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# Packing Densities of Colored and Non-Colored Patterns

Matthew R. Just
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# Packing Densities of Colored and Non-Colored Permutations

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Department of Mathematics

April 24, 2015

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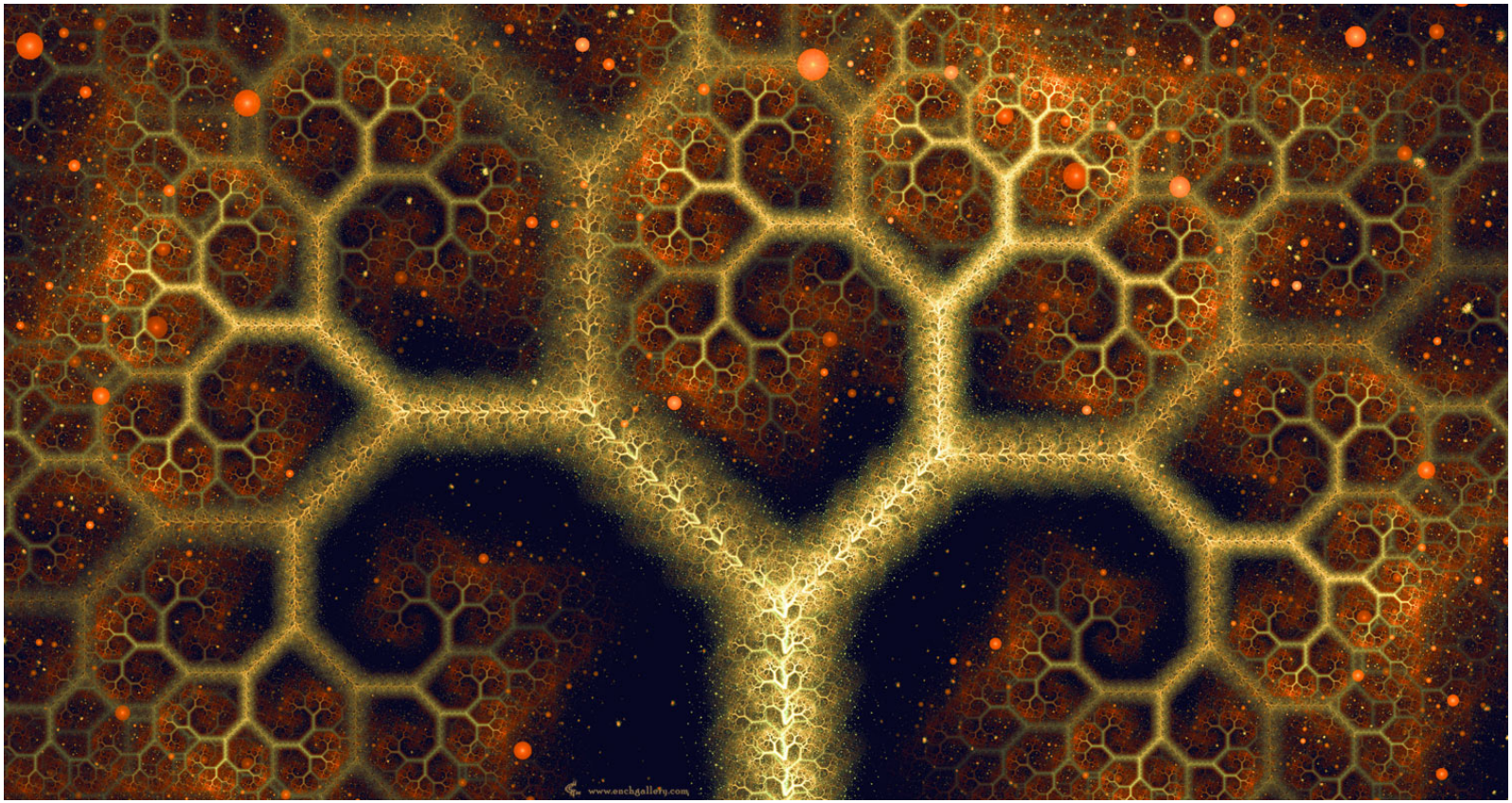

