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Radio Graceful Labelling of Graphs

Laxman Saha
Balurghat College, laxman.iitkgp@gmail.com

Alamgir Rahaman Basunia
alam.basunia@gmail.com

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Abstract

Radio labelling problem of graphs have its roots in communication problem known as *Channel Assignment Problem.* For a simple connected graph $G = (V(G), E(G))$, a radio labeling is a mapping $f\colon V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u)-f(v)| \geq \text{diam}(G)+1-d(u,v)$ for each pair of distinct vertices $u, v \in V(G)$, where diam(G) is the diameter of G and $d(u, v)$ is the distance between u and v. A radio labeling f of a graph G is a *radio graceful labeling* of G if $f(V(G)) = \{0, 1, \ldots, |V(G)| - 1\}$. A graph for which a radio graceful labeling exists is called *radio graceful.* In this article, a necessary and sufficient condition for radio graceful graphs are presented. Also some consequences of radio graceful graphs are given in terms of some new graph parameters.

1 Introduction

In wireless networks, an important task is the management of the radio spectrum. *The Channel Assignment Problem* (CAP) assigns frequencies to the transmitters in a network in such a way so that it avoids interference and uses the spectrum very efficiently. Due to rapid growth in the use of wireless communication services and the corresponding scarcity and the high cost of radio spectrum bandwidth, CAP is becoming highly important. In 1980, Hale [7] has modeled CAP as a Graph labelling problem (in particular as a generalized graph coloring problem) and radio labelling is a variation of CAP.

Inspired by FM channel assignments, a new model, namely the radio labelling problem was introduced in [3, 4] and studied further in [8, 9, 10, 11, 12, 13, 14, 15, 16]. For a simple connected graph $G = (V(G), E(G))$, a radio labeling is a mapping $f\colon V(G) \to \{0, 1, 2, \ldots\}$ such that

$$|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v) \tag{1}$$

for each pair of distinct vertices $u, v \in V(G)$, where diam(G) is the diameter of G and $d(u, v)$ is the distance between u and v. The span of f, denoted by *span*(f), is the maximum integer assigned by f to some vertex v. The radio number $rn(G)$ of G is the number $\min\{\text{span}(f)\}$, where the minimum is taken over all possible radio labeling of G. A radio labeling f of G is called minimal if $\text{span}(f) = rn(G)$. Without loss of generality, for a minimal radio labeling f we assume that $\min\limits_{v\in V(G)} f(v) = 0$, otherwise the span of f can be reduced further by subtracting the integer $\min\limits_{v\in V(G)} f(v)$ from all the labels of the vertices of the graph.

A radio labeling f of a graph G is a *radio graceful labeling* of G if

$$f(V(G)) = \{0,\ 1,\ \ldots,\ \ |V(G)| - 1\}.$$

A graph for which a radio graceful labeling exists is called *radio graceful.* Other related definitions have been given for other radio labelings, including full colorings and no-hole colorings for $L(2, 1)$-labeling (Fishburn, Roberts in [5] and [6]). This study has direct connections to radio graceful labeling when $\text{diam}(G) = 2$, to which we do not limit ourselves. Recently, Niedzialomski [2] has shown that the some Hamming graphs (Cartesian product of complete graphs) are radio graceful.

In this article, a necessary and sufficient conditions for radio graceful graphs are presented. Also some consequences of radio graceful graphs are given in terms of some new graph parameters. The rest of the article is organized as follows. Section 2 deals with notations and definitions. Also in this section, we define a new parameters namely, triameter of graphs and study some properties of this parameter. Finally in Section 3, we discuss the results on radio graceful labelling of an arbitrary graph G in terms of triameter and other graph parameters.

