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# Long path lemma concerning connectivity and independence number

Shinya Fujita\*    Alexander Halperin†    Colton Magnant‡

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## Abstract

We show that, in a  $k$ -connected graph  $G$  of order  $n$  with  $\alpha(G) = \alpha$ , between any pair of vertices, there exists a path  $P$  joining them with  $|P| \geq \min \left\{ n, \frac{(k-1)(n-k)}{\alpha} + k \right\}$ . This implies that, for any edge  $e \in E(G)$ , there is a cycle containing  $e$  of length at least  $\min \left\{ n, \frac{(k-1)(n-k)}{\alpha} + k \right\}$ . Moreover, we generalize our result as follows: for any choice  $S$  of  $s \leq k$  vertices in  $G$ , there exists a tree  $T$  whose set of leaves is  $S$  with  $|T| \geq \min \left\{ n, \frac{(k-s+1)(n-k)}{\alpha} + k \right\}$ .

## 1 Introduction

In this work, we present a tool which we believe will be useful in many applications. Much work has been devoted to finding long paths and cycles in graphs. In particular, in [4], O, West and Wu recently proved a conjecture by Fouquet and Jolivet [3] stated as follows.

**Theorem 1** ([4]) *Let  $k \geq 2$  and let  $G$  be a  $k$ -connected graph of order  $n$  with  $\alpha(G) = \alpha$ . Then there is a cycle in  $G$  of length at least  $\min \left\{ n, \frac{k(n+\alpha-k)}{\alpha} \right\}$ .*

In various situations including this work, it often becomes necessary to find a long path between a chosen pair of vertices. For this reason, O, West and Wu proved the following theorem which they used in their proof of the conjecture.

**Theorem 2** ([4]) *Let  $G$  be a  $k$ -connected graph for  $k \geq 1$ . If  $H \subseteq G$  and  $u$  and  $v$  are distinct vertices in  $G$ , then  $G$  contains a  $u, v$ -path  $P$  such that  $V(H) \subseteq V(P)$  or  $\alpha(H - P) \leq \alpha(H) - (k - 1)$ .*

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We also use this theorem and, following the proofs presented in [4], we prove the following lemma which is our main result.

**Lemma 1** *Let  $k \geq 1$  be an integer and let  $G$  be a graph of order  $n$  with  $\kappa(G) = k$  and  $\alpha(G) = \alpha$ . Then for any pair of vertices  $u, v$  in  $G$ , there exists a  $u, v$ -path of order at least  $\min\{n, \frac{(k-1)(n-k)}{\alpha} + k\}$ .*

Our hope is that this lemma may be applied to produce other results like Theorem 3, which follows immediately from Lemma 1 by choosing  $u$  and  $v$  to be the ends of  $e$ .

**Theorem 3** *Let  $k \geq 2$  be an integer and let  $G$  be a  $k$ -connected graph of order  $n$  with  $\alpha(G) = \alpha$ . Then for any edge  $e \in E(G)$ , there exists a cycle of length at least  $\min\{n, \frac{(k-1)(n-k)}{\alpha} + k\}$  in  $G$  containing the edge  $e$ .*

Lemma 1 can be generalized to the following result concerning large trees with specified sets of leaves. Let  $\ell(T)$  denote the set of leaves in a tree  $T$ .

**Theorem 4** *Let  $k$  and  $s$  be integers with  $2 \leq s \leq k$  and let  $G$  be a  $k$ -connected graph of order  $n$  with  $\alpha(G) = \alpha$ . Then for any set of  $s$  vertices  $V_s = \{v_1, \dots, v_s\} \subseteq G$ , there exists a tree  $T \subseteq G$  with  $V_s = \ell(T)$  and  $|T| \geq \min\{n, \frac{(k-s+1)(n-k)}{\alpha} + k\}$ .*

The proofs of Lemma 1 and Theorem 4 are presented in Section 3. As we will observe in Section 4, our results are all best possible.