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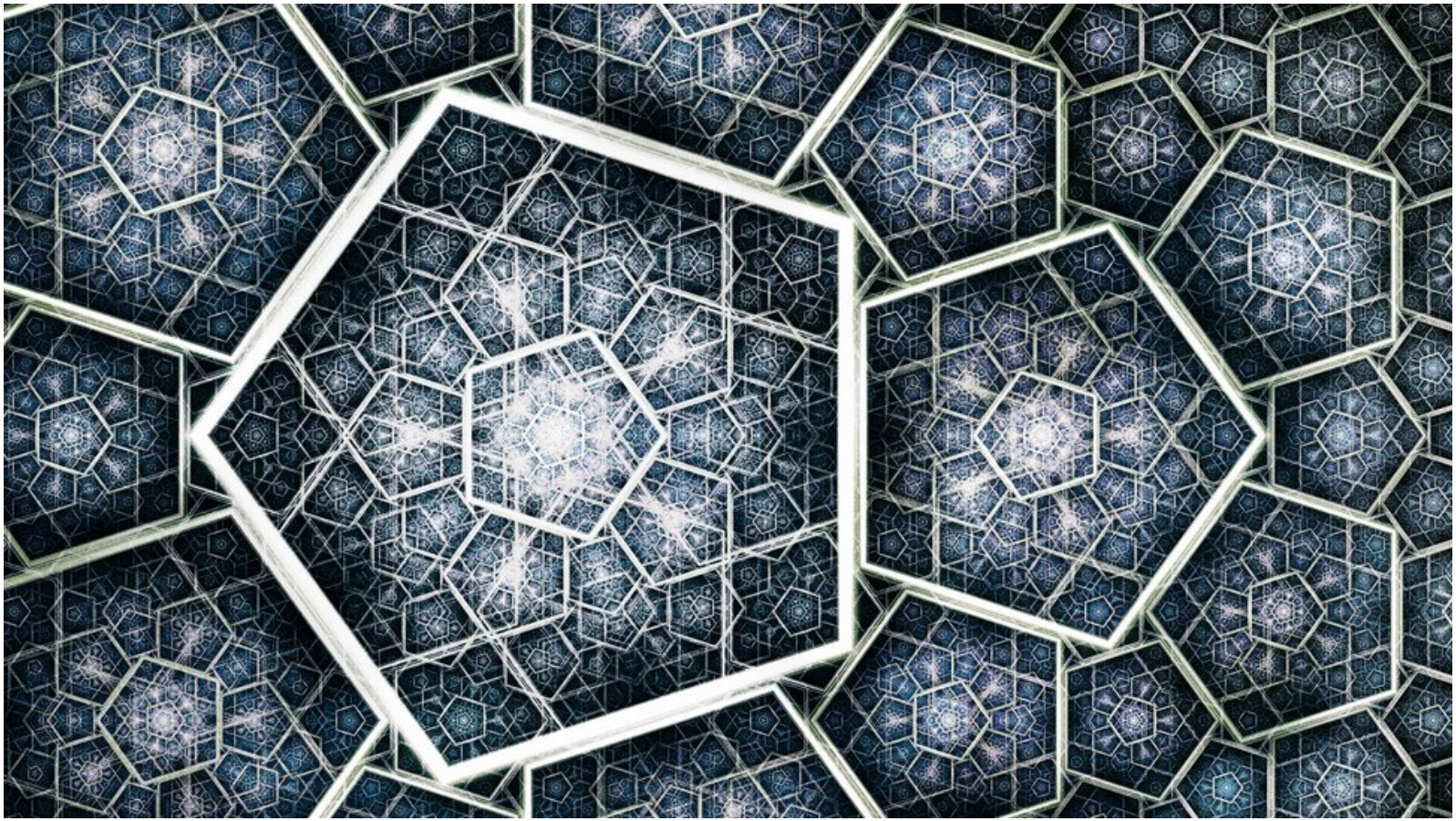