2 Triameter and k-distance complement of graphs

All the graphs considered in this paper are simple and connected, unless otherwise stated. The complete graph and the path, both on n vertices, are denoted by K_n and P_n, respectively. A Hamiltonian path in a graph G is a path through all of the vertices of the graph, visiting each vertex once and only once. For a connected graph G, we define the distance $d_G(u,v)$ between two vertices u and v as the length of a shortest path from u to v in G. The *diameter* of G is the number $\operatorname{diam}(G) = \max\{d_G(u,v)\colon u,v \in V(G)\}$.

Definition 2.1. The *complement* $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ and two vertices are adjacent in $\bar{G}$ if they are not adjacent in G. A graph G is called *self-complementary* if $\bar{G}$ is isomorphic to G.

Definition 2.2. For a positive integer k, $1 \leqslant k \leqslant \operatorname{diam}(G)$, the *$k$-distance complement* of a connected graph $G = (V(G), E(G))$ is the graph $\bar{G}^k$ on the same vertex set $V(G)$ and two distinct vertices of $\bar{G}^k$ are adjacent if they are at distance k in G. For $k = 1$, the *k-distance complement* reduces to the complement of a graph.

The following lemma represents a relationship between diameters of connected graphs G and $\bar{G}$.

Lemma 2.1. *Let $G = (V(G), E(G))$ be a graph such that both G and $\bar{G}$ are connected. Then G and its complement $\bar{G}$ cannot have both diameter larger than 3.*

Now a new parameter, named its *triameter*, is introduced for any given graph.

Definition 2.3. Let $G = (V(G), E(G))$ be a simple connected graph with at least 3 vertices. The *triameter* of G, denoted by $\operatorname{tr}(G)$, of the graph G is defined as the smallest positive integer M such that $d(u,v) + d(v,w) + d(w,u) \leqslant M$ for every triplet u, v and w in $V(G)$. From the definition, it follows that $\operatorname{tr}(G)$ is always greater than or equal to 3. Now, we investigate other bounds on $\operatorname{tr}(G)$.

Lemma 2.2. *For a graph G, $\operatorname{tr}(G) = 3$ if and only if G is complete graph.*

Theorem 2.1. *For any connected graph G, $2\operatorname{diam}(G) \leqslant \operatorname{tr}(G) \leqslant 3\operatorname{diam}(G)$ and the bounds are tight.*

Proof. Let d be the diameter of G. Since

$$\max\{d(u,v) + d(v,w) + d(w,u)\colon \text{for all } u,v,w \in V(G)\} \leqslant 3d$$

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and $\text{tr}(G)$ is the smallest integer such that $d(u,v)+d(v,w)+d(w,u) \leqslant M$ for all $u,v,w \in V(G)$, we have $\text{tr}(G) \leqslant 3d$. If the vertices u and v are chosen in such a way that $d(u,v)=d$, then from the triangle inequality $d(v,w)+d(w,u) \geqslant d(u,v)=d$. Therefore $d(u,v)+d(v,w)+d(w,u) \geqslant 2d$. Since $tr(G)$ is the smallest positive integer M such that $d(u,v)+d(v,w)+d(w,u) \leqslant M$ for every triplet u, v and w in $V(G)$, we have $\text{tr}(G) \geqslant 2\ \text{diam}(G)$. □

The result below determines a relationship between $\text{tr}(G)$ and $\text{tr}(\bar{G})$ for connected graphs G and $\bar{G}$.

Lemma 2.3. *Let $G=(V(G),E(G))$ be a graph such that both G and $\bar{G}$ are connected. If $\text{tr}(G)>9$, then $\text{tr}(\bar{G}) \leqslant 6$.*