## 2 Preliminaries

In our proof, we use the following corollary to break the problem into cases. We also state and prove a path version of Theorem 6. Both of these results come from [4].

**Corollary 5** ([4]) *If a graph  $G$  admits no vertex partition  $(V_1, V_2)$  such that  $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$ , then  $G$  is 2-connected or  $G \in \{K_1, K_2\}$ . Also, for distinct vertices  $u, v \in G$ , there is a  $u, v$ -path  $P$  such that  $\alpha(G - P) < \alpha(G)$ .*

**Theorem 6** ([4]) *Let  $k$  be an integer greater than 1. If  $C$  is a cycle with size at least  $k$  in a  $k$ -connected graph  $G$ , then for any non-empty subgraph  $H \subseteq G - C$ , there exists a cycle  $C'$  in  $G$  such that  $|C - C'| \leq \frac{|C|}{k} - 1$  and  $\alpha(H - C') \leq \alpha(H) - 1$ .*

We will also make use of the following classical result of Chvátal and Erdős [2]. A graph is said to be *hamiltonian connected* if, between any pair of vertices, there exists a path covering the entire graph.

**Theorem 7** ([2]) *For any graph  $G$ , if  $\kappa(G) > \alpha(G)$ , then  $G$  is hamiltonian connected.*

Following the notation of [4], let  $P$  be a path and  $u$  and  $v$  be vertices in  $P$ . Define  $P(u, v)$  to be the subpath of  $P$  strictly between (not including)  $u$  and  $v$ . Also, for a vertex  $v$  and a set of vertices or subgraph  $A$ , define a  $(v, A)$   $k$ -fan to be a set of  $k$  paths from  $v$  to  $A$  which are all pairwise vertex disjoint except at  $v$ . All other standard notation comes from [1].

### 3 Proofs of our Main Results

We begin by proving a key lemma used to obtain our main result. The main idea of the proof is based on that of Theorem 6.

**Lemma 2** *Let  $k \geq 2$  be an integer, and suppose  $G$  is a  $k$ -connected graph containing vertices  $u, v$ . If  $P$  is a  $u, v$ -path of order at least  $k$  in  $G$ , then for any non-empty subgraph  $H \subseteq G \setminus P$ , there is a  $u, v$ -path  $P'$  in  $G$  such that  $|P \setminus P'| \leq \frac{|P|-k}{k-1}$  and  $\alpha(H \setminus P') \leq \alpha(H) - 1$ .*

**Proof:** Suppose there exists a subgraph  $H$  for which there is no desired path  $P'$  and choose  $H$  to be the smallest such subgraph. By Corollary 5, either

- (1)  $H$  can be bipartitioned into non-empty subgraphs  $H_1$  and  $H_2$  so that  $\alpha(H) = \alpha(H_1) + \alpha(H_2)$ , or
- (2)  $H$  is 2-connected or  $H \in \{K_1, K_2\}$ . Also, for any distinct vertices  $x, y \in H$ , there exists an  $x, y$ -path  $P_{xy}$  in  $H$  such that  $\alpha(H \setminus P_{xy}) < \alpha(H)$ .

If (1) holds, we simply apply Lemma 2 on  $H_1$  (since  $H$  was the smallest counterexample) and obtain a path  $P'$  satisfying the desired conditions. Hence we may assume (2) holds.

