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Packing Densities of Colored and Non-Colored Patterns

Matthew R. Just *Georgia Southern University*, mj00788@georgiasouthern.edu

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Packing Densities of Colored and Non-Colored Permutations

Matthew Just



Georgia Southern University Department of Mathematics

April 24, 2015

Matthew Just 999999

Why Study Patterns?

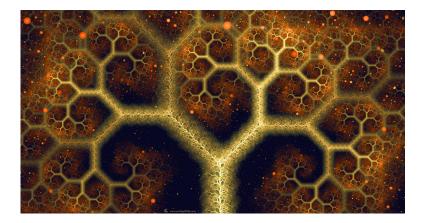


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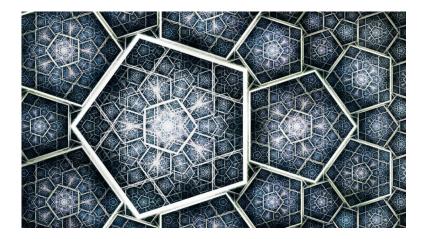
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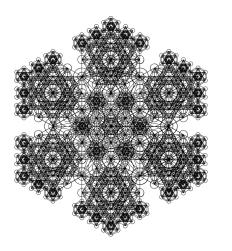
Why Study Patterns?



Why Study Patterns?



Why Study Patterns?



What is a Permutation? Sets vs. Multisets Permutations of Sets vs. Multisets

2 Patterns in Permutations

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

3 Colored Permutations

Combining sets and multisets Colored blocks Concluding remarks

Sets vs. Multisets Permutations of Sets vs. Multisets

Sets

A set is a collection of objects that are all distinct.

Sets vs. Multisets Permutations of Sets vs. Multisets

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 $\{1,2,3,4,5,6,7,8,9\}$

Sets vs. Multisets Permutations of Sets vs. Multisets

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$\{ ullet, ullet, ullet, ullet, ullet, ullet, ullet \}$

Sets vs. Multisets Permutations of Sets vs. Multisets

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$\{ ullet, ullet, ullet, ullet, ullet, ullet, ullet \}$

$\{0,1\}$

Matthew Just

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$\{ \bigcirc, \bigcirc, \bigcirc, \bigcirc \} \mapsto \{ \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc \}$

$\{0,1\} \mapsto \{ 01001000 \ 01100101 \ 01101100 \ 01101100 \ 01101111 \\ 01010111 \ 01101111 \ 01110010 \ 01101100 \ 01100100 \}$

Sets vs. Multisets Permutations of Sets vs. Multisets

Multisets

A **multiset** is a collection of objects generated by repeatedly selecting objects from a set.

$\{ \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet} \} \mapsto \{ \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet}, \textcircled{\bullet} \}$

$\{0,1\} \mapsto \{ 01001000 \ 01100101 \ 01101100 \ 01101100 \ 01101111 \\ 01010111 \ 01101111 \ 01110010 \ 01101100 \ 01100100 \}$

"Hello World"

Sets vs. Multisets Permutations of Sets vs. Multisets

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Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be a set. Then the following sequences are permutations of S:

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$$\begin{split} S_1 &= \{1,4,5,6,2,7,8,9,3\} \quad S_2 &= \{3,2,1,6,5,4,9,8,7\} \\ S_3 &= \{7,8,9,4,5,6,1,2,3\} \quad S_4 &= \{9,8,7,6,5,4,3,2,1\} \end{split}$$

Sets vs. Multisets Permutations of Sets vs. Multisets

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How many total permutations of S?

Sets vs. Multisets Permutations of Sets vs. Multisets

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$$9! = 362,880$$

Sets vs. Multisets Permutations of Sets vs. Multisets

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Sets vs. Multisets Permutations of Sets vs. Multisets

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How many total permutations of M?

$$\binom{9}{4,2,2,1} = 3,780$$

What is a pattern?

Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

Patterns and Pattern Occurence

A **pattern** is just a permutation.

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- $\{3, 2, 1\}$ is a length-3 pattern.
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Patterns and Pattern Occurence

A pattern is just a permutation.