Proof. Let G be a graph with $\text{tr}(G)>9$. Then, from Theorem 2.1 and Lemma 2.1, $\text{diam(G)} \geqslant 4$ and $\text{diam}(\bar{\text{G}}) \leqslant 3$. Now we show that $\text{diam}(\bar{\text{G}}) \leqslant 2$. If possible, let $\text{diam}(\bar{\text{G}})=3$. Then there is a path P of length 3 in $\bar{G}$. Let u and v are end vertices of P. Then the graph G will surely contain the edge uv. In G any vertex different from u and v is adjacent to u or to v because there is no path of length 2 connecting u and v in $\bar{G}$. Let x and y be any two vertices different from u and v. If they share u or v as a common neighbor in G, then xuy or xvy is a path connecting them in G. Otherwise, $xvuy$ or $xuvy$ is a path connecting them in G. In any case x and y are at distance at most 3. Thus the diameter of G is at most 3, which is a contradiction and this contradiction implies that $\text{diam}(\bar{\text{G}}) \leqslant 2$. Using this and Lemma 2.1, we have $\text{tr}(\bar{G}) \leqslant 6$. □

The *Cartesian product* $G\Box H$ of graphs G and H is the graph whose vertex set is the Cartesian product $V(G)\times V(H)$, and any two vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if either $u_1=u_2$ and v_1 is adjacent to v_2 in H or $v_1=v_2$ and u_1 is adjacent to u_2 in G. The following proposition can be proved easily.

Proposition 2.1. If (g,h) and (g',h') are vertices of $G\Box H$, then

$$d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h').$$

Lemma 2.4. For simple connected graphs G and H, $\text{diam}(G\Box H)=\text{diam}(G)+\text{diam}(H)$.

Proof. For any two vertices u,v of $G\Box H$, Proposition 2.1 gives $d(u,v) \leqslant \text{diam}(G)+\text{diam}(H)$. Now, if we choose $u{=}(g,h)$ and $v{=}(g',h')$ with $d_G(g,g')=\text{diam}(G)$ and $d_H(h,h')=\text{diam}(H)$, then the same proposition gives $d(u,v)=d_G(g,g')+d_H(h,h')=\text{diam}(G)+\text{diam}(H)$. Therefore, $\text{diam}(G\Box H)=\text{diam}(G)+\text{diam}(H)$. □

Now we prove that a similar result exists for the triameter of simple connected graphs.

Lemma 2.5. If $d_G(x,y)+d_G(y,z)+d_G(z,x) \leqslant M_1$ and $d_H(x',y')+d_H(y',z')+d_H(z',x') \leqslant M_2$ for every three vertices x,y,z of G and x',y',z' of H, then $d(u,v)+d(v,w)+d(w,u) \leqslant M_1+M_2$ for every three vertices u,v,w of $G\Box H$.

Proof. Let $u=(x_1,y_1), v=(x_2,y_2)$ and $w=(x_3,y_3)$ be any three vertices of $G\Box H$, where $x_i \in V(G)$ and $y_i \in V(H)$ for $i=1,2,3$. From Proposition 2.1, we have

$$\begin{aligned} d(u,v) &= d_G(x_1,x_2)+d_H(y_1,y_2) \quad (2)\\ d(v,w) &= d_G(x_2,x_3)+d_H(y_2,y_3) \quad (3)\\ d(w,u) &= d_G(x_3,x_1)+d_H(y_3,y_1). \quad (4) \end{aligned}$$

Adding (2) – (4), we get

$$\begin{aligned} d(u,v)+d(v,w)+d(w,u) &= d_G(x_1,x_2)+d_G(x_2,x_3)+d_G(x_3,x_1) \\ &\quad + d_H(y_1,y_2)+d_H(y_2,y_3)+d_H(y_3,y_1) \\ &\leqslant M_1+M_2. \end{aligned}$$

Therefore $d(u,v)+d(v,w)+d(w,u) \leqslant M_1+M_2$ for every three vertices u, v, w of $G\square H$. □

Corollary 2.1. For any two simple connected graphs G and H, $\mathrm{tr}(G\square H) = \mathrm{tr}(G) + \mathrm{tr}(H)$.

3 Results on graceful radio labelling of graphs

In this section, we determine the conditions for which an arbitrary graph is radio graceful. The conditions depend on the triameter and the existence of a Hamiltonian path in the k-distance complement graph. The following theorem is very important to determine a radio graceful labelling for complement graph of an arbitrary graph.

