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Nilpotent Graph

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Abstract

In this article, we introduce the concept of nilpotent graph of a finite commutative ring. The set of all non nilpotent elements of a ring is taken as the vertex set and two vertices are adjacent if and only if their sum is nilpotent. We discuss some graph theoretic properties of nilpotent graph.

1 Introduction

In this article, rings are finite commutative rings with non zero identity. The set of nilpotent elements of a ring R and the $n \times n$ matrix ring over R are denoted by $Nil(R)$ and $M_n(R)$ respectively. Here, by graph, we mean simple undirected graph. For a graph G, the set of vertices and the set of edges are denoted by $V(G)$ and $E(G)$ respectively. For a positive integer n, P. Grimaldi[4] defined and studied various properties of the unit graph $G(\mathbb{Z}_n)$, of the ring of integers modulo n, with vertex set $\mathbb{Z}_n$ and two distinct vertices are adjacent if and only if their sum is a unit. Further in [1], Ashrafi et al. generalized $G(\mathbb{Z}_n)$ to unit graph $G(R)$, where R is an arbitrary associative ring with non zero identity.

In this article we have introduce the nilpotent graph $G(R)$ associated with a finite ring R. We define the nilpotent graph $G(R)$ of a ring R by taking $R \setminus Nil(R)$ as the vertex set and two vertices x and y are adjacent if and only if $x + y$ is a nilpotent element in R. Properties like girth, clique number, chromatic index, dominating number, spectrum, Laplacian spectrum and signless Laplacian spectrum of $G(R)$ are studied.

For this article, we mention some preliminaries about graph theory. Let G be a graph. The degree of the vertex $v \in G$ is the number of edges incident to v, denoted by $deg(v)$. A graph G is said to be connected if for any two distinct vertices of G, there is a path in G connecting them. Also the girth of G, denoted by $gr(G)$ is the length of the shortest cycle in G and if there is no cycle in G, then $gr(G) = \infty$. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A bipartite graph G is a graph whose vertices can be divided into two disjoint parts V_1 and V_2, such that $V(G) = V_1 \cup V_2$ and every edge in G has the form $e = (x, y) \in E(G)$, where $x \in V_1$ and $y \in V_2$. A complete bipartite graph is a graph where every vertex of V_1 is connected to every vertex of V_2, denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. A complete bipartite graph $K_{1,n}$ is called star graph. The neighbour set of $x \in V(G)$, is denoted and defined by $N_G(x) := \{y \in V(G) | \, y \text{ is adjacent to } x\}$ and the set $N_G[x] = N_G(x) \cup \{x\}$.

A clique is a subset of vertices of a graph such that its induced subgraph is complete. A clique having n number of vertices is called n-clique. The maximal clique of a graph is a clique such that there is no clique with more vertices. The clique number of a graph G is denoted by $\omega(G)$ and defined by the number of vertices in the maximal clique of G. An edge colouring of a graph G is a map $\xi : E(G) \to S$, where S is a set of colours such that for all $e_1, e_2 \in E(G)$ and if e_1, e_2 are incident, then $\xi(e_1) \neq \xi(e_2)$. The *chromatic index* of a graph G is denoted by $\chi'(G)$ and is defined as the minimum number of colours needed for a proper colouring of G.

For a graph G, let $A(G)$ and $D(G)$ are respectively be the adjacency matrix and degree matrix of G. Then the Laplacian and signless Laplacian matrix are given by $L(G) =$

$D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ respectively. The collection of eigen values of adjacency matrix, Laplacian matrix and signless Laplacian matrix are called spectrum, Laplacian spectrum and signless Laplacian spectrum respectively. We denote spectrum, Laplacian spectrum and signless Laplacian spectrum of G by $Spec(G)$, $L\text{-}Spec(G)$ and $Q\text{-}Spec(G)$ respectively. We write $Spec(G) = \{(a_1)^{l_1}, (a_2)^{l_2}, \cdots, (a_p)^{l_p}\}$, $L\text{-}Spec(G) = \{(b_1)^{m_1}, (b_2)^{m_2}, \cdots, (b_q)^{m_q}\}$ and $Q\text{-}Spec(G) = \{(c_1)^{n_1}, (c_2)^{n_2}, \cdots, (c_r)^{n_r}\}$, where $a_1, a_2, \cdots, a_p$ are eigenvalues of $A(G)$ with multiplicities $l_1, l_2, \cdots, l_p$; $b_1, b_2, \cdots, b_q$ are eigenvalues of $L(G)$ with multiplicities $m_1, m_2, \cdots, m_q$ and $c_1, c_2, \cdots, c_r$ are eigenvalues of $Q(G)$ with multiplicities $n_1, n_2, \cdots, n_r$ respectively.

