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Recursive Formulas for Beans Functions of Graphs

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Recursive Formulas for Beans Functions of Graphs

Cover Page Footnote

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Abstract

In this paper, we regard each edge of a connected graph G as a line segment having a unit length, and focus on not only the “vertices” but also any “point” lying along such a line segment. So we can define the distance between two points on G as the length of a shortest curve joining them along G . The beans function $B_G(x)$ of a connected graph G is defined as the maximum number of points on G such that any pair of points have distance at least $x > 0$. We shall show a recursive formula for $B_G(x)$ which enables us to determine the value of $B_G(x)$ for all $x \leq 1$ by evaluating it only for $1/2 < x \leq 1$. As applications of this recursive formula, we shall propose an algorithm for computing $B_G(x)$ for a given value of $x \leq 1$, and determine the beans functions of the complete graphs K_n .

1 Introduction

A graph $G = (V(G), E(G))$ is usually defined as a composite structure of a non-empty and finite set $V(G)$ and a family $E(G)$ of 2-element subsets in $V(G)$, possibly empty. Each element in $V(G)$ is called a *vertex* while each element in $E(G)$ is called an *edge*. To visualize this structure, we often draw a figure consisting of points and line segments, which correspond to vertices and edges, respectively. In this paper, we regard a graph G as such a 1-dimensional figure and consider that the line segments are mutually disjoint except at their endpoints. Then, there are infinitely many points on each edge, and an endpoint of an edge lies at a vertex. Note that we treat a “vertex” and a “point” as different objects. Furthermore, each edge is assumed to have a unit length.

Under this setting, we can define the distance $d_G(p, q)$ between two points p and q as the length of a shortest curve joining them along a connected graph G and make G be a metric space having the distance $d_G(\cdot, \cdot)$. Of course, p and q may be not only intermediate points on line segments but also points at vertices. Now a graph G should be regarded as the set of points lying over its vertices and edges rather than a combinatorial structure.

In this situation, Negami [10] defined “beans functions” as follows. Let $x > 0$ be a positive real number. A set S of points on a connected graph G is called an x -set if for any points $p, q \in S$ one has $d(p, q) \geq x$. In particular, if S has the maximum cardinality among the x -sets in G , then S is said to be *maximum*. We denote the cardinality of a maximum x -set of G by $B_G(x)$. That is, $B_G(x) = |S|$ for a maximum x -set S . Then, we can regard $B_G(x)$ as a non-increasing function $B_G : \mathbb{R}^+ \rightarrow \mathbb{N}$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and call it the *beans function* of G .

For example, it is easy to see that $B_{C_n}(x) = \max\{\lfloor n/x \rfloor, 1\}$ for the cycle C_n of length n ; it suffices to put points at equal intervals of length x . Similarly, we have $B_{P_{n+1}}(x) = \lfloor n/x \rfloor + 1$ for the path of order $n + 1$. Such points arranged at equal intervals can be compared to a row of beans. That is why the function is called the “beans function”.

The beans function is related to the following two interesting invariants of graphs. Let k be a non-negative integer. A subset $I \subseteq V(G)$ is k -independent if any two vertices in I have distance at least $k + 1$. The k -independence number of G , denoted by $\alpha_k(G)$, is the number of vertices in a maximum k -independent set of G . A subset $X \subseteq E(G)$ is *distance- k matching* if any two edges in X have distance at least k . (The distance between two edges

$x = u_1u_2$ and $y = v_1v_2$ is defined as the length of a shortest u_i-v_j path for $i, j \in \{1, 2\}$.) The *distance- k matching number* of G , denoted by $\mu_k(G)$, is the number of edges in a maximum distance- k matching of G . Note that the values $\alpha_0(G)$, $\mu_0(G)$, $\alpha_1(G)$ and $\mu_1(G)$ are equal to $|V(G)|$, $|E(G)|$, the independence number of G and the matching number of G , respectively. Moreover, a distance-2 matching is usually called an *induced matching*. These invariants have been studied in various papers and are closely related to many other concepts (see [1, 2, 6, 7, 8, 9, 11, 12] for examples). The magnitude relation between the beans function $B_G(x)$ for a given value of $x \geq 1$ and each of the two invariants will be explained in detail in Section 2.