Theorem 3.1. If G is a simple connected graph with $\Delta(G) < \frac{|V(G)|}{2}$, then the diameter of complement graph $\bar{G}$ must be 2.

Proof. From the definition of $\bar{G}$ it is clear that $d_{\bar{G}}(u,v) = 1$ for any two non-adjacent vertices u and v in G. Thus, to prove the theorem we show that $d_{\bar{G}}(u,v) = 2$ for any two adjacent vertices u and v in G. For this we need to show that there exist a vertex w in G such that $d_G(u,w) \geqslant 2$ and $d_G(u,w) \geqslant 2$ for any two adjacent vertices u and v in G. Suppose, on the contrary, that there exists no vertex w satisfying $d_G(u,w) \geqslant 2$ and $d_G(u,w) \geqslant 2$ for two adjacent vertices u and v in G. Then either u or v is adjacent to every vertex in $V(G) - \{u,v\}$. Therefore, the degree sum of u and v is

$$d(u)+d(v) \geqslant n-2+2 = n \tag{5}$$

which contradicts the fact that $d(u) < \frac{n}{2}$ and $d(v) < \frac{n}{2}$. This contradiction says that there exist a vertex w in G such that $d_G(u,w) \geqslant 2$ and $d_G(u,w) \geqslant 2$ for any two adjacent vertices u and v in G. Thus in $\bar{G}$, $d_{\bar{G}}(u,w) = 1$ and $d_{\bar{G}}(v,w) = 1$ that is, $d_{\bar{G}}(u,v) = 2$ for every two adjacent vertices u and v in G. □

The theorem below gives a necessary and sufficient condition for the existence of a radio graceful labeling.

Theorem 3.2. *Let G be an n-vertex graph with diameter 2. Then G is radio graceful if and only if $\bar{G}$ contains a Hamiltonian path.*

Proof. Let $x_0x_1\ldots x_{n-1}$ be a Hamiltonian path in $\bar{G}$. Then $d_G(x_i,x_{i+1}) = 2$ for $0 \leqslant i \leqslant n-2$. With the help of last property it can be shown that the mapping $f\colon V(G) \to \{0,1,\ldots,n-1\}$ defined by $f(x_0) = 0,\ f(x_{i+1}) = f(x_i)+1$ is a radio graceful labelling of G.

Now we consider G to be radio graceful. We have to show that $\bar{G}$ has a Hamiltonian path. Since G is radio graceful, G has a radio labeling $f\colon V(G) \to \{0,1,\ldots,n-1\}$. Let $x_0,x_1,\ldots,x_{n-1}$ be an ordering of $V(G)$ such that $f(x_{i+1}) > f(x_i)$, $0 \leqslant i \leqslant n-2$. Since f is a radio graceful labelling and G has diameter 2, $1 = f(x_{i+1}) - f(x_i) = 2+1-d(x_i,x_{i+1})$ and this implies that $d(x_i,x_{i+1}) = 2$ for each i. Thus for each i, x_i and x_{i+1} are adjacent in $\bar{G}$ and hence $x_0x_1\ldots x_{n-1}$ is a Hamiltonian path in $\bar{G}$. □

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The complement of a Hamiltonian graph may not be radio graceful. For example, the graph $G = \bar{P}_n$ is Hamiltonian but its complement graph P_n is not radio graceful. Thus an important task is the identification of Hamiltonian graphs whose complements are radio graceful. The theorem below identifies a class of Hamiltonian graphs whose complements are radio graceful.

Theorem 3.3. *Let G be an n-vertex graph with $\Delta(G) < \frac{n}{2}$. If G is Hamiltonian, then the complement graph $\bar{G}$ is always radio graceful.*

Proof. From Theorem 3.1, the diameter of $\bar{G}$ is two. Using Theorem 3.2 on the graph $\bar{G}$, we get the desired result because the graph G is Hamiltonian. □

Example 3.1. From above theorem one may conclude that the complement graphs $\bar{C}_n$ and $\bar{P}_n$ are radio graceful.