Let  $B$  be the block of  $G \setminus P$  containing  $H$ . First we assume  $|B| \geq k$ . By Menger's Theorem, there exist  $k$  vertex-disjoint paths from  $P$  to  $B$ . Choose the shortest such set of paths, meaning that each path contains exactly one vertex of  $B$  and one vertex of  $P$ . This means that there must exist a pair of these paths, say  $P_1 = p_1 \dots b_1$  and  $P_2 = p_2 \dots b_2$  for  $p_i \in V(P)$  and  $b_i \in V(B)$  such that there are at most  $\frac{|P|-k}{k-1}$  vertices between  $p_1$  and  $p_2$  on  $P$ . Since  $B$  is 2-connected, there exist vertex-disjoint paths  $P_{b_i}$  in  $B$  from  $b_i$  to  $h_i \in V(H)$  for  $i = 1, 2$ . Note that  $h_1 = h_2$  is only possible if  $|H| = 1$ . (Suppose  $P_{b_i} \cap H = h_i$ .) By (2), there is a path  $P_H$  in  $H$  from  $h_1$  to  $h_2$  for which  $\alpha(H \setminus P_H) < \alpha(H)$ . Then  $P' = (P \setminus P(p_1, p_2)) \cup (P_1 \cup P_{b_1} \cup P_H \cup P_{b_2} \cup P_2)$  is the desired path. Hence, we may assume  $|B| < k$ .

Let  $V(B) = \{b_1, \dots, b_\ell\}$ , where we have assumed  $\ell < k$ . Note that we may possibly have  $\ell = 1$ . Let  $C$  be the component of  $G \setminus P$  containing  $B$ . Let  $S = \{p_1, \dots, p_m\}$  be the set of vertices of  $P$  (in order along  $P$ ) with at least one neighbor in  $C$ . Note that, by Menger's Theorem,  $m \geq k$ .

For each edge  $e$  from  $p_i$  to  $C$ , there exists a unique vertex  $b_j \in B$  such that there is a unique path  $Q_{i,j}$  from  $b_j$  to  $p_i$  containing  $e$  with all interior vertices in  $C \setminus B$ . Let  $X_j$  be the set of vertices  $p_i$  for which such a path  $Q_{i,j}$  exists. Note that the sets  $\{X_j\}$  are not necessarily disjoint. Also note that, since  $B$  is a block,  $Q_{i,j}$  and  $Q_{i',j'}$  are internally disjoint when  $j \neq j'$ . Call a segment  $P(p_i, p_j)$  for  $i < j$  *large* if  $p_i \in X_{i'}$  and  $p_j \in X_{j'}$  for some  $i' \neq j'$ . Otherwise, as long as the segment  $P(p_i, p_j)$  is not contained in a large segment, it will be called *small*.

Using the same argument as above, the following fact is immediate.

**Fact 1** *For any large segment  $P(p_i, p_j)$ , we have*

$$|P(p_i, p_j)| > \frac{|P| - k}{k - 1}.$$

Let  $t$  be the number of segments  $P(p_i, p_{i+1})$  for  $1 \leq i \leq m$  which are large. Since large segments contain at least  $\frac{|P|-k+1}{k-1}$  vertices, we see that

$$|P| \geq t \left( \frac{|P| - k + 1}{k - 1} \right) + k,$$

which implies that  $t < k - 1$ . For each  $b_i \in B$ , there exists a  $(b_i - P)$   $k$ -fan. Choose such a fan so that each path intersects  $P$  in exactly one vertex. Let  $v_1, \dots, v_k$  (in this order on  $P$ ) be the vertices of  $P$  at the ends of this fan. For each pair  $v_j, v_{j+1}$ , we already know that  $v_j, v_{j+1} \in X_i$ , but if one of these is also in  $X_{i'}$  for some  $i' \neq i$ , then  $P(v_j, v_{j+1})$  must be a large segment of  $P$ . This means that, for each vertex in  $B$ , there are at least  $k - 1 - t$  corresponding small segments of  $P$ . Since the ends of these small segments corresponding to  $b_i$  are all in  $X_i$ , these segments must then be disjoint from all small segments corresponding to  $b_j$  for  $j \neq i$  since the ends of those segments would be in  $X_j$ . Therefore there are  $(k - 1 - t)\ell$  small segments *all* pairwise disjoint. This implies that the average order of small segments is at most

$$\frac{|P| - t \left( \frac{|P|-k+1}{k-1} \right) - k}{(k - 1 - t)\ell}.$$

By the pigeonhole principle, if we choose the shortest small segment corresponding to each vertex  $b_i \in B$ , then the sum of the orders of these shortest segments is at most

$$\frac{|P| - t \left( \frac{|P|-k+1}{k-1} \right) - k}{(k - 1 - t)} \leq \frac{|P| - k}{k - 1}.$$