On the other hand, for $x \leq 1$, Negami's and Enami's works [10, 4] suggest that there might be a repeating pattern among the values of $B_G(x)$, as follows.

Put $A_n = (1/(n+1), 1/n]$ for a positive integer n . Then, the interval $(0, 1]$ splits into these subintervals $\bigcup_{n=1}^{\infty} A_n$. Negami [10] has given the following lower and upper bounds of $B_G(x)$ for $x \in A_n$:

Theorem 1.1 (Negami [10]). *Let n be a positive integer and $x \in A_n = (1/(n+1), 1/n]$. Then*

$$n|E(G)| + t \leq B_G(x) \leq n|E(G)| + |V(G)| - 1,$$

where $t = 1$ if G is tree, and $t = 0$ otherwise.

In addition, he has shown that $B_G(1/n) = n|E(G)| + t$, that is, $B_G(x)$ attains the above lower bound when $x = 1/n$. Furthermore, Enami [4] has shown that $B_G(x)$ also attains the upper bound over every interval A_n . Roughly speaking, $B_G(x)$ seems to increase by $|E(G)|$ when n increments. In this paper, we shall conclude that this actually does also for any intermediate value x , showing the following recursive formula:

Theorem 1.2. *Let G be a connected graph and let $x \leq 1$ be a positive real number. Then,*

$$B_G\left(\frac{x}{1+x}\right) = B_G(x) + |E(G)|.$$

It should be noticed that the function $f(x) = x/(1+x)$ is monotonically increasing and induces a one-to-one correspondence between A_n and A_{n+1} . Thus, the above formula shows that the difference between $B_G(x)$'s for corresponding values in A_n and A_{n+1} is equal to $|E(G)|$. Using this, we can show the relation between the corresponding values in A_n and A_{n+k} as the following corollary. If x belongs to A_n , then $x/(1+kx)$ belongs to A_{n+k} and we have $A_{n+k} = \{x/(1+kx) : x \in A_n\}$.

Corollary 1.3. *Let $x \leq 1$ be a positive real number and k a natural number. Then,*

$$B_G\left(\frac{x}{1+kx}\right) = B_G(x) + k|E(G)|.$$

This implies that the values of $B_G(x)$ over A_1 determines all values of $B_G(x)$. The formula in the following corollary makes it clearer. Notice that if x belongs to A_n , then we have $1 = nx + \varepsilon$ with $0 \leq \varepsilon < x$ and $x/(1 - (n-1)x)$ belongs to A_1 .

Corollary 1.4. *Let $x \leq 1$ be a positive real number and put $n = \lfloor 1/x \rfloor$. Then,*

$$B_G(x) = B_G\left(\frac{x}{1 - (n-1)x}\right) + (n-1)|E(G)|.$$

In Section 3, we shall prove Theorem 1.2 and its corollaries. As applications of our recursive formulas, we shall propose an algorithm for computing $B_G(x)$ for a given value of $x \leq 1$ in Section 4, and we shall determine the beans functions of the complete graphs K_n in Section 5. Our terminology for graph theory is quite standard and can be found in [3].

2 Relation to other invariants of graphs

Let G be a connected graph. When $x \geq 1$, $B_G(x)$ is closely related to the combinatorial invariants of G mentioned in introduction, the k -independence number and the distance- k matching number.

Lemma 2.1. *Let G be a connected graph and k be a non-negative integer. Then $B_G(k+2) \leq \alpha_k(G) \leq B_G(k+1)$.*

Proof. A set of points located at each vertex in a maximum k -independent set of G is a $(k+1)$ -set. Thus, $\alpha_k(G) \leq B_G(k+1)$.