Theorem 3.2 tells that if a 2-diameteral graph is radio graceful then its complement graph contains a Hamiltonian path. Now an important target is to proof an existence of Hamiltonian path in complement graphs for a radio graceful graphs having any diameter d. The theorem below guarantees the existence of a Hamiltonian path in d-distance complement graph for a radio graceful graph G having the diameter d.

Theorem 3.4. *Let G be an n-vertex graph with diameter d. If G is radio graceful, then the d-distance complement graph $\bar{G}^d$ is connected and contains a Hamiltonian path.*

Proof. Let G be an n-vertex graph and f be a radio graceful labeling of G. Then f induces a linear ordering $x_0, x_1, \ldots, x_{n-1}$ of the vertices of G such that $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$. Now the equalities $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$ hold if $d_G(x_t, x_{t+1}) = d$ for $0 \leqslant t \leqslant n-2$. Thus $x_0 x_1 \ldots x_{n-1}$ forms a Hamiltonian path in $\bar{G}^d$. □

Example 3.2. From above theorem one may conclude that neither cycles nor trees are radio graceful

The result below determines the possible values for traimeter of an arbitrary radio graceful graph. This result is quite significant to classify the graphs which are not radio graceful.

Theorem 3.5. *For a radio graceful graph G,* $\mathrm{tr}(G) = 3d - 1$ *or* $3d$*, where d is the diameter of G.*

Proof. Let G be an n-vertex graph and f be a radio graceful labeling of G. Then f induces a linear ordering $x_0, x_1, \ldots, x_{n-1}$ of the vertices of G such that $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$. Now the equalities $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$ hold if $d_G(x_t, x_{t+1}) = d$ for $0 \leqslant t \leqslant n-2$. Again, since f is a radio labelling, $f(x_{t+2}) - f(x_t) = 2 \geqslant d + 1 - d(x_t, x_{t+1})$ and this imply $d(x_t, x_{t+2}) \geqslant d - 1$. Thus for every t with $0 \leqslant t \leqslant n-2$, $d(x_t, x_{t+1}) + d(x_{t+1}, x_{t+2}) + d(x_{t+2}, x_t) \geqslant 3d - 1$ and this implies that $\mathrm{tr}(G) \geqslant 3d - 1$. Again, Theorem 2.1 implies $\mathrm{tr}(G) \leqslant 3d$. Thus we get the desired result. □

Example 3.3. In this example, we classify the radio graceful graphs having diameter 3. For a graph G having diameter 3, Theorem 2.1 gives $\mathrm{tr}(G) \in \{6, 7, 8, 9\}$. Again applying Theorem 3.5, the 3-diameteral graph G can not be radio graceful if $\mathrm{tr}(G) = 6$ or $\mathrm{tr}(G) = 7$.

Remark 3.1. The traimeter $\mathrm{tr}(G)$ of a graph G, satisfies the inequality

$$2\ \mathrm{diam}(G) \leqslant \mathrm{tr}(G) \leqslant 3\ \mathrm{diam}(G).$$

From the above theorem it is clear that the graph G with $\mathrm{tr}(G) \leqslant 3\ \mathrm{diam}(G) - 2$ can not be radio graceful. Thus we conclude that all most all graphs are not radio graceful.

Using Remark 3.1, the following result provides an useful idea about the existence of a radio graceful labelling for product graphs.