An Eulerian path is a path in a graph that visits every edge exactly once (allowing for revisiting vertices). A graph is called Eulerian if it consists an Eulerian cycle. A Hamiltonian path is a path in a graph that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph. A graph is called planar graph if it can be embedded in the plane; that is it can be drawn in a plane in such a way that no edges cross each other.

2 Nilpotent graphs

Definition 2.1. *The nilpotent graph of a ring R denoted by $G(R)$ is defined by setting $R \setminus Nil(R)$ as the vertex set and two distinct vertices x and y are adjacent if $x + y$ is nilpotent. Here we are not considering any loop at a vertex in the graph.*

The nilpotent graphs of $\mathbb{Z}_{12}$ and $\mathbb{Z}_{18}$ are given below,

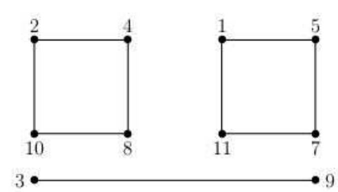

FIGURE 1. Nilpotent graph of $\mathbb{Z}_{12}$.

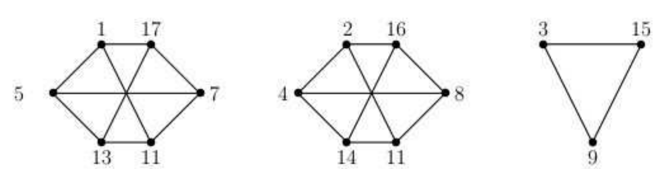

FIGURE 2. Nilpotent graph of $\mathbb{Z}_{18}$.

From the above two graphs, it is clear that $gr(G(\mathbb{Z}_{12}))$ and $gr(G(\mathbb{Z}_{18}))$ are 2 and 3 respectively.

Lemma 2.1. *Let $G(R)$ be the nilpotent graph of a ring R. Then for $x \in V(G(R))$ we have the following:*

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1. *If $2x \in Nil(R)$, then $deg(x) = |Nil(R)| - 1$.*

2. *If $2x \notin Nil(R)$, then $deg(x) = |Nil(R)|$.*

Proof. Let $x \in V(G(R))$ and $y \in Nil(R)$. As $x + R = R$, so there exists a unique $x_y \in R$ such that $x + x_y = y$. Observe that $x_y \in V(G(R))$, otherwise $x_y \in Nil(R)$ and hence $x = y - x_y \in Nil(R)$, a contradiction.
Case I: If $2x \in Nil(R)$, then there exists unique $n \in Nil(R)$ such that $x + x = n$, but we are not considering any loop, so $deg(x) = |Nil(R)| - 1$.
Case II: Let $2x \notin Nil(R)$. If $x + y \in Nil(R)$, then $x \neq y$ and hence we have $deg(x) = |Nil(R)|$. □

Corollary 2.2. *Every graph $G(\mathbb{Z}_n)$ with $n = 2^k$, where $k \geq 1$, is a complete graph.*

Proof. Here $Nil(\mathbb{Z}_n) = \{2k \,|\, 1 \leq k \leq 2^{k-1}\}$. So for any $x \in V(G(\mathbb{Z}_n))$, we have $2x \in Nil(\mathbb{Z}_n)$. Hence by Lemma 2.1, $deg(x) = |Nil(\mathbb{Z}_n)| - 1$. But $|V(G(\mathbb{Z}_n))| = |Nil(\mathbb{Z}_n)|$, so $G(\mathbb{Z}_n)$ is complete. □

Theorem 2.3. *Every graph $G(\mathbb{Z}_{\overline{n}})$, where $\overline{n} = p^n q^m$, $n > m$ and p, q are distinct primes has the following properties:*

1. *Let $p = 2$ and q be an odd prime, then*
   - *If $|Nil(\mathbb{Z}_{\overline{n}})| > 2$, then $G(\mathbb{Z}_{\overline{n}})$ is not bipartite and has a complete subgraph of order $Nil(\mathbb{Z}_{\overline{n}})$ and its complement is bipartite.*
   - *If $|Nil(\mathbb{Z}_{\overline{n}})| \leq 2$, then $G(\mathbb{Z}_{\overline{n}})$ is bipartite.*