Let $S = \{p_1, p_2, \dots, p_m\}$ be a maximum $(k+2)$ -set on G . For each point p_i for $1 \leq i \leq k$, let v_i be a nearest vertex to p_i . Since p_i and v_i lie on a common edge, which has length 1, we have $d_G(p_i, v_i) \leq 1/2$. Then, for any two distinct points p_i and p_j , we have $v_i \neq v_j$ and $d_G(v_i, v_j) \geq d_G(p_i, p_j) - 2 \cdot 1/2 \geq k+1$. Therefore, the set of vertices $I = \{v_1, v_2, \dots, v_m\}$ is a k -independent set with m vertices, and hence $B_G(k+2) \leq \alpha_k(G)$. \square

Lemma 2.2. *Let G be a connected graph and k be a non-negative integer. Then $B_G(k+2) \leq \mu_k(G) \leq B_G(k+1)$.*

Proof. A set of midpoints of edges in a maximum distance- k matching of G is a $(k+1)$ -set. Thus, $\mu_k(G) \leq B_G(k+1)$.

Let $S = \{p_1, p_2, \dots, p_m\}$ be a maximum $(k+2)$ -set on G . For each point p_i for $1 \leq i \leq k$, let e_i be an edge containing p_i . Since the length of e_i is 1, p_i is located at most 1 away from an endvertex of e_i . Then, for any two distinct points p_i and p_j , we have $e_i \neq e_j$ and $d_G(e_i, e_j) \geq d_G(p_i, p_j) - 2 \cdot 1 \geq k$. Therefore, the set of edges $I = \{e_1, e_2, \dots, e_m\}$ is distance- k matching with m edges, and hence $B_G(k+2) \leq \mu_k(G)$. \square

By these lemmas, it is easy to show the following bounds of $B_G(x)$.

Theorem 2.3. *Let G be a connected graph and x be a positive real number. Then:*

$$(1) \ B_G(x) \geq \max\{\alpha_{\lceil x \rceil - 1}(G), \mu_{\lceil x \rceil - 1}(G)\} \text{ for } x \geq 1;$$

$$(2) \ B_G(x) \leq \min\{\alpha_{\lfloor x \rfloor - 2}(G), \mu_{\lfloor x \rfloor - 2}(G)\} \text{ for } x \geq 2. \quad \square$$

3 Recursive formulas

In this section, we prove the main theorem and its corollaries.

Proof of Theorem 1.2. Put $y = x/(1+x)$. First, we shall show that

$$B_G(y) \geq B_G(x) + |E(G)|.$$

Let S be a maximum x -set. We partition S into a disjoint union $\bigcup_{e \in E(G)} S_e$. The set S_e consists of the points in S lying along an edge $e = uv$. However, if a point p in S is located at a vertex v , then we choose only one of the edges incident to v , say e , and consider that p belongs to S_e and to no others.

Suppose that S_e is not empty for an edge $e = uv$ and let p_1, \dots, p_k be the points in S_e lying along e from u in this order. Thus, p_k is the nearest point from v in S_e . Replace each point p_i with a new point p'_i satisfying the following condition:

$$d_G(p'_i, u) = \frac{1}{1+x} \cdot d_G(p_i, u)$$

After that, we add another point p'_{k+1} on e with $d_G(p'_{k+1}, v) = 1/(1+x) \cdot d_G(p_k, v)$. If there is no point in S_e , then we add a new point p'_1 at the center of e . That is, $d_G(p'_1, u) = d_G(p'_1, v) = 1/2$ exceptionally.

Let S'_e be the set of points p'_1, \dots, p'_{k+1} so obtained from S_e and put $S' = \bigcup_{e \in E(G)} S'_e$. We can evaluate the distance between each consecutive pair of points $p'_1, \dots, p'_k, p'_{k+1}$ as follows:

$$\begin{aligned} d_G(p'_{i+1}, p'_i) &= d_G(p'_{i+1}, u) - d_G(p'_i, u) \\ &= \frac{1}{1+x} \cdot (d_G(p_{i+1}, u) - d_G(p_i, u)) \\ &= \frac{1}{1+x} \cdot d_G(p_{i+1}, p_i) \geq \frac{x}{1+x} = y \\ d_G(p'_{k+1}, p'_k) &= (1 - d_G(p'_k, u)) - d_G(p'_{k+1}, v) \\ &= 1 - \left(\frac{1}{1+x} \cdot d_G(p_k, u) + \frac{1}{1+x} \cdot d_G(p_k, v) \right) \\ &= 1 - \frac{1}{1+x} = \frac{x}{1+x} = y \end{aligned}$$