We now replace each of these small segments with the corresponding  $b_i$  using the paths  $Q_{i,j}$  and  $Q_{i,j+1}$  for the appropriate choice of  $j$ . This creates a new  $u, v$ -path  $P'$  such that  $H \subseteq B \subseteq P'$  and  $|P \setminus P'| \leq \frac{|P|-k}{k-1}$ . □

Before our next lemma, we observe an easy fact without proof.

**Fact 2** *Let  $G$  be a  $k$ -connected graph for  $k \geq 2$  and let  $u$  and  $v$  be two distinct vertices in  $G$ . Then for any  $u, v$ -path  $P$  with  $|P| < k$ , there is another  $u, v$ -path  $P'$  with  $|P'| \geq k$  such that  $P \subseteq P'$ .*

**Lemma 3** *Let  $G$  be a graph with  $\kappa(G) = k$  and  $\alpha(G) = \alpha$ . If  $u, v$  are two vertices in  $G$ ,  $\ell$  is an integer satisfying  $0 \leq \ell \leq \alpha - k + 1$ , then there exists a set of  $u, v$ -paths  $P_0, \dots, P_\ell$  satisfying:*

1.  $\alpha \left( G \setminus \bigcup_{i=0}^{\ell} P_i \right) \leq \alpha - k + 1 - \ell$

$$2. \left| P_i \setminus \bigcup_{j=0}^{i-1} P_j \right| \leq \frac{|P_0| - k}{k-1} \text{ for } 1 \leq i \leq \ell$$

**Proof:** Induct on  $\ell$ . If  $\ell = 0$ , Theorem 2 gives a  $u, v$ -path  $P_0$  with  $\alpha(G \setminus P_0) \leq \alpha - k + 1$ . Now suppose we have  $u, v$ -paths  $P_0, \dots, P_{\ell-1}$  satisfying Properties 1 and 2 for  $\ell - 1$ .

Let  $H = G \setminus \bigcup_{i=0}^{\ell-1} P_i$  be so that  $\alpha(H) \leq \alpha - k + 1 - (\ell - 1)$ . Assume  $\alpha(H) \geq 1$  since otherwise we could simply set  $P_\ell = P_0$ . By Lemma 2 with  $P_0 = P$  (note that Fact 2 implies we may assume  $|P_0| \geq k$ ), there is a  $u, v$ -path  $P'$  such that  $|P_0 \setminus P'| \leq \frac{|P_0| - k}{k-1}$  and  $\alpha(H \setminus P') \leq \alpha(H) - 1 \leq \alpha - k + 1 - \ell$ .

**Case 1**  $|P'| \leq |P_0|$

Then  $|P' \setminus \bigcup_{i=0}^{\ell-1} P_i| \leq |P' \setminus P_0| \leq |P_0 \setminus P'| \leq \frac{|P_0| - k}{k-1}$ , so we can set  $P' = P_\ell$  to satisfy the desired properties.

**Case 2**  $|P'| > |P_0|$

Relabel the paths as follows:  $P'_0 = P'$  and  $P'_i = P_{i-1}$  for  $1 \leq i \leq \ell$ . This new labelling gives  $\alpha(G \setminus \bigcup_{i=0}^{\ell} P'_i) \leq \alpha - k + 1 - \ell$  so Property 1 is satisfied. For Property 2, first consider the case  $i = 1$ .  $|P'_1 \setminus P'_0| = |P_0 \setminus P'| \leq \frac{|P_0| - k}{k-1}$  as desired. For  $2 \leq i \leq \ell$ , we have

$$\left| P'_i \setminus \bigcup_{j=0}^{i-1} P'_j \right| \leq \left| P_{i-1} \setminus \bigcup_{j=0}^{i-2} P_j \right| \leq \frac{|P_0| - k}{k-1} \leq \frac{|P'_0| - k}{k-1}$$

so this labelling satisfies Properties 1 and 2, and we have our desired result.  $\square$

Using these lemmas, the proof of our main result is easy.