 $\{3,2,1\}$ is a length-3 pattern.

 $\{ \bigcirc, \bigcirc \}$ is a length-2 pattern.

A pattern **occurs** in a permutation if there is a subsequence of the permutation similar to the pattern.

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutati

Examples

Does the pattern $\{3, 2, 1\}$ occur in the following permutation?

$\{5,8,2,6,1,4,3,7,9\}$

What is a pattern? Pattern avoidance and pattern packing Pattern packing in sufficient permutations

Examples

Does the pattern $\{3, 2, 1\}$ occur in the following permutation?

$\{5, {\color{red}8}, 2, {\color{red}6}, 1, {\color{red}4}, {\color{red}3}, 7, 9\}$

What is a pattern? Pattern avoidance and pattern packin Pattern packing in set permutations Pattern packing in multiset permutat

Examples

Does the pattern $\{3, 2, 1\}$ occur in the following permutation?

$\{{\color{red}{5}}, {\color{black}{8}}, {\color{black}{2}}, {\color{black}{6}}, {\color{black}{1}}, {\color{black}{4}}, {\color{black}{3}}, {\color{black}{7}}, {\color{black}{9}}\}$

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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 $\{5,8,2,6,1,4,3,7,9\}$

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 $\{ \mathbf{O}, \mathbf{$

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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Pattern Avoidance

A permutation **avoids** a pattern if there are no occurrences.

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Pattern Avoidance

A permutation avoids a pattern if there are no occurrences.

$\{1, 4, 5, 6, 2, 7, 8, 9, 3\}$

avoids the pattern $\{3,2,1\}.$

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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Pattern Packing

What is the maximum number of occurrences of a pattern in a permutation of fixed length? This is the core question of **pattern packing**.

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Pattern Packing

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Consider the pattern $\{3,2,1\}$ and permutations of length 9:

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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 $S_2 = \{7, 8, 9, 4, 5, 6, 1, 2, 3\}, \{3, 2, 1\}$ occurs 27 times

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

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 occurs 27 times

 $S_3 = \{9, 8, 7, 6, 5, 4, 3, 2, 1\}, \quad \{3, 2, 1\} \text{ occurs 84 times}$

What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

Optimal Permutations of Patterns

An **optimal permutation** with respect to a pattern has more occurrences of the pattern than any other permutation of the same length.

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What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

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Every length-3 subsequence of S_3 is a $\{3, 2, 1\}$ pattern, thus no other permutation of length 9 has more occurrences.

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Packing Densities of Patterns

When an optimal permutation is found, the $\ensuremath{\textbf{packing density}}$ is defined

 $\delta(\{pattern\}) = \frac{\# \text{ of occurrences of } \{pattern\} \text{ in } \{optimal \text{ permutation}\}}{\text{number of subsequences in } \{optimal \text{ permutation}\}}$

What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

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What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

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What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

Packing Densities of Patterns

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If δ depends on the length of the permutation, supremum is taken.

What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

Interpretation of the Packing Density

It is straightforward to verify $0 \le \delta \le 1$ for any pattern.

What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

Interpretation of the Packing Density

- It is straightforward to verify $0 \le \delta \le 1$ for any pattern.
- Represents the **probability** a random subsequence taken from a permutation will be an occurrence of the pattern.

What is a pattern? **Pattern avoidance and pattern packing** Pattern packing in set permutations Pattern packing in multiset permutations

Interpretation of the Packing Density

- It is straightforward to verify $0 \le \delta \le 1$ for any pattern.
- Represents the **probability** a random subsequence taken from a permutation will be an occurrence of the pattern.
- Pattern packing is all about *efficiency*.

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Increasing/Decreasing Patterns

An increasing pattern: $\{1, 2, ..., k\}$. A decreasing pattern: $\{k, (k-1), ..., 1\}$.