2. *If $p, q > 2$, then $G(\mathbb{Z}_{\overline{n}})$ is bipartite.*

Proof. (i) Let $p = 2$ and consider $|Nil(\mathbb{Z}_{\overline{n}})| = k$. Now we partition our vertex set given by $P_1 = \{1, 2, \cdots, pq - 1\}$, $P_2 = \{pq + 1, pq + 2, \cdots, 2pq - 1\}$, $\cdots$, $P_{k_l} = \{pq(p^{n-1}q^{m-1} - 1) + 1, pq(p^{n-1}q^{m-1} - 1) + 2, \cdots, p^n q^m - 1\}$. Observe that $k_l = k = |Nil(\mathbb{Z}_{\overline{n}})|$. As $p = 2$, we see that $|P_1| = |P_2| = \cdots = |P_k| = pq - 1$ =odd. Since $|P_1|$ is odd so, there exists $r_1 \in P_1$ such that $2r_1 = pq$. Similarly $r_2 \in P_2$ such that $2r_2 = 3pq$ and we get $r_1 + r_2 = 2pq \in Nil(\mathbb{Z}_{\overline{n}})$. Similarly as above we can easily show that for any P_i, there exists $r_i \in P_i$ such that $2r_i = (2i - 1)pq$. Let $C = \{r_1, r_2, \cdots, r_k\}$. Then for any pair $r_i, r_j \in C$, we get $r_i + r_j = (i + j - 1)pq \in Nil(\mathbb{Z}_{\overline{n}})$. Hence induced subgraph of C forms a complete subgraph of $G(\mathbb{Z}_{\overline{n}})$ and $G(\mathbb{Z}_{\overline{n}})$ is not bipartite. Now by Lemma 2.1, the complete subgraph induced by C is not connected with any vertex in $G \setminus C$. Consider $B_i = \{rpq + i \,:\, 0 \leq r \leq (p^{n-1}q^{m-1} - 1)\}$ and $B_{-i} = \{spq - i \,:\, 0 \leq s \leq (p^{n-1}q^{m-1} - 1)\}$, where $1 \leq i < \frac{pq}{2}$. It is clear that every element of B_i is adjacent to every element of B_{-i}, also for $x \in B_i$ if $x + y \in Nil(R)$, then $y = n_1 - x = (n_1 - n_2) - i \in B_{-i}$, where $x = n_2 + i$, for some $n_1, n_2 \in Nil(R)$. Hence $B_i \cup B_{-i}$ induces a complete bipertite subgraph of $G(\mathbb{Z}_{\overline{n}})$, for $1 \leq i \leq \frac{pq}{2} - 1$. Let $A = B_1 \cup B_2 \cup \cdots \cup B_{\frac{pq}{2}-1}$ and $B = B_{-1} \cup B_{-2} \cup \cdots \cup B_{-(\frac{pq}{2}-1)}$. Observe that no two elements of A are adjacent as, $i + j \notin Nil(R)$, for $1 \leq i, j < \frac{pq}{2}$. Similarly no two elements of B are adjacent. It is a simple observation that any element of B_i, $1 \leq i \leq \frac{pq}{2} - 1$ is not adjacent with B_{-j}, where $1 \leq j \leq \frac{pq}{2} - 1$ and $i \neq j$. Similarly any element of B_{-i}, $1 \leq i \leq \frac{pq}{2} - 1$

is not adjacent with B_j, where $1 \leq j \leq \frac{pq}{2} - 1$ and $i \neq j$. Hence If $|Nil(\mathbb{Z}_{\overline{n}})| > 2$, then $G(\mathbb{Z}_{\overline{n}})$ is not bipartite and has a complete subgraph of order $Nil(\mathbb{Z}_{\overline{n}})$ and its complement is bipartite. If $|Nil(\mathbb{Z}_{\overline{n}})| \leq 2$, then clearly $G(R)$ is bipartite.

