Decomposition of Certain Complete Graphs and Complete Multipartite Graphs into Almost-bipartite Graphs and Bipartite Graphs

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Abstract

In his classical paper [14], Rosa introduced a hierarchical series of labelings called $\rho, \sigma, \beta$ and $\alpha$ labeling as a tool to settle Ringel’s Conjecture [13] which states that if $T$ is any tree with $m$ edges then the complete graph $K_{2m+1}$ can be decomposed into $2m + 1$ copies of $T$. Inspired by the result of Rosa [14] many researchers significantly contributed to the theory of graph decompositions using graph labelings. In this direction, in 2004, Blinco et al. [6] introduced $\gamma$-labeling as a stronger version of $\rho$-labeling. A function $g$ defined on the vertex set of a graph $G$ with $n$ edges is called a $\gamma$-labeling if

(i) $g$ is a $\rho$-labeling of $G$,

(ii) $G$ is a tripartite graph with vertex tripartition $(A, B, C)$ with $C = \{c\}$ and $\bar{b} \in B$ such that $\{b, c\}$ is the unique edge joining an element of $B$ to $c$,

(iii) $g(a) < g(v)$ for every edge $\{a, v\} \in E(G)$ where $a \in A$,

(iv) $g(c) - g(\bar{b}) = n$.

Further, Blinco et al. [6] proved a significant result that the complete graph $K_{2cn+1}$ can be cyclically decomposed into $c(2cn + 1)$ copies of any $\gamma$-labeled graph with $n$ edges, where $c$ is any positive integer. Recently, in 2013, Anita Pasotti [4] introduced a generalisation of graceful labeling called $d$-divisible graceful labeling as a tool to obtain cyclic decompositions in complete multipartite graphs. Let $G$ be a graph of size $e = d \cdot m$. A $d$-divisible graceful labeling of the graph $G$ is an injective function $g : V(G) \to \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that $\{|g(u) - g(v)|/\{u, v\} \in E(G)\} = \{1, 2, \ldots, d(m + 1) - 1\} \setminus \{m + 1, 2m + 1, \ldots, (d - 1)(m + 1)\}$. A $d$-divisible graceful labeling of a bipartite graph $G$ is called as a $d$-divisible $\alpha$-labeling of $G$ if the maximum value of one of the two bipartite sets is less than the minimum value of the other one. Further, Anita Pasotti [4] proved a significant result that the complete multipartite graph $K_{(\frac{e}{d}+1) \times 2dc}$ can be cyclically decomposed into copies of $d$-divisible $\alpha$-labeled graph $G$, where $e$ is the size of the graph $G$ and $c$ is any positive integer $(K_{(\frac{e}{d}+1) \times 2dc}$ contains $\frac{e}{d} + 1$ parts each of size $2dc)$. Motivated by the results of Blinco et al. [6] and Anita Pasotti [4], in this paper we prove the following results.

i) For $t \geq 2$, disjoint union of $t$ copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3, n \geq 4$ plus an edge admits $\gamma$-labeling.

ii) For $t \geq 2$, $t$-levels shadow graph of the path $P_{dn+1}$ admits $d$-divisible $\alpha$-labeling for any admissible $d$ and $n \geq 1$.

Further, we discuss related open problems.

1 Introduction

Terms which are not defined here can be found in [15]. In an attempt to settle the Ringel’s conjecture [13] which states that if $T$ is any tree with $m$ edges then the complete graph $K_{2m+1}$ can be decomposed into $2m + 1$ copies of $T$, in his classical paper [14], Rosa introduced a series of labelings called $\alpha, \beta, \sigma, \rho$-labeling.

Let $G$ be a graph with $n$ edges. A one-to-one function $g$ from $V(G)$ to $\{0, 1, 2, \ldots, n\}$ is called a $\beta$-labeling of $G$ if $\{|g(u) - g(v)|/\{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. A $\beta$-labeling $g$ of a
graph $G$ with $n$ edges is called an $\alpha$-labeling if there exists an integer $k$ such that for every edge $\{u,v\} \in E(G)$ either $g(u) \leq k < g(v)$ or $g(v) \leq k < g(u)$. Given two vertices $u$ and $v$ by $uv$ we denote the edge $\{u,v\}$.

It is clear that $\alpha$-labeling is a stronger version of $\beta$-labeling. $\beta$-labeling was later called as graceful labeling by Golomb [12] and this term is most widely used now. $\rho$-labeling is weaker version of graceful labeling. The precise definition of $\rho$-labeling is given below. Let $G$ be a graph with $n$ edges. A one-to-one function $g$ from $V(G)$ to $\{0, 1, 2, \ldots, 2n\}$ is called a $\rho$-labeling of $G$ if

$\frac{\min\{|g(u) - g(v)|, 2n + 1 - |g(u) - g(v)|\}}{\{u, v\} \in E(G)} = \{1, 2, \ldots, n\}$.

Further, Rosa [14] proved the following two significant theorems.

**Theorem 1.1.** Let $G$ be a graph with $n$ edges. Then there exists a cyclic $G$-decomposition of the complete graph $K_{2n+1}$ if and only if $G$ has a $\rho$-labeling.

**Theorem 1.2.** If $G$ is a graph with $n$ edges that has an $\alpha$-labeling, then the complete graph $K_{2cn+1}$ can be cyclically decomposed into subgraphs isomorphic to $G$, where $c$ is an arbitrary natural number.

The interesting part of $\alpha$-labeled graphs with $n$ edges is that they not only decompose complete graphs $K_{2cn+1}$ but also decompose the complete bipartite graphs $K_{an,bn}$. This interesting result proved by El-Zanati and Vanden Eynden [9] is precisely stated in the following theorem.

