+1})$.

Case 2. $d > 1$

In order to show that the edge labels of the edges of $S_t(P_{dn+1})$ are distinct, we partition the edge set of $S_t(P_{dn+1})$ into d subsets of the edge set of $S_t(P_{dn+1})$ and they are arranged as d sequences. Consequently, their corresponding edge labels are also arranged as d sequences.

When n is even then we consider the first edge sequence to be the following sequence $(v_{12}v_{11},\ v_{12}v_{21},\ v_{11}v_{22},\ v_{21}v_{22},\ v_{31}v_{22},\ \ldots,\ v_{(i-1)1}v_{i2},\ v_{i1}v_{i2},\ v_{(i+1)1}v_{i2},\ \ldots,\ v_{(t-1)(n+1)}v_{tn},\ v_{tn}v_{t(n+1)})$.

When n is odd then we consider the first edge sequence to be the following sequence $(v_{12}v_{11},\ v_{12}v_{21},\ v_{11}v_{22},\ v_{21}v_{22},\ v_{31}v_{22},\ \ldots,\ v_{(i-1)1}v_{i2},\ v_{i1}v_{i2},\ v_{(i+1)1}v_{i2},\ \ldots,\ v_{(t-1)n}v_{t(n+1)},\ v_{tn}v_{t(n+1)})$.

Then from the definition of g for both the cases we have the corresponding edge label sequence,

$S_1 : (N, N-1, N-2, \ldots, (d-1)(3t-2)n+d-1, (d-1)(3t-2)n+d)$.

When n is even then we consider the second edge sequence to be the following sequence $(v_{1(n+2)}v_{1(n+1)}, v_{1(n+2)}v_{2(n+1)}, v_{1(n+1)}v_{2(n+2)}, v_{2(n+1)}v_{2(n+2)}, v_{3(n+1)}v_{2(n+2)}, \ldots, v_{(i-1)(n+1)}v_{i(n+2)},\ v_{i(n+1)}v_{i(n+2)},\ v_{(i+1)(n+1)}v_{i(n+2)},\ \ldots,\ v_{(t-1)(2n+1)}v_{t(2n)},\ v_{t(2n)}v_{t(2n+1)})$.

When n is odd then we consider the second edge sequence to be the following sequence $(v_{1(n+1)}v_{1(n+2)}, v_{1(n+1)}v_{2(n+2)}, v_{1(n+2)}v_{2(n+1)}, v_{2(n+2)}v_{2(n+1)}, v_{3(n+2)}v_{2(n+1)}, \ldots, v_{(i-1)(n+2)}v_{i(n+1)},\ v_{i(n+2)}v_{i(n+1)},\ v_{(i+1)(n+2)}v_{i(n+1)},\ \ldots,\ v_{(t-1)(2n+1)}v_{t(2n)},\ v_{t(2n)}v_{t(2n+1)})$.

Then from the definition of g for both the cases we have the corresponding edge label sequence,

$S_2 : ((d-1)(3t-2)n+d-2, (d-1)(3t-2)n+d-3, \ldots, (d-2)(3t-2)n+d, (d-2)(3t-2)n+d-1)$.

When n is even then we consider the third edge sequence to be the following sequence $(v_{1(2n+2)}v_{1(2n+1)},\ v_{1(2n+2)}v_{2(2n+1)},\ v_{1(2n+1)}v_{2(2n+2)},\ v_{2(2n+1)}v_{2(2n+2)},\ v_{3(2n+1)}v_{2(2n+2)},\ \ldots,\ v_{(i-1)(2n+1)}v_{i(2n+2)},\ v_{i(2n+1)}v_{i(2n+2)},\ v_{(i+1)(2n+1)}v_{i(2n+2)},\ \ldots,\ v_{(t-1)(3n+1)}v_{t(3n)},\ v_{t(3n)}v_{t(3n+1)})$.