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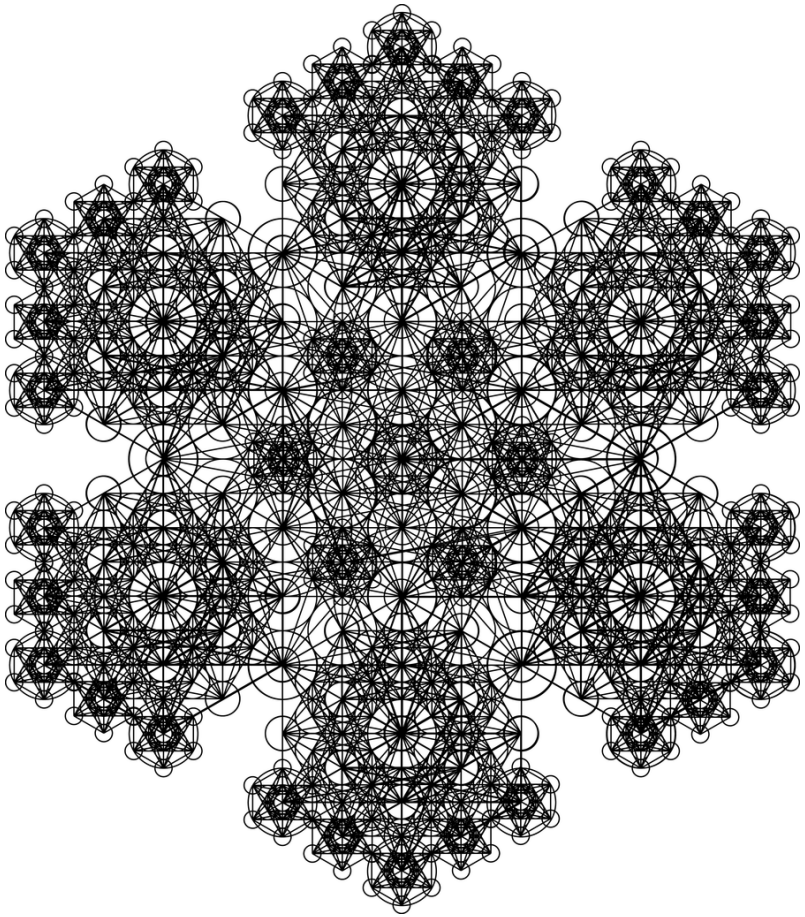

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"Hello World"

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$$9! = 362,880$$

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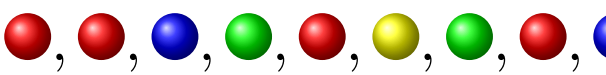
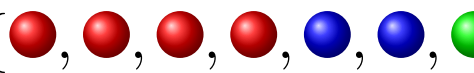
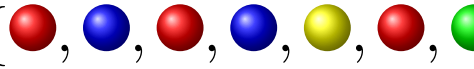
Let $M = \{$  $\}$ be a multiset. Then the following are permutations of $M$:

$M_1 = \{$  $\}$ $M_2 = \{$$\}$

$M_3 = \{$  $\}$ $M_4 = \{$$\}$

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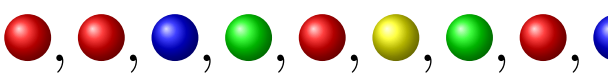
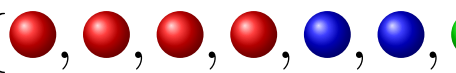
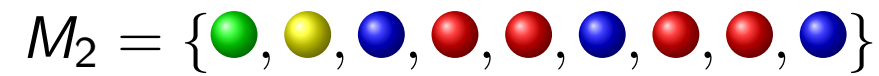
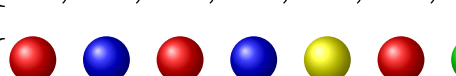
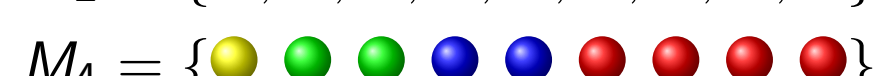
Let $M = \{$  $\}$ be a multiset. Then the following are permutations of $M$:

$M_1 =$ 
$M_2 =$ 
$M_3 =$ 
$M_4 =$ 

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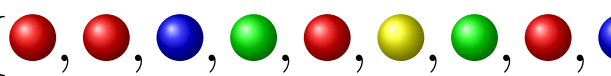
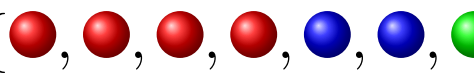
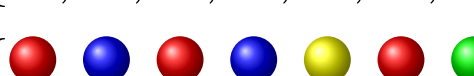
Let $M = \{$  $\}$ be a multiset. Then the following are permutations of $M$:

$M_1 = \{$  $\}$ $M_2 = \{$$\}$

$M_3 = \{$  $\}$ $M_4 = \{$$\}$

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$$\binom{9}{4,2,2,1} = 3,780$$

# Patterns and Pattern Occurence

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A pattern **occurs** in a permutation if there is a subsequence of the permutation similar to the pattern.

# Examples

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$$\{5, 8, 2, 6, 1, 4, 3, 7, 9\}$$

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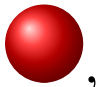
$\{$  ,  ,  ,  ,  ,  ,  ,  ,  $\}$

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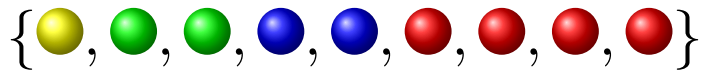
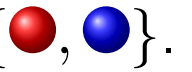


avoids the pattern  .

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Every length-3 subsequence of $S_3$ is a $\{3, 2, 1\}$ pattern, thus no other permutation of length 9 has more occurrences.

# Packing Densities of Patterns

When an optimal permutation is found, the **packing density** is defined

$$\delta(\{\text{pattern}\}) = \frac{\#\text{ of occurrences of }\{\text{pattern}\}\text{ in }\{\text{optimal permutation}\}}{\text{number of subsequences in }\{\text{optimal permutation}\}}$$

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If $\delta$ depends on the length of the permutation, supremum is taken.

# Interpretation of the Packing Density

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Pattern packing is all about *efficiency*.

# Increasing/Decreasing Patterns

An **increasing** pattern: $\{1, 2, \ldots, k\}$.
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Every subsequence of an increasing/decreasing permutation is an increasing/decreasing pattern.

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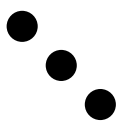

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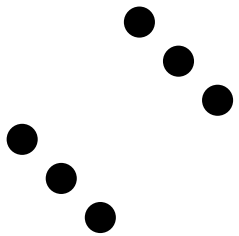

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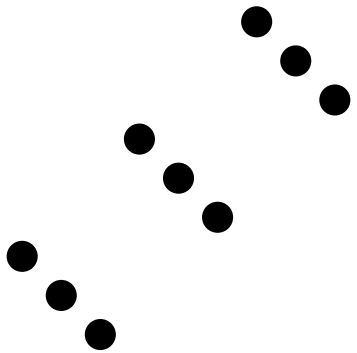

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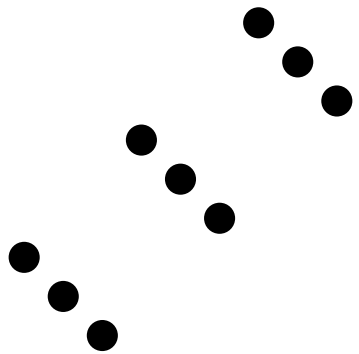

For any layered pattern, there exists an optimal permutation that is also layered [Albert, et al.].