**Theorem 1.3.** If a graph $G$ with $n$ edges has an $\alpha$-labeling then there exists a cyclic decomposition of the complete bipartite graph $K_{an,bn}$ into subgraphs isomorphic to $G$, where $a$ and $b$ are arbitrary positive integers.

These results attracted many researchers to significantly contribute in theory of graph decompositions using graph labelings. It is clear from the definition of $\alpha$-labeling that if a graph $G$ admits $\alpha$-labeling then it must be necessarily bipartite. This restriction prompted Blinco et al. [6] to introduce $\gamma$-labeling in order to achieve cyclic $G$-decompositions in $K_{2cn+1}$, where $G$ is a non-bipartite graph, $c$ is any positive integer and $n$ is the number of edges of the graph $G$. A function $g$ defined on the vertex set of a graph $G$ with $n$ edges is called a $\gamma$-labeling if

(i) $g$ is a $\rho$-labeling of $G$,

(ii) $G$ is a tripartite graph with vertex tripartition $(A, B, C)$ with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of $B$ to $c$,

(iii) $g(a) < g(v)$ for every edge $\{a, v\} \in E(G)$ where $a \in A$,

(iv) $g(c) - g(\bar{b}) = n$.

Further, in [6], Blinco et al. have proved the following significant theorem.

**Theorem 1.4.** The complete graph $K_{2cm+1}$ can be cyclically decomposed into copies of the $\gamma$-labeled graph $G$, where $m$ is the number of edges of the graph $G$ and $c$ is any positive integer.
Motivated by the above result of Blinco et al. [6], the almost-bipartite graphs $P_n + e$, $n \geq 4$, $K_{m,n} + e$, $m \geq 2$, $n > 2$, $C_{2k+1}$, $k \geq 2$, $C_{2m} + e$, $m > 2$, $C_3 \cup C_4m$, $m > 1$, $C_{2k+1} \cup C_{4n+2}$, $k \geq 1$, $n \geq 1$ are found to have $\gamma$-labeling (refer [5], [6], [7], [8], [10]). (A graph is said to be almost-bipartite if the removal of a particular edge makes the graph bipartite). For survey on $\gamma$-labeling refer the survey on graph labelings by Gallian [11]. Motivated by the results of Blinco et al. [6], in this paper we prove that for $t \geq 2$, disjoint union of $t$ copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3, n \geq 4$ plus an edge admits $\gamma$-labeling.

Recently, in 2013, Anita Pasotti [4] introduced a generalisation of graceful labeling called $d$-divisible graceful labeling as a tool to obtain cyclic $G$-decomposition in complete multipartite graphs. Let $G$ be a graph of size $e = d \cdot m$. An injective function $g : V(G) \rightarrow \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that $\{|g(u) - g(v)|/\{u,v\} \in E(G)\} = \{1, 2, \ldots, d(m + 1) - 1\}\{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}$ is called as a $d$-divisible graceful labeling of the graph $G$. A $d$-divisible graceful labeling of a graph $G$ can exist only if $d$ is a divisor of the size $e$ of $G$, hence, for this reason, any divisor $d$ of $e$ is said to be admissible for the existence of a $d$-divisible graceful labeling of $G$. A $d$-divisible graceful labeling of a bipartite graph $G$ is called as a $d$-divisible $\alpha$-labeling of $G$ if the maximum value of one of the two bipartite sets is less than the minimum value of the other one.

Further, Anita Pasotti [4] has proved the following significant theorems.

**Theorem 1.5.** (Anita Pasotti [4]) The complete multipartite graph $K_{(\frac{k}{d} + 1) \times 2d}$ can be cyclically decomposed into copies of the $d$-divisible graceful labeled graph $G$, where $e$ is the size of the graph $G$.

**Theorem 1.6.** (Anita Pasotti [4]) The complete multipartite graph $K_{(\frac{k}{d} + 1) \times 2dc}$ can be cyclically decomposed into copies of the $d$-divisible $\alpha$-labeled graph $G$, where $e$ is the size of the graph $G$ and $c$ is any positive integer.

In the literature survey [11], one can observe that a very few families of graphs are identified to have $d$-divisible $\alpha$-labeling. Anita Pasotti [4] has proved that path and star admit $d$-divisible $\alpha$-labeling for any admissible $d$. She [3] also proved that for any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_n$ admits $(2m - 1)$-divisible $\alpha$-labeling. In [1] and [2], Anna Benini and Anita Pasotti proved the following results. A hairy cycle of size $e$ admits an $e$-divisible $\alpha$-labeling if and only if it is bipartite. The hairy cycle $H(2t, \lambda)$ admits $d$-divisible $\alpha$-labeling for any admissible $d$. The ladder $L_{2k}$ has $2$-divisible $\alpha$-labeling if and only if $k$ is even.

Inspired by the decomposition theorems proved by Anita Pasotti, in this paper we prove that for $t \geq 2$, $t$-levels shadow graph of the path $P_{dn+1}$ admits $d$-divisible $\alpha$-labeling for any admissible $d$ and $n \geq 1$. $t$-levels shadow graph of a graph is defined as follows. $t$-levels shadow graph of a graph $G$, denoted $S_t(G)$ is obtained by taking $t \geq 2$ copies $G_1, G_2, \ldots, G_t$ of $G$ and joining each vertex $v_{ij}$ in $G_i$ to the copies of its adjacent vertices in $G_{i+1}$, for $1 \leq j \leq n$ and $1 \leq i \leq t - 1$, where $n = |V(G)|$.

## 2 $\gamma$-labeling of disjoint union of complete bipartite graphs plus an edge

In this section we prove that disjoint union of $t$ copies of the complete bipartite graph $K_{m,n}$, where $m \geq 3$ and $n \geq 4$ plus an edge admits $\gamma$-labeling.
**Theorem 2.1.** For \( t \geq 2 \), disjoint union of \( t \) copies of a complete bipartite graph with one part containing at least three vertices and another part containing at least four vertices, plus an edge admits \( \gamma \)-labeling.