When n is odd then we consider the third edge sequence to be the following sequence $(v_{1(2n+2)}v_{1(2n+1)},\ v_{1(2n+2)}v_{2(2n+1)},\ v_{1(2n+1)}v_{2(2n+2)},\ v_{2(2n+1)}v_{2(2n+2)},\ v_{3(2n+1)}v_{2(2n+2)},\ \ldots,\ v_{(i-1)(2n+1)}v_{i(2n+2)},\ v_{i(2n+1)}v_{i(2n+2)},\ v_{(i+1)(2n+1)}v_{i(2n+2)},\ \ldots,\ v_{(t-1)(3n)}v_{t(3n+1)},\ v_{t(3n)}v_{t(3n+1)})$.

Then from the definition of g for both the cases we have the corresponding edge label sequence,

$S_3\ :\ ((d-2)(3t-2)n+d-3,\ (d-2)(3t-2)n+d-4,\ \ldots,\ (d-3)(3t-2)n+d-1,\ (d-3)(3t-2)n+d-2)$.

In general, we consider the j^{th} edge sequence, for $4 \leq j \leq d-2$ depending on n and j.

Case i. n is even or n is odd and j is even

Then we consider the j^{th} edge sequence to be the following sequence $(v_{1(jn+2)}v_{1(jn+1)},\ v_{1(jn+2)}v_{2(jn+1)},\ v_{1(jn+1)}v_{2(jn+2)},\ v_{2(jn+1)}v_{2(jn+2)},\ v_{3(jn+1)}v_{2(jn+2)},\ \ldots,\ v_{(i-1)(jn+1)}v_{i(jn+2)},\ v_{i(jn+1)}v_{i(jn+2)},\ v_{(i+1)(jn+1)}v_{i(jn+2)},\ \ldots,\ v_{(t-1)((j+1)n)}v_{t((j+1)n+1)},\ v_{t((j+1)n)}v_{t((j+1)n+1)})$.

Case ii. n and j are odd

Then we consider the j^{th} edge sequence to be the following sequence $(v_{1(jn+1)}v_{1(jn+2)},\ v_{1(jn+1)}v_{2(jn+2)},\ v_{1(jn+2)}v_{2(jn+1)},\ v_{2(jn+2)}v_{2(jn+1)},\ v_{3(jn+2)}v_{2(jn+1)},\ \ldots,\ v_{(i-1)(jn+2)}v_{i(jn+1)},\ v_{i(jn+2)}v_{i(jn+1)},\ v_{(i+1)(jn+2)}v_{i(jn+1)},\ \ldots,\ v_{(t-1)((j+1)n+1)}v_{t((j+1)n)},\ v_{t((j+1)n)}v_{t((j+1)n+1)})$.

Then from the definition of g for all the above cases we have the corresponding edge label sequence,

$S_j : ((d-j)(3t-2)n+d-(j+1), (d-j)(3t-2)n+d-(j+2), (d-j)(3t-2)n+d-(j+3), \ldots, (d-(j+1))(3t-2)n+d-(j+2), (d-(j+1))(3t-2)n+d-(j-1), (d-(j+1))(3t-2)n+d-j)$.

Now we consider the $(d-1)^{th}$ edge sequence depending on n is even or odd.