Now let e_1 and e_2 be two edges having a common end w and let q'_i be the nearest point from w in S'_{e_i} for $i = 1, 2$. Then each of q'_i 's can be obtained as either p'_1 with $w = u$ or p'_{k+1} with $w = v$ in the previous. In either case, there is a point q_i in S_{e_i} such that $d_G(q'_i, w) = 1/(1+x) \cdot d_G(q_i, w)$. Thus we have:

$$\begin{aligned} d_G(q'_1, q'_2) &= d_G(q'_1, w) + d_G(q'_2, w) \\ &= \frac{1}{1+x} \cdot d_G(q_1, w) + \frac{1}{1+x} \cdot d_G(q_2, w) \\ &= \frac{1}{1+x} \cdot d_G(q_1, q_2) \geq \frac{x}{1+x} = y \end{aligned}$$

In the exceptional case, that is, when S_{e_1} or S_{e_2} are empty, we have $d_G(q'_1, q'_2) \geq 1/2 \geq y$. Therefore, S' becomes a y -set and $|S'| = |S| + |E(G)| = B_G(x) + |E(G)|$. This implies the desired inequality.

Secondly, we shall show that

$$B_G(y) \leq B_G(x) + |E(G)|.$$

Let S be a maximum y -set and let $S_e = \{p_1, \dots, p_{k+1}\}$ be the set of points in S lying along an edge $e = uv$ from u in this order. If S_e is empty, then S with the midpoint of e added is still an y -set since $y \leq 1/2$. This is contrary to the maximality of S . Thus, S_e contains at least one point p_1 while the point p_{k+1} is the nearest point from v in S_e . We may assume that $d_G(p_{i+1}, p_i) = y$ for $i = 1, \dots, k$ after moving the points p_i 's slightly toward p_1 .

Remove p_{k+1} from S_e and replace each point p_i in S_e for $i = 1, \dots, k$ with a point p'_i satisfying the condition

$$d_G(p'_i, u) = (1+x) \cdot d_G(p_i, u).$$

Then we have:

$$\begin{aligned} d_G(p'_{i+1}, p'_i) &= (1+x) \cdot (d_G(p_{i+1}, u) - d_G(p_i, u)) \\ &= (1+x) \cdot d_G(p_{i+1}, p_i) = (1+x) \cdot y = x \end{aligned}$$

Since $d_G(p_1, u) + ky + d_G(p_{k+1}, v) = 1$ and $(1+x)y = x$,

$$\begin{aligned} d_G(p'_k, v) &= 1 - (1+x)((k-1)y + d_G(p_1, u)) \\ &= 1 - (1+x)((k-1)y + (1 - d_G(p_{k+1}, v) - ky)) \\ &= 1 - (1+x)(1 - y - d_G(p_{k+1}, v)) \\ &= (1+x) \cdot d_G(p_{k+1}, v). \end{aligned}$$

Let S'_e be the set of points so obtained from S_e and put $S' = \bigcup_{e \in E(G)} S'_e$. Now let e_1 and e_2 be two edges having a common end v and let q_i be the nearest point from v in S_{e_i} . Combining the above equality, we can conclude that $d_G(q_1, q_2) \geq (1+x)y = x$ in all cases. Thus, S' becomes an x -set and $|S| - |E(G)| = |S'|$. This implies that $B_G(y) - |E(G)| \leq B_G(x)$ and the desired inequality follows. \square

Proof of Corollary 1.3. Put $f(x) = x/(1+x)$. Then it is easy to see that the k th iteration of f can be given as:

$$f^k(x) = \frac{x}{1+kx}$$

Applying the recursive formula given in Theorem 1.2, we can carry out the following calculation:

$$\begin{aligned} B_G\left(\frac{x}{1+kx}\right) &= B_G(f^k(x)) = B_G(f^{k-1}(x)) + |E(G)| \\ &= \dots = B_G(x) + k|E(G)| \end{aligned}$$

\square

Proof of Corollary 1.4. Put $X = x/(1 - (n - 1)x)$. Then it is easy to see that:

$$x = \frac{X}{1 + (n - 1)X}$$

Substituting this and $k = n - 1$ to the formula given in Corollary 1.3, we obtain the formula in the corollary. \square