# Examples

As a consequence, finding the packing density of layered patterns reduces to numerical optimization.

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| $\{1,3,2\}$ | 0.464 | $\{1,3,2,9,8,7,6,5,4\}$ |
| $\{1,4,3,2\}$ | 0.424 | $\{4,3,2,1,9,8,7,6,5\}$ |
| $\{2,1,4,3\}$ | 0.375 | $\{2,1,9,8,7,6,5,4,3\}$ |

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No current studies on **non-layered** patterns. Why?

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Consider the pattern {●, ●, ●}

An optimal permutation will *always* be colored in the same way.

{●, ●, ●, ●, ●, ●, ●, ●, ●}

# Proof

**Theorem**

*An optimal permutation of the pattern* $\{$  ,  ,  $\}$ *exists that is colored in the same way.*

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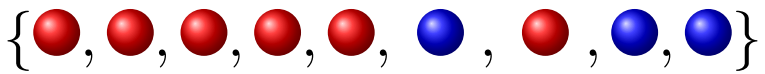

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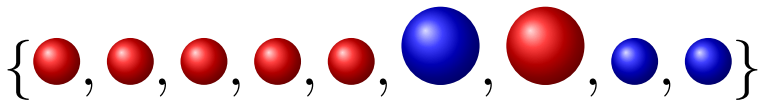

...if we switch these...

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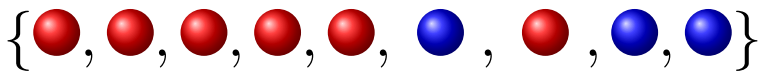
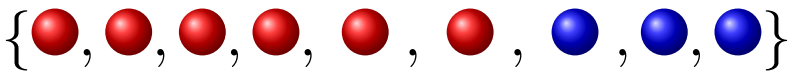
*An optimal permutation of the pattern* {●, ●, ●} *exists that is colored in the same way.*

*Proof:* Assume not! (Reductio ad absurdum)





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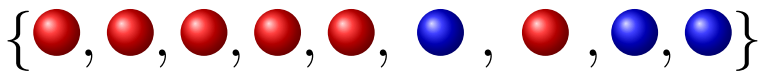
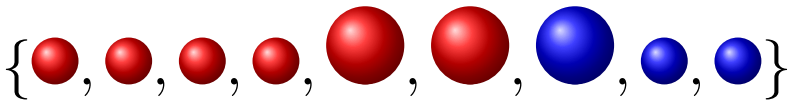
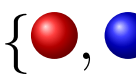




We have gained a {●, ●, ●} pattern, a contradiction! □

# Examples

| {pattern} | $\delta(\{\text{pattern}\})$ | {optimal permutation} (length 9) |
|---|---|---|
|  | 1.000 |  |
|  | 0.444 |  |
|  | 0.222 |  |
|  | 0.094 |  |

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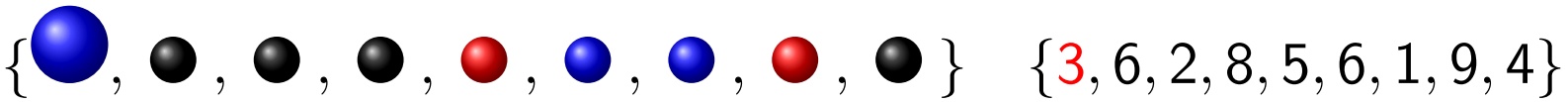


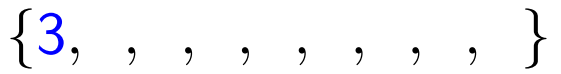

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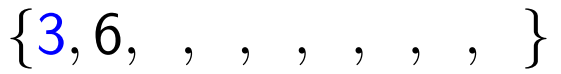

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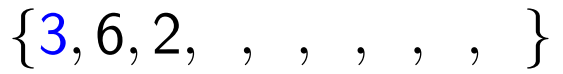
A **colored permutation** is a combination of a set and a multiset.