(ii) If p and q are two distinct odd primes, then $|P_i|$ is even for $1 \leq i \leq k$. Now in P_i we see that 1 is adjacent to $ipq - 1$, more generally $rpq + 1$ is adjacent to $spq - 1$, where $0 \leq r \leq (p^{n-1}q^{m-1} - 1)$ and $0 \leq s \leq (p^{n-1}q^{m-1} - 1)$. Let $B_1 = \{rpq + 1 : 0 \leq r \leq (p^{n-1}q^{m-1} - 1)\}$ and $B_{-1} = \{spq - 1 : 0 \leq s \leq (p^{n-1}q^{m-1} - 1)\}$, then clearly each vertex of B_1 and each vertex of B_{-1} are adjacent. But no two elements of B_1 and no two elements of B_{-1} are adjacent. For $x \in B_1$ if $x + y \in Nil(R)$, then $y = n_1 - x = (n_1 - n_2) - i \in B_{-i}$, where $x = n_2 + i$ and $n_1, n_2 \in Nil(R)$, implies $G(B_1 \cup B_{-1})$ is a subgraph which is bipartite. In general we can get $G(B_w \cup B_{-w})$ is a subgraph which is bipartite, where $1 \leq w \leq [\frac{pq}{2}]$. Now we partition $G(\mathbb{Z}_{\overline{n}})$ into two parts viz. $A_1 = B_1 \cup B_2 \cup \cdots \cup B_{[\frac{pq}{2}]}$ and $A_2 = B_{-1} \cup B_{-2} \cup \cdots \cup B_{-[\frac{pq}{2}]}$. If possible let $a, b \in A$, such that $a + b \in Nil(\mathbb{Z}_{\overline{n}})$, then $a + b = (r + s)pq + (a_1 + b_1) \in Nil(\mathbb{Z}_{\overline{n}})$, where $a_1, b_1 \leq [\frac{pq}{2}]$, which is a contradiction to the fact that $a_1 + b_1 \in Nil(\mathbb{Z}_{\overline{n}})$, as $a_1, b_1 < pq$, so no two vertices of A_1 are adjacent. Similarly we can show that no two vertices of A_2 are adjacent. Hence $G(\mathbb{Z}_{\overline{n}})$ is bipartite. □

Corollary 2.4. *If $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where p_i's are distinct odd primes, then $G(\mathbb{Z}_n)$ is bipartite and it consists of $[\frac{p_1 p_2 \cdots p_k}{2}]$ disjoint complete bipartite subgraphs $K_{|Nil(\mathbb{Z}_n)|,|Nil(\mathbb{Z}_n)|}$.*

Corollary 2.5. *If $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $p_1 = 2$ and p_i's are distinct odd primes for $2 \leq i \leq k$, then $G(\mathbb{Z}_n)$ consists of one complete subgraph of order $|Nil(R)|$ and its complement is bipartite, which is a union of disjoint complete bipartite subgraphs $K_{|Nil(\mathbb{Z}_n)|,|Nil(\mathbb{Z}_n)|}$.*

3 Girth

Lemma 3.1. *Let R be a commutative ring of odd order and $x \in R$. Then $x \in Nil(R)$ if $2x \in Nil(R)$.*

Proof. Let $2x \in Nil(R)$. There exists $n \in \mathbb{N}$ such that $2^n x^n = 0$. Since $(R, +)$ is an abelian group, as an abelian group, $O(x^n)$ divides 2^n, so $O(x^n) = 2^k$, for some $0 \leq k \leq n$. But as an abelian group, $O(x^n)$ divides $|R|$, implies $k = 0$. Hence $x^n = 0$ *i.e.* $x \in Nil(R)$. □

Lemma 3.2. *Let R be a commutative ring of even order. Then there exists $x \in R \setminus Nil(R)$ such that $2x \in Nil(R)$.*

Proof. Consider $|R| = 2^k m$, where 2 does not divide m. Let $x = m.1$, where $m.1 = (1 + 1 + \cdots + 1)(m - times)$. Observe that for $n \in \mathbb{N}$, $x^n = m^n.1$. If possible let $x = m.1 = 0$, then for any $a \in R$, $m.a = (1 + 1 + \cdots + 1).a = (m.1)a = 0$, which is a contradiction as by Cauchy's theorem there exists an element in R having order 2 in $(R, +)$. So $x \neq 0$. Suppose $x^n = 0$ for $n \in \mathbb{N}$, then $m^n.1 = 0$ implies $O(1)$ divides m^n, where $O(1)$ represents order of 1 in $(R, +)$. Also $O(1)$ divides $2^k m$, so $O(1)$ divides m *i.e.*, $m.1 = x = 0$ a contradiction. Hence $x \in R \setminus Nil(R)$. Now $(2x)^k = (2m)^k.1 = (2^k m^k).1 = m^{k-1}((2^k m).1)) = 0$ *i.e.*, $2x \in Nil(R)$. □

