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Graham's Pebbling Conjecture Holds for the Product of a Graph and a Sufficiently Large Complete Graph

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Abstract

For connected graphs G and H, Graham conjectured that $\pi(G\square H) \leq \pi(G)\pi(H)$ where $\pi(G), \pi(H)$, and $\pi(G\square H)$ are the pebbling numbers of G, H, and the Cartesian product $G\square H$, respectively. In this paper, we show that the inequality holds when H is a complete graph of sufficiently large order in terms of graph parameters of G.

1 Introduction

Throughout this paper, all graphs are considered to be finite and simple. For a graph G, we denote the order of G by $|G|$. For a positive integer n, we denote K_n to be a complete graph of n vertices. For basic definitions and terminologies not mentioned here, we refer the reader to the book of West [10].

Given two graphs G and H, the *Cartesian product* of G and H, denoted by $G\square H$, is the graph with the vertex set $V(G) \times V(H)$ and the edge set

$$\{(u, v_1)(u, v_2) : u \in V(G) \text{ and } v_1v_2 \in E(H)\} \cup \{(u_1, v)(u_2, v) : u_1u_2 \in E(G) \text{ and } v \in V(H)\}.$$

We note that $G\square H$ is connected if and only if G and H are both connected. For more detail treatments of graph products, we refer the reader to [7]. In order to study graph products practically, we need some definitions that consider the product of sets A and B. In particular, if $C \subseteq A \times B$, we define $p_1(C) = \{a : (a, b) \in C \text{ where } b \in B\}$. For a function f from a finite set I to the set $\mathbb{N} \cup \{0\}$, we recall that $\sum_{i\in I} f(i) = 0$ whenever $I = \emptyset$. And we use this convention for Lemma 2.1 and the proof of Proposition 2.1. Moreover, for graphs G and H, we denote $S\square H$ and $G\square T$ the induced subgraphs of $G\square H$ induced by $S \times V(H)$ and $V(G) \times T$, respectively, where $S \subseteq V(G)$ and $T \subseteq V(H)$.

Let G be a connected graph. A *(pebbling) configuration* on G is defined to be a function $D : V(G) \to \mathbb{N} \cup \{0\}$ or we can say that D distributes $\sum_{v\in V(G)} D(v)$ pebbles on G. A configuration D on G is said to be *moveable* if there exist two adjacent vertices u and v such that $D(u) \geq 2$. For a moveable configuration D on a graph G and adjacent vertices u and v with $D(v) \geq 2$, the *(pebbling) move* from u to v in G is defined to be the triple (D, u, v) and we denote it by $D(u \to v)$ for convenience. For a move $D(u \to v)$ in G, the configuration $D' : V(G) \to \mathbb{N} \cup \{0\}$ defined by

$$D'(x) = \begin{cases} D(x) - 2 & \text{if } x = u; \\ D(x) + 1 & \text{if } x = v; \\ D(x) & \text{otherwise} \end{cases}$$

is called the *configuration with respect* to $D(u \to v)$. Let D be a moveable configuration on a graph G. A *D-moving sequence* in G is a finite sequence of moves $D_1(u_1 \to v_1), D_2(u_2 \to v_2), \ldots, D_n(u_n \to v_n)$ such that $D = D_1$ and D_i is the moveable configuration with respect to $D_{i-1}(u_{i-1} \to v_{i-1})$ for every $i \in \{2, \ldots, n\}$ and we write $u_1 \to v_1, u_2 \to v_2, \ldots, u_n \to v_n$ for convenience. For a vertex r of G, if r appears in some D-moving sequences or $D(r) \geq 1$, we say that one *can pebble* r under a configuration D on G or we can say that D is *r-solvable* on G. Furthermore, a configuration is *solvable* whenever it is r-solvable for every vertex