**Proof.** Consider the complete bipartite graph \( K_{m,n} \), where \( m \geq 3, n \geq 4 \).

Let \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\} \) be the two parts of \( K_{m,n} \).

For any \( i = 1, 2, \ldots, t \), let \( U_i = \{u_1, u_2, \ldots, u_m\} \) and \( V_i = \{v_1, v_2, \ldots, v_n\} \) be the two parts of the \( i \)-th copy \( K_{m,n}^i \) of the complete bipartite graph \( K_{m,n} \).

Set \( U = \bigcup_{i=1}^{t} U_i \) and \( V = \bigcup_{i=1}^{t} V_i \).

Clearly, \( U \) and \( V \) are the two parts of the disjoint union of the \( t \) copies of \( K_{m,n} \), denoted by \( \bigcup_{i=1}^{t} K_{m,n}^i \).

Join the vertices \( v_{11} \) and \( v_{12} \) by an edge \( \hat{e} \).

Denote the new graph thus obtained by \( \bigcup_{i=1}^{t} K_{m,n}^i + \hat{e} \).

Observe that \( |V(\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e})| = t(m+n) \) and \( |E(\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e})| = tmn + 1 \).

Define \( g : V(\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}) \rightarrow \{0, 1, 2, \ldots, 2N\} \), where \( N = tmn + 1 \) in the following way.

First we define the labels of the vertices in the set \( U \) in the following way.

For \( 1 \leq j \leq m \), define \( g(u_{1j}) = 2(j-1) \) and \( g(u_{2j}) = 2j + 1 \).

For each \( i \), \( 3 \leq i \leq t \), define

\[
g(u_{ij}) = g(u_{i(j-1)}) + m, \quad g(u_{ij}) = (g(u_{ij-1})) + 1, \text{ for each } j, \ 2 \leq j \leq m.
\]

Now we define the labels of the vertices in the set \( V \) in the following manner.

Define \( g(v_{11}) = 2N - 1 \), \( g(v_{12}) = N - 1 \), \( g(v_{13}) = 2N \), \( g(v_{14}) = N - 2 \).

For \( 5 \leq k \leq n \), define

\[
g(v_{1k}) = \begin{cases} g(v_{1(k-1)}) - 2m + 1, & \text{if } k \text{ is odd} \\ g(v_{1(k-1)}) - 1, & \text{if } k \text{ is even} \end{cases}
\]

Define \( g(v_{21}) = \begin{cases} g(v_{1n}) - 4(r-1), & \text{if } m = 2r, r \geq 2 \text{ and } n \text{ is even} \\ g(v_{1n}) - 4(r-2), & \text{if } m = 2r + 1, r \geq 1 \text{ and } n \text{ is even} \\ g(v_{1n}) + 2, & \text{if } n \text{ is odd} \end{cases} \)

We define the labels of the vertices \( v_{2k} \), for \( 2 \leq k \leq n \) in two cases depending on \( n \) is even or odd.

**Case 1.** \( n \) is even

For \( 2 \leq k \leq n \), define

\[
g(v_{2k}) = \begin{cases} g(v_{2(k-1)}) - 1, & \text{if } k \text{ is even} \\ g(v_{2(k-1)}) - 2m + 1, & \text{if } k \text{ is odd} \end{cases}
\]
Case 2. $n$ is odd
For $2 \leq k \leq n$, define
\[
g(v_{2k}) = \begin{cases} 
g(v_{2(k-1)}) - 2m + 1, & \text{if } k \text{ is even} \\
g(v_{2(k-1)}) - 1, & \text{if } k \text{ is odd}. \end{cases}
\]
For each $i, 3 \leq i \leq t$, define the labels of the vertices $v_{ik}$, for each $k, 2 \leq k \leq n$ in the following way.
For each $i, 3 \leq i \leq t$, define
\[
g(v_{1i}) = g(v_{(i-1)n}) + m - 1,
\]
and
\[
g(v_{ik}) = g(v_{(k-1)}) - m, \text{ for each } k, 2 \leq k \leq n.
\]

Observation 1. Vertex labels of $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are distinct.

We prove that the vertex labels of the graph $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are distinct depending on $n$ is even or odd.

Case 1. $n$ is even
If the labels of the vertices of the graph $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are arranged as,
\[
g(u_{11}), g(u_{12}), g(u_{21}), g(u_{13}), g(u_{22}), g(u_{14}), \ldots, g(u_{1m}), g(u_{2(m-1)}), g(u_{2m}), (g(u_{ij}))_{i=1, j=1}^{t, m},
g(v_{1n}), g(v_{(n-1)}), g(v_{(n-2)}), \ldots, g(v_{1}), g(v_{(t-1)n}), g(v_{(t-1)(n-1)}), g(v_{(t-1)(n-2)}), \ldots, g(v_{(t-2)}), g(v_{(t-2)(n-2)}), \ldots, g(v_{(t-3)}), g(v_{(3)(n-3)}), \ldots, g(v_{(3)}), g(v_{(2)n}), g(v_{(2)(n-2)}), \ldots, g(v_{(2)}), g(v_{(1)n}), g(v_{(1)(n-1)}), g(v_{(1)(n-2)}), g(v_{(1)(n-3)}), \ldots, g(v_{14}), g(v_{12}), g(v_{11}), g(v_{13}),
\]
then it forms a monotonically increasing sequence.