Theorem 3.3. *Let R be a commutative ring of odd order. If $|Nil(R)| \geq 3$, then $gr(G(R))$ is 4 and otherwise $gr(G(R))$ is infinite.*

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Proof. If possible let $gr(G(R))$ is 3, then there exists a path $z_1 \longrightarrow z_2 \longrightarrow z_3 \longrightarrow z_1$ in $G(R)$. *i.e.* $\{2z_1, 2z_2, 2z_3\} \subset Nil(R)$, which is a contradiction by Lemma 3.1, as $z_i \notin Nil(R)$ for $1 \leq i \leq 3$.
Consider $R \setminus Nil(R) \neq \phi$, otherwise our graph is an empty graph. Let $x \in R \setminus Nil(R)$. Since $|Nil(R)| \geq 3$, so there exist elements $0, n_1, n_2 \in Nil(R)$ such that all are distinct. Clearly $x \longrightarrow n_1 - x \longrightarrow n_2 + x \longrightarrow n_2 - x \longrightarrow x$ is a cycle in $G(R)$. Observe that all four elements $x, n_1 - x, n_2 + x, n_2 - x$ are distinct, otherwise either $2x \in Nil(R)$ or $n_i = n_j$ for $i \neq j$. Let $|Nil(R)| < 3$, since $|Nil(R)|$ divides $|R|$, so $|Nil(R)| = 1$ and hence $gr(G(R))$ is infinite. □

Theorem 3.4. *Let R be a commutative ring of even order and $|Nil(R)| \geq 3$, then $gr(G(R))$ is 3.*

Proof. By Lemma 3.2, there exists an element $x \in R \setminus Nil(R)$ such that $2x \in Nil(R)$. Since $|Nil(R)| \geq 3$, so there exist elements $n_1, n_2, n_3 \in Nil(R)$ such that all are distinct. It is clear that $n_3 + x \longrightarrow n_2 + x \longrightarrow n_1 + x \longrightarrow n_3 + x$ be a cycle in $G(R)$. Clearly $n_i + x \in R \setminus Nil(R)$, for $1 \leq i \leq 3$ and all are distinct. Hence $gr(G(R)) = 3$. □

4 Clique number

In this section we derive results on clique number for any arbitrary ring.

Theorem 4.1. *Let R be a finite commutative ring*

1. *If $|R|$ is odd, then $\omega(G(R)) = 2$.*
2. *If $|R|$ is even, then $\omega(G(R)) = |Nil(R)|$ provided $|Nil(R)| \geq 2$, otherwise $\omega(G(R)) = 2$.*

Proof.

1. Clear from Theorem 3.3.
2. From definition of nilpotent graph, it is clear that $\omega(G(R)) \leq |Nil(R)|$. Next we claim that $\omega(G(R)) = |Nil(R)|$. By Lemma 3.2, there exists $x \in R \setminus Nil(R)$ such that $2x \in Nil(R)$. Let $|Nil(R)| = n$ and $a_i's$ are distinct elements of $Nil(R)$, for $0 \leq i \leq n-1$. Consider $A = \{x + a_i \ : \ a_i \in Nil(R)\}$. Observe that all pair of distinct elements of A are adjacent and $A \subseteq R \setminus Nil(R)$. Hence vertex set A induce a complete subgraph of $G(R)$. If $|Nil(R)| = 1$, then obviously $\omega(G(R)) = 2$.

□

5 Spectrum, Laplacian spectrum and signless Laplacian spectrum

It is well known that the spectrum, Laplacian spectrum and signless Laplacian spectrum of the complete bipartite graph $K_{n,n}$ are given by $\{(n)^1, (-n)^1, (0)^{2n-2}\}$, $\{(2n)^1, (0)^1, (n)^{2n-2}\}$ and $\{(2n)^1, (0)^1, (n)^{2n-2}\}$. Also the spectrum, laplacian spectrum and signless laplacian spectrum of the complete graph K_n are given by $\{(-1)^{n-1}, (n-1)^1\}$, $\{(0)^1, (n)^{n-1}\}$ and $\{(2n-2)^1, (n-2)^{n-1}\}$.