r. It is *unsolvable* otherwise. Given a configuration D on a connected graph G; we call $\sum_{v\in V(G)} D(v)$ the *size* of D and denoted by $|D|$. In a Cartesian product graph $G\square H$, $|D_x|$ denotes $\sum_{v\in V(H)} D(x,v)$ for each $x \in V(G)$. The *pebbling number* of a connected graph G, denoted by $\pi(G)$, is the smallest integer m such that D is solvable for every configuration D on G with $|D| \geq m$. We note a basic fact, mentioned by Chung [1], of pebbling number of a connected graph G that $\pi(G) \geq |G|$. For a survey of graph pebbling we refer the reader to [5], [6] and [8]. Now, we introduce a new graph pebbling parameter called the support number which is actually an extension of the pebbling number. The *support* of a configuration D on a connected graph G means the set $\{v \in V(G) : D(v) > 0\}$. For a connected graph G and a positive integer n, the *n-support number* of G is the minimum m such that D is solvable for any configuration D on G with $\sum_{v\in V(G)} \left\lfloor \frac{D(v)}{n} \right\rfloor \geq m$ if $n \leq \pi(G)$. It equals 1 otherwise. Obviously, the 1-support number is actually the pebbling number. Additionally, we denote the 2-support number of G by $\tilde{\pi}(G)$.

One of the interesting topics in recent graph pebbling is the Graham's conjecture which introduced by Chung [1]. It is about an upper bound of the pebbling number of the Cartesian product of graphs as follows:

Conjecture 1.1. *[1] If G and H are connected, then*

$$\pi(G\square H) \leq \pi(G)\pi(H).$$

Chung [1] showed that the conjecture holds when H is a complete graph and G is a graph satisfying the so-called 2-pebbling property. Such property plays an important role in verifying the conjecture for certain families of graphs. In case H is a complete graph, it is in general still open by Herscovici [4]. However, we make progress toward this work from a different perspective by focusing on the order of the complete graph H in terms of $\pi(G)$ and $|G|$ as we see in the next section.

2 Main Results

In this section, we will prove Theorem 2.4 by means of the technical Lemma 2.2 about the 2-support number.

Lemma 2.1. *Let $G = (V, E)$ be a connected graph, S be a subset of V and D be a configuration on G. Then we have*

$$\sum_{v\in V\setminus S} D(v) - n \sum_{v\in V\setminus S} \left\lfloor \frac{D(v)}{n} \right\rfloor \leq (n-1)(|G| - |S|)$$

for any positive integer n.

Proof. The inequality holds since $D(v) - n\left\lfloor \frac{D(v)}{n} \right\rfloor \leq n-1$ for each $v \in V$. □

We see that the configuration D on G defined by $D(v) = n-1$ for each $v \in V$ attains the upper bound in Lemma 2.1.

Lemma 2.2. *For a nontrivial connected graph G and a positive integer m greater than 1, we have*

$$\tilde{\pi}(G\square K_m) \leq \pi(G).$$

Proof. Let $V' = V(G\square K_m)$, D be a configuration on $G\square K_m$ with $\sum_{(x,y)\in V'} \left\lfloor \frac{D(x,y)}{2} \right\rfloor \geq \pi(G)$ and (r,t) be a vertex of $G\square K_m$. Let $M = \{(x,y) \in V' : D(x,y) > 1\}$ and let $M_x = \{z \in V(K_m) : (x,z) \in M\}$ for each $x \in p_1(M)$. Then we can pebble (x,t) with at least $\sum_{z\in M_x} \left\lfloor \frac{D(x,z)}{2} \right\rfloor$ pebbles for each $x \in p_1(M)$ since $\tilde{\pi}(K_m) = 1$. Let D' be a configuration on $G\square K_m$ after pebbling (x,t) with at least $\sum_{z\in M_x} \left\lfloor \frac{D(x,z)}{2} \right\rfloor$ pebbles for all $x \in p_1(M)$. It follows that

$$\begin{aligned}
\sum_{x\in V(G)} D'(x,t) \geq \sum_{x\in p_1(M)} D'(x,t) &\geq \sum_{x\in p_1(M)} \sum_{z\in M_x} \left\lfloor \frac{D(x,z)}{2} \right\rfloor \\
&= \sum_{(x,y)\in V'} \left\lfloor \frac{D(x,y)}{2} \right\rfloor \geq \pi(G).
\end{aligned}$$

Hence we can pebble (r,t) within the induced subgraph $G\square\{t\}$. □

Lemma 2.3. *Let G be a nontrivial connected graph with $V = V(G)$. For a positive integer n, let D be a configuration on $G\square K_n$ and (r,t) be a vertex of $G\square K_n$. If S is a proper subset of V containing r such that $\sum_{x\in V\setminus S} |D_x| \geq n(|V\setminus S|) + 2\pi(G)$, then one can pebble (r,t).*