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Increasing/Decreasing Patterns

An increasing pattern: $\{1, 2, ..., k\}$. A decreasing pattern: $\{k, (k-1), ..., 1\}$.

$$\delta(\{1, 2, \dots, k\}) = \delta(\{k, (k-1), \dots, 1\}) = 1$$

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Increasing/Decreasing Patterns

An increasing pattern: $\{1, 2, ..., k\}$. A decreasing pattern: $\{k, (k-1), ..., 1\}$.

$$\delta(\{1, 2, \dots, k\}) = \delta(\{k, (k-1), \dots, 1\}) = 1$$

Every subsequence of an increasing/decreasing permutation is an increasing/decreasing pattern.

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Layered Patterns

A layered permutation is of the form:

 $\{3,2,1,6,5,4,9,8,7\}$

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Layered Patterns

A layered permutation is of the form:

 $\{{\color{red}{\textbf{3}}}, {\color{blue}{\textbf{2}}}, {\color{blue}{\textbf{1}}}, {\color{blue}{\textbf{6}}}, {\color{blue}{\textbf{5}}}, {\color{blue}{\textbf{4}}}, {\color{blue}{\textbf{9}}}, {\color{blue}{\textbf{8}}}, {\color{blue}{\textbf{7}}}\}$

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

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A layered permutation is of the form:

 $\{3,2,1,{\color{red}{6}},{\color{blue}{5}},{\color{blue}{4}},9,8,7\}$



What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

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A layered permutation is of the form:

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What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Layered Patterns

A layered permutation is of the form:

 $\{3,2,1,6,5,4,9,8,7\}$



For any layered pattern, there exists an optimal permutation that is also layered [Albert, et al.].

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Examples

As a consequence, finding the packing density of layered patterns reduces to numerical optimization.

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

Examples

As a consequence, finding the packing density of layered patterns reduces to numerical optimization.

$\{pattern\}$	$\delta(\{pattern\})$	${optimal permutation}$ (length 9)
$\{1, 2, 3\}$	1.000	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$\{1, 3, 2\}$	0.464	$\{1, 3, 2, 9, 8, 7, 6, 5, 4\}$
$\{1, 4, 3, 2\}$	0.424	$\{4, 3, 2, 1, 9, 8, 7, 6, 5\}$
$\{2, 1, 4, 3\}$	0.375	$\{2, 1, 9, 8, 7, 6, 5, 4, 3\}$

What is a pattern? Pattern avoidance and pattern packing **Pattern packing in set permutations** Pattern packing in multiset permutations

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$\{1, 4, 3, 2\}$	0.424	$\{4, 3, 2, 1, 9, 8, 7, 6, 5\}$
$\{2, 1, 4, 3\}$	0.375	$\{2, 1, 9, 8, 7, 6, 5, 4, 3\}$

No current studies on non-layered patterns. Why?

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

Patterns in Multisets

Patterns in multisets are much easier to study.

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Consider the pattern $\{ \bigcirc, \bigcirc, \bigcirc \}$

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Patterns in Multisets

Patterns in multisets are much easier to study.

Consider the pattern $\{ \bullet, \bullet, \bullet \}$

An optimal permutation will always be colored in the same way.



What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

Proof

Theorem

An optimal permutation of the pattern $\{ \Theta, \Theta, \Theta \}$ exists that is colored in the same way.

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An optimal permutation of the pattern $\{ \odot, \odot, \odot \}$ exists that is colored in the same way.

Proof: Assume not! (Reductio ad absurdum)

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... if we switch these ...

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Proof

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An optimal permutation of the pattern $\{ \bullet, \bullet, \bullet, \bullet \}$ exists that is colored in the same way.

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$\{ \textcircled{\bullet}, \textcircled{\bullet} \}$

$\{ \bigcirc, \bigcirc \}$

What is a Permutation?What is a pattern?Patterns in Permutations
Colored PermutationsPattern avoidance and pattern packing in set permutations
Pattern packing in multiset permutations

Proof



An optimal permutation of the pattern $\{ \bullet, \bullet, \bullet \}$ exists that is colored in the same way.

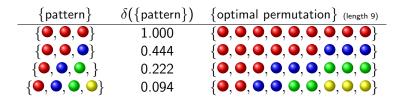
Proof: Assume not! (Reductio ad absurdum)

$\{ \bigcirc, \bigcirc \}$

We have gained a $\{ \bigcirc, \bigcirc, \bigcirc \}$ pattern, a contradiction!