4 Algorithm for computing the value of $B_G(x)$

Our recursive formulas enable us to suggest an algorithm for computing the value of $B_G(x)$ for any connected graph G and for any given value of $0 < x \leq 1$, say $x = a$

1. If $a \leq 1/2$, then translate it into the corresponding value $c = a/(1 - (n - 1)a)$ in A_1 , where $n = \lfloor 1/a \rfloor$. By Corollary 1.4, we only have to evaluate $B_G(c)$.
2. For each edge e , assign 0, 1 or 2 and prepare as many unknowns as this number. Since $c > 1/2$, each edge contains at most two points in a c -set. Then, this assignment corresponds to a possible configuration of points, and these unknowns represent the distances from one end of e to the corresponding points lying on e .
3. Set up simultaneous linear inequalities for all unknowns to bound the distances between consecutive pairs of points by c and solve them. If they have a solution, then it implies that $B_G(c) \geq s$, where s stands for the number of points contained in the given configuration.
4. Repeating the second and third step for any configurations of points and determine the maximum value of s taken over the inequalities having solutions. The maximum s is none other than $B_G(c)$. Translate it to the value of $B_G(a)$ by the recursive formula.

The computational complexity of this algorithm does not depend on the input value of $x = a$ and depends only on the graph G . Unfortunately, exponentially many configurations will be generated with no restriction. However, they should be restricted since any edge assigned 0 must be adjacent to another edge with 1 or 2 for example. In addition, by Theorem 1.1, we have known that $|E(G)| \leq B_G(c) \leq |E(G)| + |V(G)| - 1$ for the corresponding value c , and hence we need to investigate possible configurations of at least $|E(G)|$ and at most $|E(G)| + |V(G)| - 1$ points lying on G and their distances. More clever ideas might reduce the complexity of this algorithm.

Furthermore, we would like to find an algorithm for determining the formula for $B_G(x)$ over the interval A_1 , which is divided into several subintervals, like the formulas for $B_{K_m}(x)$ of the complete graph of order m , which can be given in the next section. The authors have already discussed such dividing points of A_1 at which $B_G(x)$ is discontinuous in [5].

5 The complete graphs

As one of applications of our recursive formulas for $B_G(x)$, we shall determine the beans functions of the complete graphs K_m . It is easy to see that $B_{K_1}(x) = 1$, $B_{K_2}(x) = \lfloor 1/x \rfloor + 1$ and $B_{K_3}(x) = \max\{\lfloor 3/x \rfloor, 1\}$ for all $x > 0$. Thus, we shall assume that $m \geq 3$ hereafter. If we know the values of $B_G(x)$ only for $x \in A_1$, that is, $1/2 < x \leq 1$, then we can determine all values of $B_G(x)$. So we shall investigate the values of $B_{K_m}(x)$, decomposing the interval $A_1 = (1/2, 1]$ into four parts as follows.

Lemma 5.1. *If $3/4 < x \leq 1$, then we have*

$$B_{K_m}(x) = |E(K_m)| = \frac{m(m-1)}{2}.$$

Proof. It is clear that the set of the midpoints of all edges forms an x -set for any positive real number $x \leq 1$ in general. Thus, we have $B_{K_m}(x) \geq |E(K_m)| = m(m-1)/2$.

Let \mathcal{C} be the family of cycles of length 3 in K_m and let Δ be any cycle in \mathcal{C} . Let S be a maximum x -set with $x \leq 1$. We denote the set of points of S contained in Δ by S_Δ ; that is, $S_\Delta = S \cap \Delta$. Since each edge is contained in $m-2$ cycles in \mathcal{C} and $|S_\Delta| \leq \lfloor 3/x \rfloor$, we have

$$(m-2)B_{K_m}(x) \leq \sum_{\Delta \in \mathcal{C}} |S_\Delta| \leq \binom{m}{3} \cdot \lfloor 3/x \rfloor.$$

This implies that

$$B_{K_m}(x) \leq \frac{m(m-1)}{3 \cdot 2} \cdot \lfloor 3/x \rfloor = |E(K_m)| \cdot (1/3) \cdot \lfloor 3/x \rfloor.$$