Proof. Let $V' = V(G\square K_n)$ and $S' = V(S\square K_n)$. By Lemma 2.1, we obtain that

$$\begin{aligned}
2\sum_{(x,y)\in V'\setminus S'} \left\lfloor \frac{D(x,y)}{2} \right\rfloor &\geq \left(\sum_{(x,y)\in V'\setminus S'} D(x,y)\right) - (|V'| - |S'|) \\
&= \left(\sum_{x\in V\setminus S} |D_x|\right) - (n|G| - n|S|) = \left(\sum_{x\in V\setminus S} |D_x|\right) - n(|G| - |S|) \\
&= \left(\sum_{x\in V\setminus S} |D_x|\right) - n|V\setminus S| \geq n|V\setminus S| + 2\pi(G) - n|V\setminus S| \\
&= 2\pi(G).
\end{aligned}$$

By Lemma 2.2,

$$\sum_{(x,y)\in V'} \left\lfloor \frac{D(x,y)}{2} \right\rfloor \geq \sum_{(x,y)\in V'\setminus S'} \left\lfloor \frac{D(x,y)}{2} \right\rfloor \geq \pi(G) \geq \tilde{\pi}(G\square K_n).$$

Therefore, we can pebble (r,t). □

Now, we are ready for determining an upper bound for the pebbling number of the Cartesian product of a graph and a complete graph.

Proposition 2.1. *For a positive integer n and a connected graph G, we have*

$$\pi(G\square K_n) \le n|G| + 2\pi(G) - 2.$$

Proof. Let $V = V(G)$ and $V' = V(K_n)$. If $|D_r| \ge n$, then we can pebble (r,t). In addition, we can assume that $|D_r| \le n-1$. We now consider the following two cases.
Case 1: $|D_r| \le n-2$.

Clearly,

$$\sum_{x\in V\setminus\{r\}} |D_x| = |D| - |D_r| \ge |D| - (n-2) = (n|G| + 2\pi(G) - 2) - (n-2)$$
$$= n(|G|-1) + 2\pi(G) = n|V\setminus\{r\}| + 2\pi(G).$$

By Lemma 2.3, we can pebble (r,t).
Case 2: $|D_r| = n-1$.

If $D(r, v_r) \ge 2$ for some $v_r \in V'\setminus\{t\}$, then we can pebble (r,t). So we can assume that $D(r,v_r) = 1$ for all $v_r \in V'\setminus\{t\}$. Since $n(|G|-1)+2\pi(G)-1 \ge n(|G|-1)+1$, there are at least $n(|G|-1)+1$ pebbles distributed by D on $n(|G|-1)$ vertices in $V(G\square K_n)\setminus V(\{r\}\square K_n)$. By the pigeonhole principle, $D(g,u) \ge 2$ for some $(g,u) \in V(G\square K_n)\setminus V(\{r\}\square K_n)$. Let $g = w_1, w_2, \ldots, w_m = r$ be a g,r-path in G. Obviously, $m \ge 2$ since $g \ne r$. Note that

$$\sum_{x\in V\setminus\{w_m\}} |D_x| = \sum_{x\in V\setminus\{r\}} |D_x| = |D| - |D_r| = |D| - (n-1)$$
$$= (n|G| + 2\pi(G) - 2) - (n-1) = n(|G|-1) + 2\pi(G) - 1.$$

This implies that $V\setminus\{w_m\} \ne \emptyset$ since $n(|G|-1) + 2\pi(G) - 1 \ge 2\pi(G) - 1 > 0$. In this case, we can succeed within $m-1$ steps.
Step 1.
If $D(w_{m-1}, v_{m-1}) \ge 2$ for some $v_{m-1} \in V'$, then we move

- $(w_{m-1}, v_{m-1}) \to (w_m, t) = (r,t)$ if $v_{m-1} = t$;
- $(w_{m-1}, v_{m-1}) \to (w_m, v_{m-1}), (w_m, v_{m-1}) \to (w_m, t) = (r,t)$ if $v_{m-1} \ne t$.

In addition, we can assume that $D(w_{m-1}, v_{m-1}) \le 1$ for all $v_{m-1} \in V'$, i.e., $|D_{w_{m-1}}| \le n$.