Remark 3.2. For simple connected graphs G and H, if the Cartesian product $G\Box H$ is radio graceful, then either (a) $\mathrm{tr}(G) = 3\ \mathrm{diam}(G)$, $\mathrm{tr}(H) = 3\ \mathrm{diam}(H) - 1$ or (b) $\mathrm{tr}(G) = 3\ \mathrm{diam}(G) - 1$, $\mathrm{tr}(H) = 3\ \mathrm{diam}(H)$. Again for a collection of simple connected graphs G_i $(3 \leqslant i \leqslant m)$ if the graph $G_1\Box G_2\Box \ldots \Box G_m$ is radio graceful then $\mathrm{tr}(G_i) = 3\ \mathrm{diam}(G_i)$ for each $i \in \{1, 2, \ldots, m\}$ except for one value t for which $\mathrm{tr}(G_t) = 3\ \mathrm{diam}(G_t)$ or $3\ \mathrm{diam}(G_t) - 1$. It is also noted that if $G\Box G\Box \ldots \Box G$ is radio graceful, then $\mathrm{tr}(G) = 3\ \mathrm{diam}(G)$.

The theorem below gives a necessary and sufficient conditions for the existence of a radio graceful labelling of complement graph $\bar{G}$ for any graph G with $\mathrm{tr}(G) > 9$.

Theorem 3.6. *Let $G = (V(G), E(G))$ be a graph such that both G and $\bar{G}$ are connected. Also let, $\mathrm{tr}(G) > 9$. Then $\bar{G}$ is radio graceful only if G contains a Hamiltonian path and $\mathrm{tr}(\bar{G}) = 5$ or 6.*

Proof. Here the triameter $\mathrm{tr}(G) > 9$. Thus from Lemma 2.3, $\mathrm{tr}(\bar{G}) \leqslant 6$. If $\mathrm{tr}(\bar{G}) = 3$, then Lemma 2.2 implies that $\bar{G}$ is a complete graph and hence G is disconnected. Thus $4 \leqslant \mathrm{tr}(\bar{G}) \leqslant 6$. Now if $\mathrm{tr}(\bar{G}) = 4$, then using Lemma 2.2 and Theorem 2.1, we have $\mathrm{diam}(\bar{G}) = 2$. Applying Theorem 3.5, $\bar{G}$ is not radio graceful. For $\mathrm{tr}(\bar{G}) = 5$, the diameter of $\bar{G}$ may be 2 or 3. If $\mathrm{tr}(\bar{G}) = 5$ and $\mathrm{diam}(\bar{G}) = 3$, then $\bar{G}$ is not radio graceful due to Theorem 3.5. Again if $\mathrm{tr}(\bar{G}) = 5$ and $\mathrm{diam}(\bar{G}) = 2$, then from Theorem 3.2, $\bar{G}$ is radio graceful if and only if G contains a Hamiltonian path. Lastly, we consider $\mathrm{tr}(\bar{G}) = 6$. In this case if $\mathrm{diam}(\bar{G}) = 2$, then $\bar{G}$ radio graceful if and only if G contains a Hamiltonian path. Again, if we consider $\mathrm{diam}(\bar{G}) \neq 2$, then G can not be radio graceful due to Theorem 3.5. □

Theorem 3.2 gives a necessary and sufficient conditions for radio graceful labelling of a graph G having diameter two. The theorem below gives the same for a graph G having diameter three.

Theorem 3.7. *Let G be an n-vertex graph with diameter 3. Then G is radio graceful if and only if the 3-distance complement $\bar{G}^3$ of G contains a Hamiltonian path P_n, with $E(P_n^2) \cap E(G)$ empty.*

Proof. Suppose G is radio graceful and let f be a radio graceful labeling of G. Then f induces a linear ordering $x_0, x_1, \ldots, x_{n-1}$ of the vertices of G such that $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$. Now the equalities $f(x_{t+1}) = f(x_t) + 1$, $0 \leqslant t \leqslant n-2$ hold if $d_G(x_t, x_{t+1}) = 3$ for $0 \leqslant t \leqslant n-2$. Thus the 3-distance complement $\bar{G}^3$ of G contains a Hamiltonian path $P_n :\ x_0x_1 \ldots x_{n-1}$. Again, since f is radio labelling, $f(x_{t+2}) - f(x_t) = 2 \geqslant 3 + 1 - d_G(x_t, x_{t+1})$ and this imply $d_G(x_t, x_{t+2}) \geqslant 2$. Hence for this Hamiltonian path P_n, $E(P_n^2) \cap E(G)$ is empty.