Case 2. $n$ is odd
If the labels of the vertices of the graph $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are arranged as,
\[
g(u_{11}), g(u_{12}), g(u_{21}), g(u_{13}), g(u_{22}), g(u_{14}), \ldots, g(u_{1m}), g(u_{2(m-1)}), g(u_{2m}), (g(u_{ij}))_{i=1, j=1}^{t, m},
g(v_{1n}), g(v_{(n-1)}), g(v_{(n-2)}), \ldots, g(v_{1}), g(v_{(t-1)n}), g(v_{(t-1)(n-1)}), g(v_{(t-1)(n-2)}), \ldots, g(v_{(t-2)}), g(v_{(t-2)(n-2)}), \ldots, g(v_{(t-3)}), g(v_{(3)(n-3)}), \ldots, g(v_{(3)}), g(v_{(2)n}), g(v_{(2)(n-2)}), \ldots, g(v_{(2)}), g(v_{(1)n}), g(v_{(1)(n-1)}), g(v_{(1)(n-2)}), g(v_{(1)(n-3)}), \ldots, g(v_{14}), g(v_{12}), g(v_{11}), g(v_{13}),
\]
then it forms a monotonically increasing sequence.

Hence the vertex labels of the graph $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are distinct.

Observation 2. Edge labels of $\bigcup_{i=1}^{t} K_{m,n}^i + \hat{e}$ are distinct.

The edge $v_{11}v_{12}$ has the label $N$. 
We prove that the edge labels of $\bigcup_{i=1}^{t} K_{m,n}^i$ are distinct in two cases depending on $n$ is even or odd.

**Case i.** $n$ is even

The labels of the edges in the first copy $K_{m,n}^1$ can be arranged as a sequence,

$$S_{11} : ((N - 1, N - 2, N - 3, \ldots, N + 2m - 1 - mn, N + 2m - 2mn), (2m, 2m - 1, \ldots, 2, 1)).$$

For each $i, 2 \leq i \leq t$, the labels of the edges in the $i^{th}$ copy $K_{m,n}^i$ can be arranged as a sequence,

$$S_{1i} : (N + 2m - (i - 1)mn - 1, N + 2m - (i - 1)mn - 2, \ldots, N + 2m - imn + 2, N + 2m - imn + 1, N + 2m - imn).$$

The labels of the edges in the above sequences together with the label of the edge $v_1v_2$, $|g(v_{11}) - g(v_{12})| = N$ can be rearranged as a monotonic decreasing sequence

$$S : (N, N - 1, N - 2, \ldots, 3, 2, 1).$$

Thus the edge labels are distinct when $n$ is even.

**Case ii.** $n$ is odd

The labels of the edges in the first copy $K_{m,n}^1$ can be arranged as a sequence,

$$S_{21} : ((N - 1, N - 2, N - 3, \ldots, N + 2m - mn + 2, N + 2m - mn + 1, N + 2m - mn),$$

$$(N + 3m - mn - 1, N + 3m - mn - 3, N + 3m - mn - 5, \ldots, N + m - mn + 3, N + m - mn + 1), (2m, 2m - 1, \ldots, 2, 1)).$$

The labels of the edges in the second copy $K_{m,n}^2$ can be arranged as a sequence,

$$S_{22} : (N + 3m - mn - 2, N + 3m - mn - 4, N + 3m - mn - 6, \ldots, N + m - mn + 2, N + m - mn, N + m - mn - 1, N + m - mn - 2, N + m - mn - 3, \ldots, N + 2m - 2mn + 2, N + 2m - 2mn + 1, N + 2m - 2mn).$$

For each $i, 3 \leq i \leq t$, the labels of the edges in the $i^{th}$ copy $K_{m,n}^i$ can be arranged as a sequence,

$$S_{2i} : (N + 2m - (i - 1)mn - 1, N + 2m - (i - 1)mn - 2, \ldots, N + 2m - imn + 2, N + 2m - imn + 1, N + 2m - imn).$$

The labels of the edges in the above sequences together with the label of the edge $v_1v_2$, $|g(v_{11}) - g(v_{12})| = N$ can be rearranged as a monotonic decreasing sequence

$$S : (N, N - 1, N - 2, \ldots, 3, 2, 1).$$

Thus the edge labels are distinct when $n$ is odd.

Hence the edge labels of the graph $\bigcup_{i=1}^{t} K_{m,n}^i + e$ are distinct.
Observation 3. $g$ is a $\gamma$-labeling.

In order to prove that $g$ is a $\gamma$-labeling, we partition the vertex set $V((\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e})$ as $(A, B, C)$, where $A = U, B = V \setminus \{v_{11}\}$ and $C = \{v_{11}\}$. Then, by the above labeling we have $g(u_{ij}) < g(v_{ik})$ for any $u_{ij} \in A$ and for any $v_{ik} \in B \cup C$. The label of the edge $v_{11}v_{12} = N = (2N - 1 - (N - 1))$. Hence, the graph $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ admits $\gamma$-labeling. □

Illustration

$\gamma$-labeling that is defined as in the proof of Theorem 2.1 for the disjoint union of three copies of the complete bipartite graph $K_{4,5}$ plus an edge, $(\bigcup_{i=1}^{3} K_{4,5}^i) + \hat{e}$ and the disjoint union of two copies of the complete bipartite graph $K_{3,4}$ plus an edge, $(\bigcup_{i=1}^{2} K_{3,4}^i) + \hat{e}$ are given in Figure 1 and Figure 2 respectively.

![Figure 1: $\gamma$-labeling of $(\bigcup_{i=1}^{3} K_{4,5}^i) + \hat{e}$](image)
The following corollary is an immediate implication of Blinco et al.’s theorem, Theorem 1.4.

**Corollary 2.2.** The complete graph $K_{2cr+1}$ can be cyclically decomposed into copies of the graph $(\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e}$, where $c$ is any positive integer, $m \geq 3, n \geq 4, t \geq 2$ and $r = |E((\bigcup_{i=1}^{t} K_{m,n}^{i}) + \hat{e})|$.