What is a pattern? Pattern avoidance and pattern packing Pattern packing in set permutations Pattern packing in multiset permutations

Examples



Combining sets and multisets Colored blocks

Colored Permutations

Combining sets and multisets Colored blocks

Colored Permutations



Combining sets and multisets Colored blocks

Colored Permutations



Combining sets and multisets Colored blocks

Colored Permutations



$$\{3, 6, , , , , , , \}$$

Combining sets and multisets Colored blocks

Colored Permutations



$$\{3, 6, 2, , , , , \}$$

Combining sets and multisets Colored blocks

Colored Permutations

A colored permutation is a combination of a set and a multiset.



 $\{3, 6, 2, 8, , , , \}$

Combining sets and multisets Colored blocks

Colored Permutations



Combining sets and multisets Colored blocks

Colored Permutations

A colored permutation is a combination of a set and a multiset.



$\{3, 6, 2, 8, 5, 6, , \}$

Combining sets and multisets Colored blocks

Colored Permutations



$$\{3, 6, 2, 8, 5, 6, 1, , \}$$

Combining sets and multisets Colored blocks

Colored Permutations

A colored permutation is a combination of a set and a multiset.



$\{3, 6, 2, 8, 5, 6, 1, 9, \}$

Combining sets and multisets Colored blocks

Colored Permutations

$$\{ \bullet, \bullet \} \{3, 6, 2, 8, 5, 6, 1, 9, 4\}$$

$$\{3, 6, 2, 8, 5, 6, 1, 9, 4\}$$

Combining sets and multisets Colored blocks

Colored Permutations

A colored permutation is a combination of a set and a multiset.



 $\{{\color{red}{\textbf{3}}}, {\color{black}{\textbf{6}}}, {\color{black}{\textbf{2}}}, {\color{black}{\textbf{8}}}, {\color{black}{\textbf{5}}}, {\color{black}{\textbf{6}}}, {\color{black}{\textbf{1}}}, {\color{black}{\textbf{9}}}, {\color{black}{\textbf{4}}}\}$

Colored permutations can be thought of as a deck of cards.

Combining sets and multisets Colored blocks

Colored Blocks

Every colored permutation can be split into colored blocks:

Combining sets and multisets Colored blocks

Colored Blocks

Every colored permutation can be split into colored blocks:

312|6|54|879



Matthew Just 999999

Combining sets and multisets Colored blocks

Colored Blocks

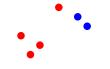
Every colored permutation can be split into colored blocks:



Combining sets and multisets Colored blocks

Colored Blocks

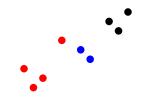
Every colored permutation can be split into colored blocks:



Combining sets and multisets Colored blocks

Colored Blocks

Every colored permutation can be split into colored blocks:



Optimal Permutations of Colored Patterns

Theorem (Just, Wang)

For every colored pattern consisting of two or three colored blocks there exists an optimal permutation that has the same number of colored blocks, each colored in the same way as the pattern.

Optimal Permutations of Colored Patterns

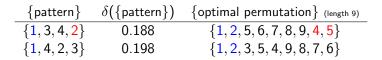
Theorem (Just, Wang)

For every colored pattern consisting of two or three colored blocks there exists an optimal permutation that has the same number of colored blocks, each colored in the same way as the pattern.

$$\{1, 3, 4, 2\}$$
 $\{1, 2, 5, 6, 7, 8, 9, 4, 5\}$

Combining sets and multisets Colored blocks

Examples



Combining sets and multisets Colored blocks

Further Research

Any better way to tackle non-layered patterns?

Combining sets and multisets Colored blocks

Further Research

Any better way to tackle non-layered patterns?

Studying more than two or three blocks in colored permutations?

Combining sets and multisets Colored blocks

Further Research

Any better way to tackle non-layered patterns?

Studying more than two or three blocks in colored permutations?

Combining pattern avoidance with pattern packing?

Combining sets and multisets Colored blocks

Thank You!1

 1 This work was partially supported by grants from the Simons Foundation (#245307).