If $3/4 < x \leq 1$, then we have $\lfloor 3/x \rfloor = 3$ and $B_{K_m}(x) \leq |E(K_m)|$. Therefore, we have $B_{K_m}(x) = |E(K_m)|$. \square

Consider the partition $S = \bigcup_{e \in E(K_m)} S_e$ of a maximum x -set in G as in the previous section and let E_i be the set of edges e of G such that $|S_e| = i$. Notice that $|S_e|$ may not coincide with the number of points in S lying along e since a point in G placed at a vertex v belongs to exactly one of S_e 's for edges e incident to v . Here we assume that $x > 1/2$. Then each edge contains at most two points of S and hence $E(G) = E_0 \cup E_1 \cup E_2$. Thus we have $|E(G)| = |E_0| + |E_1| + |E_2|$ and

$$|S| = B_G(x) = |E_1| + 2|E_2| = |E(G)| + |E_2| - |E_0|$$

in general. We shall use these notations throughout the proofs below.

A set of edges of a graph G is called a *matching* if no two edges in the set have a common end. A *maximum matching* in G is a matching of the maximum cardinality among all matchings in G and its cardinality is denoted by $\mu(G)$. (As remarked in Section 2, $\mu(G) = \mu_1(G)$.) It is clear that $\mu(G) = \lfloor m/2 \rfloor$ for the complete graph K_m .

Lemma 5.2. *If $2/3 < x \leq 3/4$, then we have*

$$B_{K_m}(x) = |E(K_m)| + 1 = \frac{m(m-1)}{2} + 1.$$

Proof. First, we shall show an example of an x -set S in K_m for $x \in (2/3, 3/4]$. Choose one edge $e = uv$ of K_m with endpoints u and v , and take the following points as those in S :

- (i) Two points p and q lying on e with $d_{K_m}(p, u) = 1/8$ and $d_{K_m}(q, v) = 1/8$
- (ii) A point r lying on each edge incident to u (or v) with $d_{K_m}(r, u) = 5/8$ (or $d_{K_m}(r, v) = 5/8$)
- (iii) The midpoints of all edges other than e and edges incident to u or v

The set S consists of these $|E(K_m)| + 1$ points and it is easy to see that it forms an x -set if $x \leq 3/4$. This implies that $B_{K_m}(x) \geq |E(K_m)| + 1$.

Now let S be any maximum x -set in K_m . Then we have $B_{K_m}(x) = |E(K_m)| + |E_2| - |E_0|$. Since $B_{K_m}(x) \geq |E(K_m)| + 1$, we have $|E_2| - |E_0| \geq 1$ and hence $|E_2| > |E_0|$. If there are two edges in E_2 which have a common end, then they form a path of length 2 and the four points in S divide the path into at least three segments. One of these segments must have length at most $2/3$. This implies that $x \leq 2/3$, which is contrary to our assumption in the lemma. Thus, E_2 forms a matching in K_m .

Assume that E_2 contains at least two edges, say u_1v_1 and u_2v_2 . Since E_2 is a matching, they have no common end and hence form two cycles $u_1v_1u_2v_2$ and $u_1v_1v_2u_2$ in K_m . All edges in these cycles rather than u_1v_1 and u_2v_2 belong to $E_0 \cup E_1$. For each cycle, if none of the edges belongs to E_0 , then this cycle of length 4 contains six points in S and hence there is a pair of points p and q among the six with $d_{K_m}(p, q) \leq 4/6 = 2/3$. This is contrary to our assumption in the lemma, again. Thus, any two edges in E_2 are joined by at least two edge in E_0 . This implies that $|E_0| \geq 2 \binom{|E_2|}{2} \geq |E_2|$, which is contrary to $|E_2| > |E_0|$. Thus, E_2 contains only one edges and hence $|E_2| = 1 > |E_0| = 0$. Therefore, $B_{K_m}(x) = |E(K_m)| + 1$ if $2/3 < x \leq 3/4$. \square