### 3 $d$-divisible $\alpha$-labeling of $t$-levels shadow graph of path

In this section we prove that for $t \geq 2$, $t$-levels shadow graph of the path $P_{dn+1}$, $S_{t}(P_{dn+1})$ with $d \geq 1, n \geq 1$ admits $d$-divisible $\alpha$-labeling for all $d \geq 1$.

**Theorem 3.1.** For $t \geq 2$, the $t$-levels shadow graph of the path $P_{dn+1}$, $S_{t}(P_{dn+1})$ with $d \geq 1$ and $n \geq 1$ admits $d$-divisible $\alpha$-labeling for all $d \geq 1$.

**Proof.** Consider the path $P_{dn+1}$, where $d \geq 1, n \geq 1$.

For the convenience, we let $P_{dn+1} : v_{1}, v_{2}, \ldots, v_{dn}, v_{dn+1}$, $n \geq 1, d \geq 1$.

Suppose $G_{1}, G_{2}, \ldots, G_{t}$ are the $t$ copies of $P_{dn+1}$.

Let $V_{i} = \{v_{i1}, v_{i2}, \ldots, v_{i(dn+1)}\}$ be the vertex set of the $i^{th}$ copy $G_{i}$ of $P_{dn+1}$.

Then the $t$-levels shadow graph of the path $P_{dn+1}$, $S_{t}(P_{dn+1})$ has the vertex set $W = \bigcup_{i=1}^{t} V_{i}$.

Therefore, $|V(S_{t}(P_{dn+1}))| = t|V(P_{dn+1})| = t(dn + 1)$.

By the definition of the $t$-levels shadow graph of the path $P_{dn+1}$, the graph $S_{t}(P_{dn+1})$ can be visualised as $t$ copies of the path $P_{dn+1}$ and a pair of $t - 1$ copies of $P_{dn+1}$ which connect the vertices of the copies $G_{i}$ and $G_{i+1}$ of the path $P_{dn+1}, 1 \leq i \leq t - 1$.

Therefore, $|E(S_{t}(P_{dn+1}))| = tdn + 2(t - 1)dn = (3t - 2)dn$.

Since the path $P_{dn+1}$ is bipartite, the $i^{th}$ copy of $P_{dn+1}$, $G_{i}$ is also bipartite having the bipartition $(V_{i1}, V_{i2})$, where $V_{i1} = \{v_{ij} / 1 \leq j \leq dn + 1$ and $j$ odd$\}$ and
Let $V_2 = \{v_{ij}/1 \leq j \leq dn + 1 \text{ and } j \text{ even} \}$, for $1 \leq i \leq t$.

Let $N = d(3t - 2)n + 1 - 1$.

Define $g : V(S_t(P_{dn+1})) \rightarrow \{0, 1, 2, \ldots, N\}$ in the following way.

For $2 \leq i \leq t$, $g(v_i) = i - 1$.

For all the remaining vertices of $S_t(P_{dn+1})$ we define $g$ depending on $d = 1$ and $d > 1$.

**When $d = 1$ define $g$ as follows.**

For $1 \leq j \leq \ell$, $g(v_{1(2j+1)}) = g(v_{1(2j-1)}) + 3t - 2$, where

\[
\ell = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even}, \\
\frac{n-1}{2}, & \text{if } n \text{ is odd.}
\end{cases}
\]

For $2 \leq j \leq k$, $g(v_{1(2j)}) = g(v_{1(2j-2)}) - (3t - 2)$, where

\[
k = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even}, \\
\frac{n+1}{2}, & \text{if } n \text{ is odd.}
\end{cases}
\]

For $3 \leq j \leq dn + 1$, $g(v_{ij}) = \begin{cases} 
g(v_{i(j+1)}) + 1, & \text{for } j \text{ odd and } 2 \leq i \leq t, \\
g(v_{i(j-1)}) - 2, & \text{for } j \text{ even and } 2 \leq i \leq t.
\end{cases}$

**When $d > 1$ then define $g$ in two cases depending on $n$ is even or $n$ is odd.**

**Case a. $n$ is even**

$g(v_{1(2j+1)}) = g(v_{1(2j-1)}) + 3t - 2, \ 1 \leq j \leq \frac{dn}{2}$,

$g(v_{1(2j)}) = \begin{cases} 
g(v_{1(2j-2)}) - (3t - 2), & 2 \leq j \leq \frac{dn}{2} \text{ and } j \neq \frac{kn+2}{2}, k = 1, 2, \ldots, d - 1 \\
g(v_{1(kn)}) - (3t - 1), & \text{for } j = \frac{kn+2}{2}, k = 1, 2, \ldots, d - 1.
\end{cases}$

**Case b. $n$ is odd**

$g(v_{1(2j+1)}) = \begin{cases} 
g(v_{1(2j-1)}) + 3t - 2, & 1 \leq j \leq \ell, \ell = \frac{dn}{2} \text{ if } d \text{ is even, } \\
\ell = \frac{dn-1}{2} \text{ if } d \text{ is odd and } j \neq \frac{kn+1}{2}, 1 \leq k \leq d - 1 \text{ and } k \text{ odd, } \\
g(v_{1(kn)}) + 3t - 1, & \text{for } j = \frac{kn+1}{2}, 1 \leq k \leq d - 1 \text{ and } k \text{ odd}
\end{cases}$