Now we prove the converse. That is, if the 3-distance complement $\bar{G}^3$ of G contains a Hamiltonian path P_n with $E(P_n^2) \cap E(G) = \emptyset$, empty set, then G is radio graceful. Let us consider $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. Since P_n is a Hamiltonian path in the 3-distance complement $\bar{G}^3$ of G, $d_G(v_t, v_{t+1}) = 3$ for $0 \leqslant t \leqslant n-2$. Again $E(P_n^2) \cap E(G) = \emptyset$ implies that $d_G(v_t, v_{t+2}) \geqslant 2$. Now we define a labelling $f\colon V(G) \to \{0, 1, \ldots, n-1\}$ as

$$\begin{aligned} f(v_0) &= 0; \\ f(v_{t+1}) &= f(v_t) + 1, \quad 0 \leqslant t \leqslant n-2. \end{aligned}$$

Here for each t, $d_G(v_t, v_{t+1}) = 3$, $d_G(v_t, v_{t+2}) \geqslant 2$. Since the diameter of G is 3, f is a radio graceful labelling of G. □

As we are aware of the fact the Theorems 3.2 and 3.7 provides us the necessary and sufficient conditions for radio gracefulness of a graph having diameters two and three, respectively. The establishment of the necessary and sufficient conditions for radio graceful labelling of a graph G having a diameter d is a gruelling task. In the theorem below we give a necessary and sufficient for radio graceful labelling of a graph G having any diameter d.

Theorem 3.8. *Let G be an n-vertex graph with diameter d. Then G is radio graceful if and only if the d-distance complement $\bar{G}^d$ of G contains a Hamiltonian path P_n with*

$$\bigcup_{k=t}^{d} E(\bar{G}^d) \supseteq E(P_n^{d+1-t}) \setminus E(P_n^{d-t}) \quad \textit{for all } t = 2, 3, \ldots, d-1.$$

Proof. First we consider the graph G to be radio graceful and let f be a radio graceful labeling of G. Then f induces a linear ordering $x_0, x_1, \ldots, x_{n-1}$ of the vertices of G such that $f(x_{l+1}) = f(x_l) + 1$, $0 \leqslant l \leqslant n-1$. Now the equalities $f(x_{l+1}) = f(x_l) + 1$, $0 \leqslant l \leqslant n-1$ hold if $d_G(x_l, x_{l+1}) = d$ for $0 \leqslant l \leqslant n-1$. Thus the d-distance complement $\bar{G}^d$ of G contains a Hamiltonian path $P_n : \; x_0 x_1 \ldots x_{n-1}$. Let $i, j \in \{0, 1, \ldots, n-1\}$ with $j > i$ and say $j = i + l$. Then, since f is radio labelling, $l = f(x_j) - f(x_i) \geqslant d + 1 - d_G(x_i, x_j)$ and this imply that $d_G(x_i, x_j) \geqslant d + 1 - l$. Let $e = (x_i, x_j) \in E(P_n^{d+1-t}) \setminus E(P_n^{d-t})$ with $j > i$. Then $l = j - i = d + 1 - t$ and $d_G(x_i, x_j) \geqslant d + 1 - l$ implies $d_G(x_i, x_j) \geqslant t$. Since $d_G(x_i, x_j) \geqslant t$, $e = (x_i, x_j) \in E(\bar{G}^k)$ for some $k \geqslant t$. Thus $\bigcup_{k=t}^{d} E(\bar{G}^k) \supseteq E(P_n^{d+1-t}) \setminus E(P_n^{d-t})$.