- If $D(w_{m-1}, v_{m-1}) = 1$ for all $v_{m-1} \in V'$, then $|D_{w_{m-1}}| = n$ and so

$$\sum_{x\in V\setminus\{w_{m-1},w_m\}} |D_x| = \left(\sum_{x\in V\setminus\{w_m\}} |D_x|\right) - |D_{m-1}|$$
$$= \left(\sum_{x\in V\setminus\{w_m\}} |D_x|\right) - n$$
$$= (n(|G|-1) + 2\pi(G) - 1) - n$$
$$= n(|G|-2) + 2\pi(G) - 1.$$

This implies $V\setminus\{w_{m-1}, w_m\} \ne \emptyset$ since $\sum_{x\in V\setminus\{w_{m-1},w_m\}} |D_x| \ge n(|G|-2) + 2\pi(G) - 1 \ge 2\pi(G) - 1 \ge 2|G| - 1 > 0$. So $|G| \ge 3$ and we go to Step 2.

- If $D(w_{m-1}, v_{m-1}) = 0$ for some $v_{m-1} \in V'$, then $|D_{w_{m-1}}| \leq n-1$ and so

$$\begin{aligned}
\sum_{x \in V \setminus \{w_{m-1}, w_m\}} |D_x| &= \left(\sum_{x \in V \setminus \{w_m\}} |D_x| \right) - |D_{w_{m-1}}| \\
&\geq \left(\sum_{x \in V \setminus \{w_m\}} |D_x| \right) - (n-1) \\
&= (n(|G|-1) + 2\pi(G) - 1) - (n-1) \\
&= n(|G|-2) + 2\pi(G) \\
&= n|V \setminus \{w_{m-1}, w_m\}| + 2\pi(G).
\end{aligned}$$

By Lemma 2.3, we can pebble (r, t).

Step i ($1 \leq \text{i} \leq m-2$).
If $D(w_{m-i}, v_{m-i}) \geq 2$ for some $v_{m-i} \in V'$, then we move

- $(w_{m-i}, v_{m-i}) \to (w_{m-i+1}, v_{m-i}), \ldots, (w_{m-2}, v_{m-i}) \to (w_{m-1}, v_{m-i}), (w_{m-1}, v_{m-i}) \to (w_m, t) = (r, t)$ if $v_{m-i} = t$;

- $(w_{m-i}, v_{m-i}) \to (w_{m-i+1}, v_{m-i}), \ldots, (w_{m-2}, v_{m-i}) \to (w_{m-1}, v_{m-i}), (w_{m-1}, v_{m-i}) \to (w_m, v_{m-i}), (w_m, v_{m-i}) \to (w_m, t) = (r, t)$ if $v_{m-i} \neq t$.

In addition, we can assume that $D(w_{m-i}, v_{m-i}) \leq 1$ for all $v_{m-i} \in V'$, i.e., $|D_{w_{m-i}}| \leq n$.

- If $D(w_{m-i}, v_{m-i}) = 1$ for all $v_{m-i} \in V'$, then $|D_{w_{m-i}}| = n$ and so

$$\begin{aligned}
\sum_{x \in V \setminus \{w_{m-i}, \ldots, w_{m-1}, w_m\}} |D_x| &= \left(\sum_{x \in V \setminus \{w_{m-i+1}, \ldots, w_{m-1}, w_m\}} |D_x| \right) - |D_{w_{m-1}}| \\
&= \left(\sum_{x \in V \setminus \{w_{m-i+1}, \ldots, w_{m-1}, w_m\}} |D_x| \right) - n \\
&= (n(|G|-i) + 2\pi(G) - 1) - n \\
&= n(|G| - (i+1)) + 2\pi(G) - 1.
\end{aligned}$$

This implies $V \setminus \{w_{m-i}, \ldots, w_{m-1}, w_m\} \neq \emptyset$ since $\sum_{x \in V \setminus \{w_{m-i}, \ldots, w_{m-1}, w_m\}} |D_x| \geq n(|G| - (i+1)) + 2\pi(G) - 1 \geq 2\pi(G) - 1 \geq 2|G| - 1 > 0$. So $|G| \geq i+2$ and we go to Step i+1.