$g(v_{1(2j)}) = \begin{cases} 
g(v_{1(2j-2)}) - (3t - 2), & 2 \leq j \leq \ell, \ell = \frac{dn+1}{2} \text{ if } d \text{ is odd, } \\
\ell = \frac{dn}{2} \text{ if } d \text{ is even and } j \neq \frac{kn+2}{2}, 2 \leq k \leq d - 1 \text{ and } k \text{ even, } \\
g(v_{1(kn)}) - (3t - 1), & \text{for } j = \frac{kn+2}{2}, 2 \leq k \leq d - 1 \text{ and } k \text{ even}
\end{cases}$

For both the cases, for $3 \leq j \leq dn + 1$, define

$g(v_{ij}) = \begin{cases} 
g(v_{i(j-1)}) + 1, & \text{for } j \text{ odd and } 2 \leq i \leq t, \\
g(v_{i(j-1)}) - 2, & \text{for } j \text{ even and } 2 \leq i \leq t.
\end{cases}$
From the definition of $g$ if the labels of the vertices of $S_t(P_{dn+1})$ are arranged as,
\begin{align*}
g(v_1), & \, g(v_2), \ldots, g(v_{n}), \\
g(v_3), & \, g(v_{2n}), \ldots, g(v_{2n+1}), \\
g(v_4), & \, g(v_{2n+1}), \ldots, g(v_{3n+1}), \\
& \vdots \\
g(v_{1(s-2)}), & \, g(v_{2(s-2)}), \ldots, g(v_{ts-2}).
\end{align*}
then the above sequence forms a strictly increasing sequence. Hence the vertex labels of $S_t(P_{dn+1})$ are distinct.

From the above arrangement of vertex labels observe that
\[ \max \{g(u)/u \in V_1, 1 \leq i \leq t \} = g(v_{tn}) \]
\[ < \min \{g(v)/v \in V_2, 1 \leq i \leq t \} = g(v_{tn+1}), \] when $dn + 1$ is even;
while when $dn + 1$ is odd, \[ \max \{g(u)/u \in V_1, 1 \leq i \leq t \} = g(v_{tn+1}) \]
\[ < \min \{g(v)/v \in V_2, 1 \leq i \leq t \} = g(v_{tn+1}), \]

We prove that the edge labels of $S_t(P_{dn+1})$ are distinct depending on $d = 1$ and $d > 1$.

**Case 1.** $d = 1$

When $n$ is even, the edges of the graph $S_t(P_{dn+1})$ can be arranged as the following sequence,
\begin{align*}
(v_{12}v_{11}, & \, v_{12}v_{21}, \, v_{11}v_{22}, \, v_{21}v_{22}, \, v_{31}v_{22}, \ldots, \, v_{(i-1)1}v_{12}, \, v_{i1}v_{12}, \, v_{(i+1)1}v_{12}, \ldots, \, v_{(t-1)(n+1)1}v_{tn}, \\
v_{tn}v_{tn+1}).
\end{align*}

When $n$ is odd, the edges of $S_t(P_{dn+1})$ can be arranged as the following sequence,
\begin{align*}
(v_{12}v_{11}, & \, v_{12}v_{21}, \, v_{11}v_{22}, \, v_{21}v_{22}, \ldots, \, v_{(i-1)1}v_{12}, \, v_{i1}v_{12}, \, v_{(i+1)1}v_{12}, \ldots, \, v_{(t-1)n}v_{tn+1}, \\
v_{tn}v_{tn+1}).
\end{align*}

Then from the definition of $g$ for both the cases we have the corresponding edge label sequence,
\begin{align*}
(N, & \, N-1, N-2, \ldots, 3, 2, 1).
\end{align*}
Hence, it is clear that the edge labels are distinct.
Therefore, when $d = 1$, $g$ is a 1-divisible $\alpha$-labeling of $S_t(P_{dn+1})$. That is, $g$ is an $\alpha$-labeling of the graph $S_t(P_{dn+1})$.

**Case 2.** $d > 1$

In order to show that the edge labels of the edges of $S_t(P_{dn+1})$ are distinct, we partition the edge set of $S_t(P_{dn+1})$ into $d$ subsets of the edge set of $S_t(P_{dn+1})$ and they are arranged as $d$ sequences. Consequently, their corresponding edge labels are also arranged as $d$ sequences.
When $n$ is even then we consider the first edge sequence to be the following sequence 
\[(v_{12}v_{11}, v_{12}v_{21}, v_{11}v_{22}, v_{21}v_{22}, v_{31}v_{22}, \ldots, v_{(i-1)}v_{i2}, v_{i1}v_{i2}, v_{(i+1)}v_{i2}, \ldots, v_{(t-1)(n+1)}v_{tn}, v_{tn}v_{(n+1)}).\]

When $n$ is odd then we consider the first edge sequence to be the following sequence 
\[(v_{12}v_{11}, v_{12}v_{21}, v_{11}v_{22}, v_{21}v_{22}, v_{31}v_{22}, \ldots, v_{(i-1)}v_{i2}, v_{i1}v_{i2}, v_{(i+1)}v_{i2}, \ldots, v_{(t-1)n}v_{tn+1}, v_{tn}v_{(n+1)}).\]

Then from the definition of $g$ for both the cases we have the corresponding edge label sequence,

\[S_1 : (N, N - 1, N - 2, \ldots, (d - 1)(3t - 2)n + d - 1, (d - 1)(3t - 2)n + d).\]

When $n$ is even then we consider the second edge sequence to be the following sequence 
\[(v_{1(n+2)}v_{1(n+1)}, v_{1(n+2)}v_{2(n+1)}, v_{1(n+2)}v_{2(n+2)}, v_{2(n+1)}v_{2(n+2)}, \ldots, v_{(i-1)(n+1)}v_{i(n+2)}, v_{i(n+1)}v_{i(n+2)}, \ldots, v_{(t-1)(2n+1)}v_{rt(2n+1)}).\]