For the converse part, let $\bar{G}^d$ contain a Hamiltonian path $P_n : \; x_0 x_1 \ldots x_{n-1}$ with $\bigcup_{k=t}^{d} E(\bar{G}^k) \supseteq E(P_n^{d+1-t}) \setminus E(P_n^{d-t})$ for all $t = 2, 3, \ldots, d-1$. Here it is clear that $d_G(x_l, x_{l+1}) = d$ for $0 \leqslant l \leqslant n-1$. Now we define a mapping $f\colon V(G) \to \{0, 1, \ldots, n-1\}$ defined by

$$\begin{aligned} f(x_0) &= 0 \\ f(x_{l+1}) &= f(x_l) + 1, \; 0 \leqslant l \leqslant n-2. \end{aligned}$$

We show that f is a radio graceful labelling of G. Let $i, j \in \{0, 1, \ldots, n-1\}$ with $j > i$. Then $f(x_j) - f(x_i) = j - i$. If $j - i \geqslant d$, then nothing to prove. Thus we consider $j - i \leq d - 1$ and $j - i = l$. Since $d_{P_n}(x_i, x_j) = l$, $(x_i, x_j) \in E(P_n^l) \setminus E(P_n^{l-1})$. Let $l = d + 1 - t$. Then from given condition $(x_i, x_j) \in \bigcup_{k=t}^{d} E(G^{kc})$ and this implies that $d_G(x_i, x_j) \geqslant t = d + 1 - l$. Thus $d + 1 - d_G(x_i, x_j) \leqslant l$ and consequently $f(x_j) - f(x_i) \geqslant d + 1 - d_G(x_i, x_j)$. This proves that f is a radio graceful labelling of G. □

References

[1] First Author, Second Author, and Third Author. Title of cited paper. *Abbrev. Journal Name.*, Volume(Issue):Page–Page, Year.

[2] A. Niedzialomski. Radio graceful Hamming graphs. *Discuss. Math. Graph Theory*, 36:1007–1020, 2016.

[3] G. Chartrand, D. Erwin, F. Harary, and P. Zhang. Radio labelings of graphs. *Bull. Inst. Combin. Appl.*, 33:77–85, 2001.

[4] G. Chartrand, D. Erwin, and P. Zhang. A graph labeling problem suggested by FM channel restrictions. *Bull. Inst. Combin. Appl.*, 43:43–57, 2005.

[5] P. C. Fishburn and F.S. Roberts. No-hole $L(2,1)$-colorings. *Discrete Appl. Math.*, 130:513–519, 2003.

[6] P. C. Fishburn and F.S. Roberts. Full color theorems for $L(2,1)$-colorings. *SIAM J. Discrete Math.*, 20:428–443, 2006.

[7] W. K. Hale. Frequency assignment: theory and application. *Proc. IEEE*, 68:1497–1514, 1980.

[8] R. Khennoufa and O. Togni. The radio antipodal and radio numbers of the hypercube. *Ars Combin.*, 102:447–461, 2011.

[9] D. D. -F. Liu. Radio number for trees. *Discrete Math.*, 308:1153–1164, 2008.

[10] D. D. -F. Liu and X. Zhu. Multi-level distance labelings for paths and cycles. *SIAM J. Discrete Math.*, 19:610–621, 2005.

[11] D. D. -F. Liu and M. Xie. Radio Number for Square Paths. *Ars Combin.*, 90:307–319, 2009.

[12] D. D. -F. Liu and M. Xie. Radio number for square of cycles. *Congr. Numer.*, 169:105–125, 2004.

[13] X. Li, V. Mak and S. Zhou. Optimal radio labellings of complete m-ary trees. *Discrete Appl. Math.*, 158:507–515, 2010.

[14] J. P. Ortiz, P. Martinez, M. Tomova, and C. Wyels. Radio numbers of some generalized prism graphs. *Discuss. Math. Graph Theory*, 31(1): 45–62, 2011.

[15] L. Saha and P. Panigrahi. On the Radio number of Toroidal grids. *Australian J. of Combin.*, 55:273–288, 2013.

[16] L. Saha and P. Panigrahi. A lower bound for radio k-chromatic number. *Discrete Appl. Math.*, 192:87–100, 2015.