Lemma 5.3. *If $3/5 < x \leq 2/3$, then we have*

$$B_{K_m}(x) = |E(K_m)| + \mu(K_m) = \frac{m(m-1)}{2} + \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. First, we shall show an example of an x -set in G for $x \leq 2/3$. Let M be any maximum matching in K_m , which has $\lfloor m/2 \rfloor$ edges. Place two points p and q on each edge uv in M so that $d_{K_m}(p, u) = d_{K_m}(q, v) = 1/6$. Add the midpoint of each of the other edges not belonging to M . It is easy to see that these $|E(K_m)| + \lfloor m/2 \rfloor$ points form an x -set if $x \leq 2/3$. Thus, $B_{K_m}(x) \geq |E(K_m)| + \lfloor m/2 \rfloor$.

Let S be any maximum x -set and suppose that $|S| = B_{K_m}(x) > |E(K_m)| + \lfloor m/2 \rfloor$. Then we have $|E_2| - |E_0| > \lfloor m/2 \rfloor = \mu(K_m)$. Since $|E_2| > \mu(K_m)$, E_2 contains a pair of edges which have a common end. If there exist three edges in E_2 which form a path of length 3, then the path contains six points in S and is divided into at least five segments by these points. This implies that there are a pair of points p and q in S with $d_{K_m}(p, q) \leq 3/5$, which is contrary to our assumption in the lemma. Thus, the subgraph $\langle E_2 \rangle$ induced by E_2 in K_m contains no path of length 3 and hence each of its components is isomorphic to the star $K_{1,s}$ for some $s \geq 1$.

If the edge between two vertices of degree 1 in such a component belonged to E_1 , then there would be a cycle of length 3 which contains five points and it would contain a pair of

pints p and q in S with $d_{K_m}(p, q) \leq 3/5$, contrary to our assumption, again. Therefore, any pair of the s vertices of degree 1 in the component isomorphic to $K_{1,s}$ with $s \geq 2$ are joined by an edge in E_0 and hence there are s edges in E_2 and $s(s-1)/2$ such edges in E_0 around it. Thus, each component of $\langle E_2 \rangle$ contributes to $|E_2| - |E_0|$ by a non-positive number if $s \geq 3$ and by 1 if $s = 1, 2$. We can choose one edge from each component of $\langle E_2 \rangle$ to form a matching in K_m and hence the number of components of $\langle E_2 \rangle$ does not exceed $\mu(K_m)$.

In this situation, if there were a component with $s \geq 2$, $|E_2| - |E_0|$ would be less than $\mu(K_m)$. Thus, each component of $\langle E_2 \rangle$ consists of a single edge with $s = 1$. This implies that E_2 must be a matching, but it is contrary to $|E_2| > \mu(K_m)$. Therefore we have $|S| = |E(K_m)| + \mu(K_m)$ and the lemma follows. \square

Lemma 5.4. *If $1/2 < x \leq 3/5$, then we have*

$$B_{K_m}(x) = \frac{m(m-1)}{2} + m - 1.$$

Proof. Choose any vertex v of K_m and take two points p and q on each edge incident to v with $d_{K_m}(p, v) = 3/10$ and $d_{K_m}(q, v) = 9/10$. Add the midpoints of all edges not incident to v . Then it is easy to see that these $|E(K_m)| + m - 1$ points form an x -set if $x \leq 3/5$. Thus, $|E(K_m)| + m - 1$ gives a lower bound for $B_{K_m}(x)$. By Theorem 1.1, this gives an upper bound for it, too. However, we can show it easily as follows.

Let S be a maximum x -set in K_m and suppose that $|S| = B_{K_m}(x) > |E(K_m)| + m - 1$. Then we have $|E_2| - |E_0| > m - 1$ and hence $|E_2| \geq m = |V(K_m)|$ in particular. In this case, the subgraph induced by E_2 in K_m contains a cycle C and the cycle C contains $2|C|$ points in S . These points divides C into $2|C|$ segments and hence there is a pair of points p and q in S with $d_{K_m}(p, q) \leq |C|/2|C| = 1/2$. This implies that $x \leq 1/2$, contrary to our assumption in the lemma. Therefore, we have $B_{K_m}(x) \leq |E(K_m)| + m - 1$ if $1/2 < x \leq 3/5$ and the lemma follows. \square