When $n$ is odd then we consider the second edge sequence to be the following sequence 
\[(v_{1(n+1)}v_{1(n+2)}, v_{1(n+2)}v_{2(n+1)}, v_{1(n+2)}v_{2(n+2)}, v_{2(n+1)}v_{2(n+2)}, \ldots, v_{(i-1)(n+1)}v_{i(n+2)}, v_{i(n+1)}v_{i(n+2)}, \ldots, v_{(t-1)(2n+1)}v_{rt(2n+1)}).\]

Then from the definition of $g$ for both the cases we have the corresponding edge label sequence,

\[S_2 : ((d - 1)(3t - 2)n + d - 2, (d - 1)(3t - 2)n + d - 3, \ldots, (d - 2)(3t - 2)n + d, (d - 2)(3t - 2)n + d - 1).\]

When $n$ is even then we consider the third edge sequence to be the following sequence 
\[(v_{1(2n+2)}v_{1(2n+1)}, v_{1(2n+2)}v_{2(2n+1)}, v_{1(2n+2)}v_{2(2n+2)}, v_{2(2n+1)}v_{2(2n+2)}, \ldots, v_{(i-1)(2n+1)}v_{i(2n+2)}, v_{i(2n+1)}v_{i(2n+2)}, \ldots, v_{(t-1)(3n+1)}v_{rt(3n+1)}).\]

When $n$ is odd then we consider the third edge sequence to be the following sequence 
\[(v_{1(2n+1)}v_{1(2n+2)}, v_{1(2n+2)}v_{2(2n+1)}, v_{1(2n+2)}v_{2(2n+2)}, v_{2(2n+1)}v_{2(2n+2)}, \ldots, v_{(i-1)(2n+1)}v_{i(2n+2)}, v_{i(2n+1)}v_{i(2n+2)}, \ldots, v_{(t-1)(3n+1)}v_{rt(3n+1)}).\]

Then from the definition of $g$ for both the cases we have the corresponding edge label sequence,

\[S_3 : (((d - 2)(3t - 2)n + d - 3, (d - 2)(3t - 2)n + d - 4, \ldots, (d - 3)(3t - 2)n + d - 1, (d - 3)(3t - 2)n + d).\]

In general, we consider the $j^{th}$ edge sequence, for $4 \leq j \leq d - 2$ depending on $n$ and $j$.

**Case i.** $n$ is even or $n$ is odd and $j$ is even

Then we consider the $j^{th}$ edge sequence to be the following sequence 
\[(v_{1(jn+2)}v_{1(jn+1)}, v_{1(jn+2)}v_{2(jn+1)}, v_{1(jn+2)}v_{2(jn+2)}, v_{2(jn+1)}v_{2(jn+2)}, \ldots, v_{(i-1)(jn+1)}v_{i(jn+2)}, v_{i(jn+1)}v_{i(jn+2)}, v_{(i+1)(jn+1)}v_{i(jn+2)}, \ldots, v_{(t-1)(j+1)n}v_{rt((j+1)n+1)}).\]

**Case ii.** $n$ and $j$ are odd

Then we consider the $j^{th}$ edge sequence to be the following sequence 
\[(v_{1(jn+1)}v_{1(jn+2)}, v_{1(jn+2)}v_{2(jn+1)}, v_{1(jn+2)}v_{2(jn+2)}, v_{2(jn+1)}v_{2(jn+2)}, \ldots, v_{(i-1)(jn+2)}v_{i(jn+1)}, v_{i(jn+2)}v_{i(jn+1)}, v_{(i+1)(jn+2)}v_{i(jn+1)}, \ldots, v_{(t-1)(j+1)n}v_{rt((j+1)n)}).\]

Then from the definition of $g$ for all the above cases we have the corresponding edge label sequence,

\[S_j : ((d - j)(3t - 2)n + d -(j+1), (d - j)(3t - 2)n + d -(j+2), (d - j)(3t - 2)n + d -(j+3), \ldots, (d - j)(3t - 2)n + d -(j+2), (d - j)(3t - 2)n + d -(j+1), (d - (j+1))(3t - 2)n + d -(j+1), (d - (j+1))(3t - 2)n + d - j).\]

Now we consider the $(d - 1)^{th}$ edge sequence depending on $n$ is even or odd.
Case I. $n$ is even

Then we consider the $(d - 1)^{th}$ edge sequence to be the following sequence

$$v_1((d-2)n+1)v_1((d-2)n+1), v_1((d-2)n+2)v_2((d-2)n+1), v_1((d-2)n+1)v_2((d-2)n+2), v_2((d-2)n+1)v_2((d-2)n+2),
\text{v}_3((d-2)n+1)v_2((d-2)n+2), \ldots, \text{v}_{(i-1)}((d-2)n+1)v_i((d-2)n+2), \text{v}_i((d-2)n+1)v_i((d-2)n+2),
\text{v}_{(i+1)}((d-2)n+1)v_i((d-2)n+2), \ldots, \text{v}_{(t-1)}((d-1)n+1)v_t((d-1)n), \text{v}_t((d-1)n)v_t((d-1)n+1).$$

Case II. $n$ is odd

Then we consider the $(d - 1)^{th}$ edge sequence in the following subcases depending on $d - 1$ is even or odd.