Theorem 5.1. *For $G(\mathbb{Z}_n)$, if $|Nil(\mathbb{Z}_n)| = t$, then the following hold:*

1. *If $n = 2^{r_0}p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes, then*
   - $Spec(G(\mathbb{Z}_n)) = \{(t)^{p_1p_2\cdots p_k-1}, (-t)^{p_1p_2\cdots p_k-1}, (0)^{(2t-2)(p_1p_2\cdots p_k-1)}, (-1)^{t-1}, (t-1)^1\}$.
   - $L\text{-}Spec(G(\mathbb{Z}_n)) = \{(2t)^{p_1p_2\cdots p_k-1}, (0)^{p_1p_2\cdots p_k-1}, (t)^{(2t-2)(p_1p_2\cdots p_k-1)}, (0)^1, (t)^{t-1}\}$
   - $Q\text{-}Spec(G(\mathbb{Z}_n)) = \{(2t)^{p_1p_2\cdots p_k-1}, (0)^{p_1p_2\cdots p_k-1}, (t)^{(2t-2)(p_1p_2\cdots p_k-1)}, (2t-2)^1, (t-2)^{t-1}\}$.
2. *If $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes, then*
   - $Spec(G(\mathbb{Z}_n)) = \{(t)^{[\frac{p_1p_2\cdots p_k}{2}]}, (-t)^{[\frac{p_1p_2\cdots p_k}{2}]}, (0)^{(2t-2)([\frac{p_1p_2\cdots p_k}{2}])}\}$.
   - $L\text{-}Spec(G(\mathbb{Z}_n)) = \{\{(2t)^{[\frac{p_1p_2\cdots p_k}{2}]}, (0)^{[\frac{p_1p_2\cdots p_k}{2}]}, (t)^{(2t-2)([\frac{p_1p_2\cdots p_k}{2}])}\}\}$
   - $Q\text{-}Spec(G(\mathbb{Z}_n)) = \{\{(2t)^{[\frac{p_1p_2\cdots p_k}{2}]}, (0)^{[\frac{p_1p_2\cdots p_k}{2}]}, (t)^{(2t-2)([\frac{p_1p_2\cdots p_k}{2}])}\}\}$.

Proof. 1. Let $s = 2p_1p_2\cdots p_k$ and consider the partition of $V(G(\mathbb{Z}_n))$, $P_1 = \{1, 2, \cdots, s-1\}$, $P_2 = \{s+1, s+2, \cdots, 2s-1\}$, $\cdots$ $P_l = \{n-s+1, n-s+2, \cdots, n-1\}$. Observe that $|Nil(\mathbb{Z}_n)| = l$. Let us collect $\frac{s}{2}, 3\frac{s}{2}, \cdots(2l-1)\frac{s}{2}$ from the partition $P_1, P_2, \cdots, P_l$ respectively and say $L = \{\frac{s}{2}, 3\frac{s}{2}, \cdots(2l-1)\frac{s}{2}\}$. Consider $Q_i = P_i \setminus \{(2i-1)\frac{s}{2}\}$, for $1 \le i \le l$ and so $|Q_i|$ is even. Since every pair of elements of L are adjacent and $2a \in Nil(\mathbb{Z}_n)$, for all $a \in L$, hence L induces a complete subgraph of $G(\mathbb{Z}_n)$. Next for $1 \le i \le s-1$, define $B_{-i} = \{m - i\ ;\ m \in Nil(\mathbb{Z}_n)\}$ and $B_{+i} = \{m + i\ ;\ m \in Nil(\mathbb{Z}_n)\}$. Then proceeding similar to the proof of Theorem 2.3(ii), we conclude that if x and y are adjacent and $x \in B_{+j}$ for some $1 \le j \le s-1$, then $y \in B_{-j}$. Hence we get $G(\mathbb{Z}_n)$ consists of $s-1$ complete bipartite subgraphs $K_{|Nil(\mathbb{Z}_n)|,|Nil(\mathbb{Z}_n)|}$ and one complete subgraph $K_{|Nil(\mathbb{Z}_n)|}$ and all subgraphs mentioned are disjoint. Hence the result follows.

2. It follows from Corollary 2.4.

□

Next we study the spectrum, Laplacian spectrum and signless Laplacian spectrum of a ring of odd order.