Theorem 5.5. *Let K_m be the complete graph of order $m \geq 3$ and let $n \geq 1$ be any natural number. Then:*

$$B_{K_m}(x) = \begin{cases} \frac{n \cdot m(m-1)}{2} + m - 1 & \left(\frac{1}{n+1} < x \leq \frac{3}{3n+2}\right) \\ \frac{n \cdot m(m-1)}{2} + \left\lfloor \frac{m}{2} \right\rfloor & \left(\frac{3}{3n+2} < x \leq \frac{2}{2n+1}\right) \\ \frac{n \cdot m(m-1)}{2} + 1 & \left(\frac{2}{2n+1} < x \leq \frac{3}{3n+1}\right) \\ \frac{n \cdot m(m-1)}{2} & \left(\frac{3}{3n+1} < x \leq \frac{1}{n}\right) \\ m - 1 & \left(1 < x \leq \frac{3}{2}\right) \\ \left\lfloor \frac{m}{2} \right\rfloor & \left(\frac{3}{2} < x \leq 2\right) \\ 1 & (2 < x) \end{cases}$$

Proof. We have evaluated the value of $B_{K_m}(x)$ for $1/2 < x \leq 1$ in Lemmas 5.1, 5.2, 5.3 and 5.4. Our recursive formulas given by Theorem 1.2 and its corollaries enable us to determine the value of $B_{K_m}(x)$ for a positive real number $x \in A_n = (1/(n+1), 1/n]$, as follows.

Any formula in the lemmas can be expressed by the form $B_{K_m}(x) = |E(K_m)| + \alpha(m)$ for $x \in A_1 = (1/2, 1]$, where $\alpha(m)$ stands for a function depending only on m . Since $X = x/(1 - (n - 1)x)$ belongs to A_1 , we have

$$B_{K_m}(x) = B_{K_m}(X) + (n - 1)|E(K_m)| = n|E(K_m)| + \alpha(m)$$

by Corollary 1.4. The function $f^{n-1}(x) = x/(1 + (n - 1)x)$ maps A_1 to A_n bijectively and translates $1/2$, $3/5$, $2/3$, $3/4$ and 1 into $1/(n + 1)$, $3/(3n + 2)$, $2/(2n + 1)$, $3/(3n + 1)$ and $1/n$, respectively. Thus, the first four cases in the formula correspond to the four cases given in Lemmas 8, 7 6 and 5 in order.

Now, we shall determine the value of $B_{K_m}(x)$ for $x > 1$. It is obvious that a pair of the farthest points lie at the midpoints of two non-adjacent edges in K_m and that they have distance 2. Thus, we have $B_{K_m}(x) = 1$ for $x > 2$. Notice that each edge contains at most one of points in any x -set for $x > 1$.

Let M a maximum matching in K_m and take the midpoint of each edge in M . Then, these $\lfloor m/2 \rfloor$ points form an x -set if $x \leq 2$. Let S be an x -set for $x \leq 2$ in K_m . If $B_{K_m}(x) > \lfloor m/2 \rfloor = \mu(K_m)$, then the edges containing points in S does not form a matching. Thus, there are two edges among them which have a common end and form a cycle of length 3 with another edge. Since such a cycle contains at least two points in S , then we have $x \leq 3/2$. Therefore, $B_{K_m}(x) = \lfloor m/2 \rfloor$ if $3/2 < x \leq 2$.

Choose a vertex v of G and take the $m - 1$ points p lying on edges incident to v so that $d_{K_m}(p, v) = 3/2$. It is clear that these points form an x -set if $x \leq 3/2$ and hence $B_{K_m}(x) \geq m - 1$. Let S be a maximum x -set in K_m . If $B_{K_m}(x) > m - 1$, then there is a cycle C consisting of some edges each of which contains one point in S . That is, C contains exactly $|C|$ points in S and we conclude that $x \leq 1$. Therefore, we have $B_{K_m}(x) = m - 1$ if $1 < x \leq 3/2$. \square

Notice that the theorem holds for K_3 of course. However, the second and third cases in the formula give the same value $3n + 1$ and hence they should be unified to one case.

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