- If $D(w_{m-i}, v_{m-i}) = 0$ for some $v_{m-i} \in V'$, then $|D_{w_{m-i}}| \leq n-1$ and so

$$\begin{aligned}\sum_{x\in V\setminus\{w_{m-i},\dots,w_{m-1},w_m\}} |D_x| &= \left(\sum_{x\in V\setminus\{w_{m-i+1},\dots,w_{m-1},w_m\}} |D_x|\right) - |D_{w_{m-i}}| \\ &\geq \left(\sum_{x\in V\setminus\{w_{m-i+1},\dots,w_{m-1},w_m\}} |D_x|\right) - (n-i) \\ &= (n(|G|-i) + 2\pi(G) - 1) - (n-i) \\ &= n(|G| - (i+1)) + 2\pi(G) \\ &= n|V \setminus \{w_{m-i},\dots,w_{m-1},w_m\}| + 2\pi(G).\end{aligned}$$

By Lemma 2.3, we can pebble (r,t).

Step m-1.
Since $D(w_1,u) = D(g,u) \geq 2$, we can move

- $(w_1,u) \to (w_2,u), \dots, (w_{m-2},u) \to (w_{m-1},u), (w_{m-1},u) \to (w_m,u) = (r,t)$ if $u = t$;
- $(w_1,u) \to (w_2,u), \dots, (w_{m-2},u) \to (w_{m-1},u), (w_{m-1},u) \to (w_m,u), (w_m,u) \to (w_m,t)$ $= (r,t)$ if $u \neq t$.

□

It is easy to establish the sharpness of the upper bound stated in Proposition 2.1, by considering $G = K_1$ together with the fact that $\pi(K_1 \square K_n) = \pi(K_n) = n$.

In the following result, we obtain an alternative sufficient condition for the Cartesian product of a graph and a complete graph to satisfy Graham's conjecture.

Theorem 2.4. *For a positive integer n and a connected graph G, if $\pi(G) > |G|$ and $n \geq \frac{2(\pi(G)-1)}{\pi(G)-|G|}$, then*

$$\pi(G \square K_n) \leq \pi(G)\pi(K_n).$$

Proof. If $\pi(G) > |G|$ then $n \geq \frac{2(\pi(G)-1)}{\pi(G)-|G|}$ implies $n|G| + 2\pi(G) - 2 \leq n\pi(G) = \pi(K_n)\pi(G)$ so the results follows from Proposition 2.1. □

We note that the condition in Theorem 2.4 does not imply the 2-pebbling property of G as one can see in the following counter example. For a positive integer k, Gao and Yin [2] not only proved that the graph L_k (see Fig. 1) does not satisfy the 2-pebbling property, but they also showed that $\pi(L_k) = 2^{k+3}$.

However, L_k satisfies the condition of G in Theorem 2.4 for each k with a sufficiently large n. And we obtain the following partial result of Gao and Yin [3].

Corollary 2.5. *For positive integers k and n, if $\frac{2}{n} + \frac{4k+7}{2^{k+3}-1} \leq 1$, then*

$$\pi(L_k \square K_n) \leq \pi(L_k)\pi(K_n).$$

for positive

Proof. By mathematical induction on k, $\pi(L_k) = 2^{k+3} > 4k+8 = |L_k|$. Furthermore, we can derive $\frac{2}{n} + \frac{4k+7}{2^{k+3}-1} \leq 1$ from $n \geq \frac{2(\pi(L_k)-1)}{\pi(L_k)-|L_k|} = \frac{2(2^{k+3}-1)}{2^{k+3}-4k-8}$. Hence the result follows by Theorem 2.4. □

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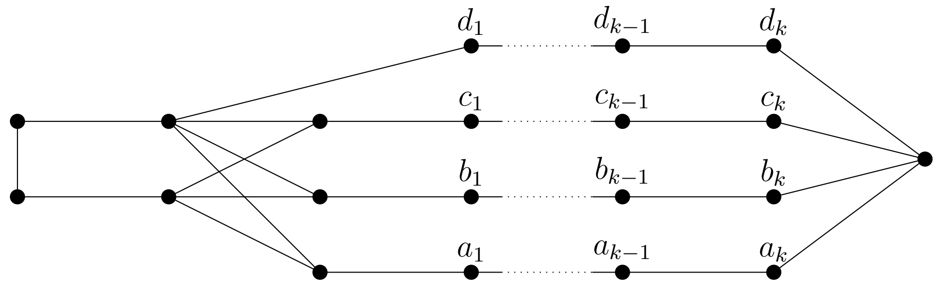

Figure 1: The graph L_k.

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