Theorem 5.2. *If R is a ring of odd order, then the graph $G(R)$ is bipartite having disjoint complete bipartite subgraphs.*

Proof. Clearly $|Nil(R)|$ is odd as R is a commutative ring. Let $Nil(R) = \{n_1, n_2, \cdots, n_k\}$. For $x \in R \setminus Nil(R)$, consider $A_{x^+} = \{n_i + x\ :\ 1 \le i \le k\}$ and $A_{x^-} = \{n_i - x\ :\ 1 \le i \le k\}$. Observe that every element of A_{x^+} is adjacent to every element of A_{x^-} and no two elements of A_{x^+} or no two elements of A_{x^-} are adjacent by Lemma 3.1. So $A_{x^+} \cup A_{x^-}$ induces a complete bipartite subgraph of $G(R)$. Let $y \in R \setminus Nil(R)$ with $x \ne y$.
Case I: Let $x + y \in Nil(R)$, then $y = n_i - x$, for some $1 \le i \le k$. So we get $A_{x^+} = A_{y^-}$ and $A_{x^-} = A_{y^+}$.
Case II: Let $x + y \in R \setminus Nil(R)$. If $x - y \in Nil(R)$, then $x = n_j + y$, for some $1 \le i \le k$ and we get $A_{x^+} = A_{y^+}$ and $A_{x^-} = A_{y^-}$. If $x - y \in R \setminus Nil(R)$, then $(A_{x^+} \cup A_{x^-}) \cap (A_{y^+} \cup A_{y^-}) = \emptyset$. So for distinct $x, y \in R \setminus Nil(R)$, subgraph induced by $A_{x^+} \cup A_{x^-}$ and subgraph induced by $A_{y^+} \cup A_{y^-}$ are either identical or disjoint. Hence the graph $G(R)$ is bipartite having disjoint complete bipartite subgraphs. □

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Theorem 5.3. *If R be a ring of odd order, then the number of disjoint complete bipartite subgraphs of $G(R)$ is $\frac{|R|-|Nil(R)|}{2|Nil(R)|}$.*

Proof. Let $x \in R \setminus Nil(R)$ and $Nil(R) = \{n_1, n_2, \cdots, n_k\}$. From the above proof it is enough to show $|A_{x^+} \cup A_{x^-}| = 2|Nil(R)|$. Clearly $n_i + x = n_j + x$ or $n_i - x = n_j - x$ implies $i = j$. Also $n_i - x = n_j + x$ implies $2x \in Nil(R)$, a contradiction by Lemma 3.1. Hence the result follows. □

Theorem 5.4. *If R is a ring of odd order, then*

- $Spec(G(R)) = \{(|Nil(R)|)^m, (-|Nil(R)|)^m, (0)^{2m(|Nil(R)|-1)}\}$
- $L\text{-}Spec(G(R)) = \{(2|Nil(R)|)^m, (0)^m, (|Nil(R)|)^{2m(|Nil(R)|-1)}\}$
- $Q\text{-}Spec(G(R)) = \{(2|Nil(R)|)^m, (0)^m, (|Nil(R)|)^{2m(|Nil(R)|-1)}\}$.

Where $m = \frac{|R|-|Nil(R)|}{2|Nil(R)|}$.

Proof. Proof follows from Theorem 5.2 and Theorem 5.3, as $Spec(K_{n,n}) = \{(n)^1, (-n)^1, (0)^{2n-2}\}$, $L\text{-}Spec(K_{n,n}) = \{(2n)^1, (0)^1, (n)^{2n-2}\}$ and $Q\text{-}Spec(K_{n,n}) = \{(2n)^1, (0)^1, (n)^{2n-2}\}$. □

Corollary 5.5. *If R is a ring of odd order, then $G(R)$ can not be Eulerian. In fact no components of $G(R)$ is Eulerian.*

Proof. Since $G(R)$ is not connected, so $G(R)$ is not Eulerian. By Theorem 5.2 and Theorem 5.3, each component of $G(R)$ is of the form $K_{|Nil(R)|,|Nil(R)|}$. Since $|Nil(R)|$ is odd so $K_{|Nil(R)|,|Nil(R)|}$ can not be Eulerian. □

Corollary 5.6. *If R is a ring of odd order with $|Nil(R)| > 3$, then $G(R)$ can not be Hamiltonian. But all components of $G(R)$ are Hamiltonian.*