$\{3, 6, 2, 8, 5, 6, 1, 9, 4\}$



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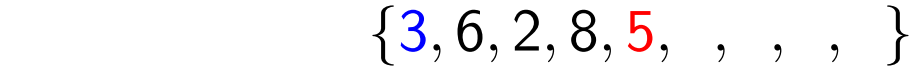

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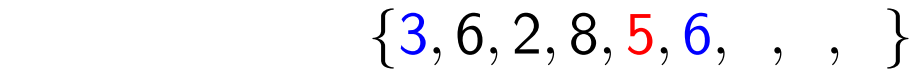

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Colored permutations can be thought of as a deck of cards.

# Colored Blocks

Every colored permutation can be split into **colored blocks**:

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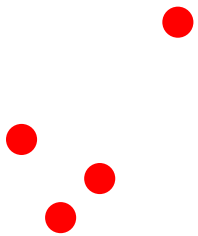

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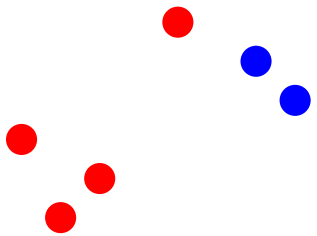

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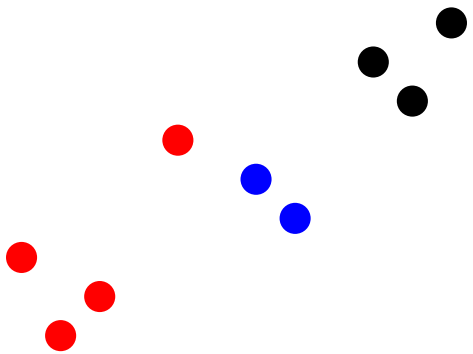

# Optimal Permutations of Colored Patterns

**Theorem (Just, Wang)**

*For every colored pattern consisting of two or three colored blocks there exists an optimal permutation that has the same number of colored blocks, each colored in the same way as the pattern.*

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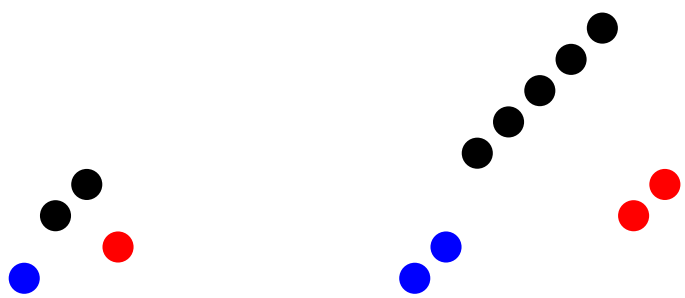
$\{1, 3, 4, 2\}$ $\{1, 2, 5, 6, 7, 8, 9, 4, 5\}$



# Examples

| $\{\text{pattern}\}$ | $\delta(\{\text{pattern}\})$ | $\{\text{optimal permutation}\}$ (length 9) |
|---|---|---|
| $\{1,3,4,2\}$ | 0.188 | $\{1,2,5,6,7,8,9,4,5\}$ |
| $\{1,4,2,3\}$ | 0.198 | $\{1,2,3,5,4,9,8,7,6\}$ |

# Further Research

Any better way to tackle non-layered patterns?

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Studying more than two or three blocks in colored permutations?

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Studying more than two or three blocks in colored permutations?

Combining pattern avoidance with pattern packing?

# Thank You![1]

[1]This work was partially supported by grants from the Simons Foundation (#245307).