Case I. n is even
Then we consider the $(d-1)^{th}$ edge sequence to be the following sequence
$(v_{1((d-2)n+2)}v_{1((d-2)n+1)},\ v_{1((d-2)n+2)}v_{2((d-2)n+1)},\ v_{1((d-2)n+1)}v_{2((d-2)n+2)},\ v_{2((d-2)n+1)}v_{2((d-2)n+2)},$
$v_{3((d-2)n+1)}v_{2((d-2)n+2)},\ \ldots,\ v_{(i-1)((d-2)n+1)}v_{i((d-2)n+2)},\ v_{i((d-2)n+1)}v_{i((d-2)n+2)},$
$v_{(i+1)((d-2)n+1)}v_{i((d-2)n+2)},\ \ldots,\ v_{(t-1)((d-1)n+1)}v_{t((d-1)n)},\ v_{t((d-1)n)}v_{t((d-1)n+1)})$.
Case II. n is odd
Then we consider the $(d-1)^{th}$ edge sequence in the following subcases depending on $d-1$ is even or odd.
Case IIa. $d-1$ is even
Then we consider the $(d-1)^{th}$ edge sequence to be the following sequence
$(v_{1((d-2)n+1)}v_{1((d-2)n+2)},\ v_{1((d-2)n+1)}v_{2((d-2)n+2)},\ v_{1((d-2)n+2)}v_{2((d-2)n+1)},\ v_{2((d-2)n+2)}v_{2((d-2)n+1)},$
$v_{3((d-2)n+2)}v_{2((d-2)n+1)},\ \ldots,\ v_{(i-1)((d-2)n+2)}v_{i((d-2)n+1)},\ v_{i((d-2)n+2)}v_{i((d-2)n+1)},$
$v_{(i+1)((d-2)n+2)}v_{i((d-2)n+1)},\ldots,\ v_{(t-1)((d-1)n+1)}v_{t((d-1)n)},\ v_{t((d-1)n+1)}v_{t((d-1)n)})$.
Case IIb. $d-1$ is odd
Then we consider the $(d-1)^{th}$ edge sequence to be the following sequence
$(v_{1((d-2)n+2)}v_{1((d-2)n+1)},\ v_{1((d-2)n+2)}v_{2((d-2)n+1)},\ v_{1((d-2)n+1)}v_{2((d-2)n+2)},\ v_{2((d-2)n+1)}v_{2((d-2)n+2)},$
$v_{3((d-2)n+1)}v_{2((d-2)n+2)},\ \ldots,\ v_{(i-1)((d-2)n+1)}v_{i((d-2)n+2)},\ v_{i((d-2)n+1)}v_{i((d-2)n+2)},$
$v_{(i+1)((d-2)n+1)}v_{i((d-2)n+2)},\ \ldots,\ v_{(t-1)((d-1)n)}v_{t((d-1)n+1)},\ v_{t((d-1)n)}v_{t((d-1)n+1)})$.
Then from the definition of g for all the above cases we have the corresponding edge label sequence,
$S_{d-1}: (2(3t-2)n+1,\ 2(3t-2)n,\ 2(3t-2)n-1,\ \ldots,\ (3t-2)n+3,\ (3t-2)n+2)$.