Proof. Since $|Nil(R)| > 3$, so $G(R)$ is not connected as $G(R)$ has atleast two components by Theorem 5.2 and Theorem 5.3, hence $G(R)$ is not Hamiltonian. By Theorem 5.2 and Theorem 5.3, each component of $G(R)$ is of the form $K_{|Nil(R)|,|Nil(R)|}$. But every complete bipartite graph is Hamiltonian, so the theorem follows. □

Corollary 5.7. *Let R be a ring of odd order with $|Nil(R)| = 3$, then we have the following:*

1. *If $\frac{|R|}{|Nil(R)|} = 3$, then $G(R)$ is Hamiltonian.*
2. *If $\frac{|R|}{|Nil(R)|} > 3$, then $G(R)$ is not Hamiltonian.*

Proof. Proof follows from Theorem 5.2 and Theorem 5.3. □

Corollary 5.8. *Let R be a ring of odd order and $|Nil(R)| \geq 3$. If $\frac{|R|}{|Nil(R)|} > 1$, then $G(R)$ can not be a planar graph.*

Proof. Since a bipartite graph can not be planar if it has $K_{3,3}$ minor. But from Theorem 5.2 and Theorem 5.3, $G(R)$ has atleast one component which contains a $K_{3,3}$ minor, as $|Nil(R)| \geq 3$ and $\frac{|R|}{|Nil(R)|} > 1$. Hence the result follows. □

6 Dominating number

For a graph G, a subset D of the vertex set of G is said to be a dominating set of G if every vertex not in D is adjacent to at least one member of D. The dominating number is the number of vertices in the minimal dominating set for G. In this section we study the dominating number of $G(\mathbb{Z}_n)$ and $G(R)$, where R is a ring of odd order.

Theorem 6.1. *For $G(\mathbb{Z}_n)$, the following hold:*

1. *If n is odd, then the dominating number is $\frac{n-|Nil(\mathbb{Z}_n)|}{2}$.*
2. *If n is even, then the dominating number is $\frac{n-2|Nil(\mathbb{Z}_n)|}{2}+1$.*

Proof.
1. From Corollary 2.4, if n is odd, then $G(\mathbb{Z}_n)$ is a disjoint union of $\frac{n-|Nil(\mathbb{Z}_n)|}{2|Nil(\mathbb{Z}_n)|}$ complete bipartite subgraphs $K_{|Nil(\mathbb{Z}_n)|,|Nil(\mathbb{Z}_n)|}$. Hence the result follows.
2. Similarly it follows from Corollary 2.5.

□

Theorem 6.2. *If R is a finite commutative ring of odd order, then the dominating number is $\frac{|R|-|Nil(R)|}{2}$.*

Proof. Proof follows from Theorem 5.2 and Theorem 5.3. □

7 Chromatic index

Vizings Theorem [3] says that $\triangle \leq \chi'(G) \leq \triangle + 1$, where $\triangle$ is the maximum vertex degree of G. Graphs satisfying $\chi'(G) = \triangle$ are called graphs of class 1 and those with $\chi'(G) = \triangle+1$ are called graphs of class 2.

Theorem 7.1. *Nilpotent graph of any ring R is of class* 1.

Proof. Put colour $x+y$ for an edge xy of $G(R)$. Let $C = \{x+y \,:\, xy \text{ is an edge of } G(R)\}$ then C is the set of colours of $G(R)$. Therefore $G(R)$ has a $|C|$-*edge* colouring, so $\chi'(G(R)) \leq |C|$. But $C \subseteq Nil(R)$ and $\chi'(G(R)) \leq |C| \leq |Nil(R)|$. Also $\triangle \leq |Nil(R)|$, hence by Vizings Theorem we get $\chi'(G(R)) = |Nil(R)| = \triangle$. □

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References

[1] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi. Unit graphs associated with rings. *Comm. Algebra*, 38:2851–2871, 2010.

[2] D. K. Basnet and J. Bhattacharyya. Nil clean graphs of rings. *Algebra Colloq.*, 24(3):481–492, 2017.

[3] R. Diestel. Graph Theory. Springer-Verlag, New York 1997.

[4] R. P. Grimaldi. Graphs from rings. *Proceedings of the 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Atlantic University*, 71:95–103, 1990.