Case IIa. $d - 1$ is even

Then we consider the $(d - 1)^{th}$ edge sequence to be the following sequence

$$v_1((d-2)n+1)v_1((d-2)n+1), v_1((d-2)n+2)v_2((d-2)n+1), v_1((d-2)n+1)v_2((d-2)n+2), v_2((d-2)n+1)v_2((d-2)n+2),
\text{v}_3((d-2)n+1)v_2((d-2)n+2), \ldots, \text{v}_{(i-1)}((d-2)n+1)v_i((d-2)n+2), \text{v}_i((d-2)n+1)v_i((d-2)n+2),
\text{v}_{(i+1)}((d-2)n+1)v_i((d-2)n+2), \ldots, \text{v}_{(t-1)}((d-1)n+1)v_t((d-1)n), \text{v}_t((d-1)n)v_t((d-1)n+1).$$

Then from the definition of $g$ for all the above cases we have the corresponding edge label sequence,

$S_{d-1} : (2(3t-2)n + 1, 2(3t-2)n, 2(3t-2)n-1, \ldots, (3t-2)n + 3, (3t-2)n + 2).$

Finally, we consider the $d^{th}$ edge sequence depending on $n$ is even or odd.

Case I. $n$ is even

Then we consider the $d^{th}$ edge sequence to be the following sequence

$$v_1((d-1)n+1)v_1((d-1)n+1), v_1((d-1)n+2)v_2((d-1)n+1), v_1((d-1)n+1)v_2((d-1)n+2), v_2((d-1)n+1)v_2((d-1)n+2),
\text{v}_3((d-1)n+1)v_2((d-1)n+2), \ldots, \text{v}_{(i-1)}((d-1)n+1)v_i((d-1)n+2), \text{v}_i((d-1)n+1)v_i((d-1)n+2),
\text{v}_{(i+1)}((d-1)n+1)v_i((d-1)n+2), \ldots, \text{v}_{(t-1)}((d-1)n)v_t((d-1)n), \text{v}_t((d-1)n)v_t((d-1)n+1).$$

Case 2. $n$ is odd

Then we consider the $d^{th}$ edge sequence in the following subcases depending on $d$ is even or odd.

Case 2a. $d$ is even

Then we consider the $d^{th}$ edge sequence to be the following sequence

$$v_1((d-1)n+1)v_1((d-1)n+1), v_1((d-1)n+2)v_2((d-1)n+1), v_1((d-1)n+1)v_2((d-1)n+2), v_2((d-1)n+1)v_2((d-1)n+2),
\text{v}_2((d-1)n+1)v_2((d-1)n+2), \ldots, \text{v}_{(i-1)}((d-1)n+2)v_i((d-1)n+1), \text{v}_i((d-1)n+2)v_i((d-1)n+1),
\text{v}_{(i+1)}((d-1)n+2)v_i((d-1)n+1), \ldots, \text{v}_{(t-1)}((d-1)n)v_t((d-1)n), \text{v}_t((d-1)n)v_t((d-1)n+1).$$

Then from the definition of $g$ for all the above cases we have the corresponding edge label sequence,

$S_d : ((3t-2)n, (3t-2)n-1, (3t-2)n-2, \ldots, 3, 2, 1).$

Using all the above defined edge label sequences $S_1, S_2, S_3, \ldots, S_j, \ldots, S_{d-1}, S_d$, we form a combined edge label sequence in the order as $S : (S_1, S_2, S_3, \ldots, S_j, \ldots, S_{d-1},$ $S_d).$
Then we observe that $S$ forms a monotonically decreasing sequence. Also observe that none of the terms $(d-1)((3t-2)n+1)$, $(d-2)((3t-2)n+1)$, ..., $3((3t-2)n+1)$, $2((3t-2)n+1)$, $(3t-2)n+1$ appear in the combined sequence $S$. Thus, $g$ is a $d$-divisible $\alpha$-labeling of $S_t(P_{dn+1})$ for any admissible $d > 1$. Therefore the graph $S_t(P_{dn+1})$ admits $d$-divisible $\alpha$-labeling for any admissible $d$. \[ \square \]

Illustration

The 4-divisible $\alpha$-labeling, 3-divisible $\alpha$-labeling and 2-divisible $\alpha$-labeling that are defined as in the proof of Theorem 3.1 for the graphs $S_4(P_5)$, $S_4(P_{10})$, $S_4(P_9)$ are given in Figures 3, 4, 5 respectively.

![Figure 3](image-url)  
Figure 3: 4-divisible $\alpha$-labeling of $S_4(P_5)$

![Figure 4](image-url)  
Figure 4: 3-divisible $\alpha$-labeling of $S_4(P_{10})$
The following corollary is an immediate implication of Anita Pasotti’s theorem, Theorem 1.6.

**Corollary 3.2.** The multipartite graph $K_{(\frac{t}{2}+1)\times 2dn}$ can be cyclically decomposed into copies of the $t$-levels shadow graph of the path $P_{dn+1}$, $S_t(P_{dn+1})$, where $e = |E(S_t(P_{dn+1}))|$, $t \geq 2$, $d \geq 1$, $n \geq 1$ and $m$ is any positive integer.

### 4 Discussion

In this section we pose two open problems for further research.

In Theorem 2.1 we have proved that for $t \geq 2$, disjoint union of $t$ copies of the complete bipartite graph $K_{m,n}$ plus an edge, $(\bigcup_{i=1}^{t} K_{m,n}^i) + \hat{e}$ admits $\gamma$-labeling. In this direction investigating the following question will be useful for achieving a generalised result.

**Is it true that disjoint union of $t$ copies of an $\alpha$-labeled graph $G$ plus an edge, $t \geq 2$, admits $\gamma$-labeling?**

In Theorem 3.1 we have proved that for $t \geq 2$, the $t$-levels shadow graph of the path $P_{dn+1}$ with $d \geq 1, n \geq 1$ admits $d$ divisible $\alpha$-labeling for all $d \geq 1$. It is evident that the path $P_{dn+1}$ admits $\alpha$-labeling for all $d \geq 1, n \geq 1$. This observation tempts us to ask the following question to understand $d$-divisible $\alpha$-labeled graphs.

**What are the $\alpha$-labeled graphs whose $t$-levels shadow graph admits $d$ divisible $\alpha$-labeling for all values of $d$?**

### References