Finally, we consider the d^{th} edge sequence depending on n is even or odd.
Case 1. n is even
Then we consider the d^{th} edge sequence to be the following sequence
$(v_{1((d-1)n+2)}v_{1((d-1)n+1)},\ v_{1((d-1)n+2)}v_{2((d-1)n+1)},\ v_{1((d-1)n+1)}v_{2((d-1)n+2)},\ v_{2((d-1)n+1)}v_{2((d-1)n+2)},$
$v_{3((d-1)n+1)}v_{2((d-1)n+2)},\ \ldots,\ v_{(i-1)((d-1)n+1)}v_{i((d-1)n+2)},\ v_{i((d-1)n+1)}v_{i((d-1)n+2)},$
$v_{(i+1)((d-1)n+1)}v_{i((d-1)n+2)},\ \ldots,\ v_{(t-1)(dn+1)}v_{t(dn)},\ v_{t(dn)}v_{t(dn+1)})$.
Case 2. n is odd
Then we consider the d^{th} edge sequence in the following subcases depending on d is even or odd.
Case 2a. d is even
Then we consider the d^{th} edge sequence to be the following sequence
$(v_{1((d-1)n+1)}v_{1((d-1)n+2)},\ v_{1((d-1)n+1)}v_{2((d-1)n+2)},\ v_{1((d-1)n+2)}v_{2((d-1)n+1)},\ v_{2((d-1)n+2)}v_{2((d-1)n+1)},$
$v_{2((d-1)n+1)}v_{3((d-1)n+2)},\ \ldots,\ v_{(i-1)((d-1)n+2)}v_{i((d-1)n+1)},\ v_{i((d-1)n+2)}v_{i((d-1)n+1)},$
$v_{(i+1)((d-1)n+2)}v_{i((d-1)n+1)},\ \ldots,\ v_{(t-1)(dn+1)}v_{t(dn)},\ v_{t(dn)}v_{t(dn+1)})$.
Case 2b. d is odd
Then we consider the d^{th} edge sequence to be the following sequence
$(v_{1((d-1)n+2)}v_{1((d-1)n+1)},\ v_{1((d-1)n+2)}v_{2((d-1)n+1)},\ v_{1((d-1)n+1)}v_{2((d-1)n+2)},\ v_{2((d-1)n+1)}v_{2((d-1)n+2)},$
$v_{3((d-1)n+1)}v_{2((d-1)n+2)},\ \ldots,\ v_{(i-1)((d-1)n+2)}v_{i((d-1)n+1)},\ v_{i((d-1)n+2)}v_{i((d-1)n+1)},$
$v_{(i+1)((d-1)n+2)}v_{i((d-1)n+1)},\ \ldots,\ v_{(t-1)(dn)}v_{t(dn+1)},\ v_{t(dn)}v_{t(dn+1)})$.
Then from the definition of g for all the above cases we have the corresponding edge label sequence,
$S_d: ((3t-2)n,\ (3t-2)n-1,\ (3t-2)n-2,\ \ldots,\ 3,\ 2,\ 1)$.

Using all the above defined edge label sequences $S_1,\ S_2,\ S_3,\ \ldots,\ S_j,\ \ldots,\ S_{d-1},\ S_d$, we form a combined edge label sequence in the order as $S:\ (S_1,\ S_2,\ S_3,\ \ldots,\ S_j,\ \ldots,\ S_{d-1},$

S_d). Then we observe that S forms a monotonically decreasing sequence. Also observe that none of the terms $(d-1)((3t-2)n+1)$, $(d-2)((3t-2)n+1)$, $\ldots$, $3((3t-2)n+1)$, $2((3t-2)n+1)$, $(3t-2)n+1$ appear in the combined sequence S. Thus, g is a d-divisible α-labeling of $S_t(P_{dn+1})$ for any admissible $d > 1$. Therefore the graph $S_t(P_{dn+1})$ admits d-divisible α-labeling for any admissible d. □

Illustration

The 4-divisible α-labeling, 3-divisible α-labeling and 2-divisible α-labeling that are defined as in the proof of Theorem 3.1 for the graphs $S_4(P_5)$, $S_4(P_{10})$, $S_4(P_9)$ are given in Figures 3, 4, 5 respectively.

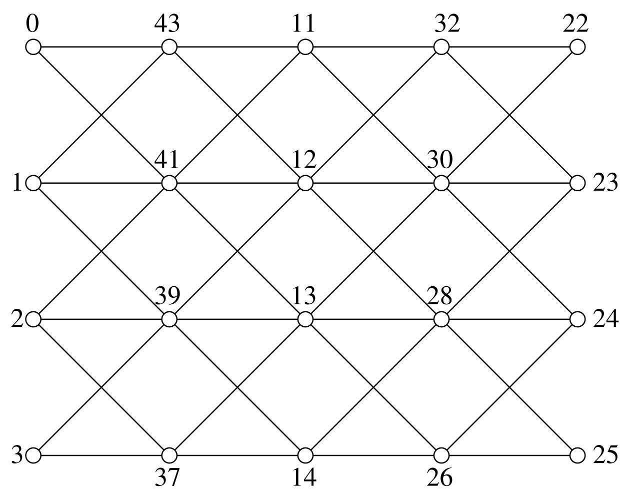

Figure 3: 4-divisible α-labeling of $S_4(P_5)$

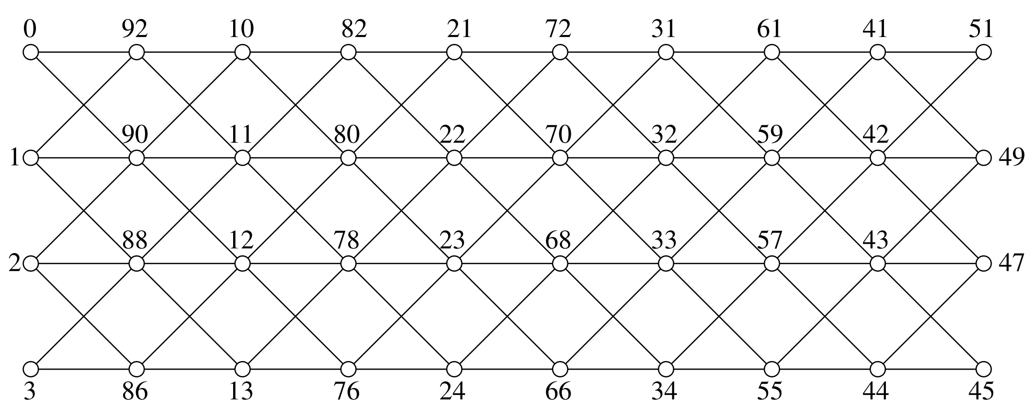

Figure 4: 3-divisible α-labeling of $S_4(P_{10})$

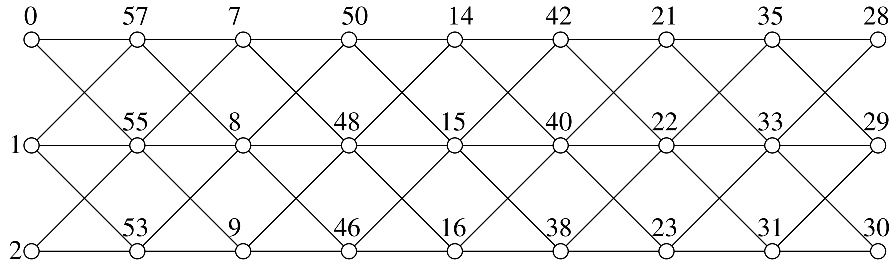

Figure 5: 2-divisible α-labeling of $S_3(P_9)$

The following corollary is an immediate implication of Anita Pasotti's theorem, Theorem 1.6.

Corollary 3.2. *The multipartite graph $K_{(\frac{e}{d}+1)\times 2dm}$ can be cyclically decomposed into copies of the t-levels shadow graph of the path P_{dn+1}, $S_t(P_{dn+1})$, where $e = |E(S_t(P_{dn+1}))|$, $t \geq 2$, $d \geq 1$, $n \geq 1$ and m is any positive integer.*

4 Discussion

In this section we pose two open problems for further research.

In Theorem 2.1 we have proved that for $t \geq 2$, disjoint union of t copies of the complete bipartite graph $K_{m,n}$ plus an edge, $(\bigcup_{i=1}^{t} K^i_{m,n}) + \hat{e}$ admits γ-labeling. In this direction investigating the following question will be useful for achieving a generalised result.

Is it true that disjoint union of t copies of an α-labeled graph G plus an edge, $t \geq 2$, admits γ-labeling?

In Theorem 3.1 we have proved that for $t \geq 2$, the t-levels shadow graph of the path P_{dn+1} with $d \geq 1, n \geq 1$ admits d divisible α-labeling for all $d \geq 1$. It is evident that the path P_{dn+1} admits α-labeling for all $d \geq 1, n \geq 1$. This observation tempts us to ask the following question to understand d-divisible α-labeled graphs.

What are the α-labeled graphs whose t-levels shadow graph admits d divisible α-labeling for all values of d?

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