**Proof of Lemma 1:** For  $k = 1$ , the result is trivial so we will assume  $k \geq 2$ . When  $k > \alpha$ , the assertion holds by Theorem 7. Thus, we may also assume  $\alpha \geq k$ .

Set  $\ell = \alpha - k + 1$  and apply Lemma 3. By Property 1, the set of paths  $P_0, \dots, P_\ell$  must cover all of  $V(G)$ . Using Property 2, this implies

$$n = |P_0| + \sum_{i=1}^{\ell} \left| P_i \setminus \bigcup_{j=0}^{i-1} P_j \right| \leq |P_0| + (\alpha - k + 1) \left( \frac{|P_0| - k}{k-1} \right).$$

Solving for  $|P_0|$ , we get the desired result  $|P_0| \geq \frac{(k-1)(n-k)}{\alpha} + k$ .  $\square$

**Proof of Theorem 4:** This proof is by induction on  $s$ . If  $s = 2$ , the result follows immediately from Lemma 1. Now suppose  $s > 3$  and consider  $G \setminus v_s$ . This graph has  $\kappa(G \setminus v_s) \geq k - 1$  and we will assume  $\alpha(G \setminus v_s) = \alpha(G)$  (otherwise a stronger result is possible). By induction on  $s$ , there exists a tree  $T_{s-1} \subseteq G$  with  $\ell(T_{s-1}) = \{v_1, \dots, v_{s-1}\}$  and

$$\begin{aligned}
|T_{s-1}| &\geq \min \left\{ n-1, \frac{(k-s+1)(n-k)}{\alpha} + k-1, \frac{(k-s+2)(n-k-1)}{\alpha} + k \right\} \\
&\geq \min \left\{ n-1, \frac{(k-s+1)(n-k)}{\alpha} + k-1 \right\}
\end{aligned}$$

as long as  $n \geq 2k + 2 - s - \alpha$ . Otherwise, if we assume  $n < 2k + 2 - s - \alpha$ , then since  $n \geq k + 1$ , if we let  $H = G \setminus \{v_3, v_4, \dots, v_s\}$ , we have  $\kappa(H) \geq \alpha + 1$ . By Theorem 7, this means that  $H$  is hamiltonian connected so we can find a path  $P$  from  $v_1$  to  $v_2$  using all of  $H$ . Since  $G$  is  $k$ -connected, each vertex  $v_i$  for  $3 \leq i \leq s$  has at least  $k$  paths to  $P$ . Since  $k \geq s$ , there is an edge from each  $v_i$  to  $P \setminus \{v_1, v_2\}$ , forming the desired tree of order  $n$ . Hence, we may suppose the above inequality holds.

In  $G$ , there are  $k$  disjoint (except at  $v_s$ ) paths from  $v_s$  to  $T_{s-1}$  so there is at least one such path  $Q$  which avoids the set  $\{v_1, \dots, v_{s-1}\}$ . Hence, the tree  $T = T_{s-1} \cup Q$  is the desired tree with  $|T| \geq |T_{s-1}| + 1$ .  $\square$

## 4 Conclusion

The results contained in this work are all sharp by the following example. Let  $C = K_k$  and let  $H_i = K_{\frac{n-k}{\alpha}}$  for  $1 \leq i \leq \alpha$  where we assume  $\alpha$  divides  $n - k$ . Let  $G = C + (\cup H_i)$  where  $+$  is the standard join operation such that  $V(A + B) = V(A) \cup V(B)$  and  $E(A + B) = E(A) \cup E(B) \cup \{u, v : u \in A, v \in B\}$ . Choose  $u, v \in C$  and let  $P$  be a  $u, v$ -path that uses all vertices of  $C$  and all of  $H_1, \dots, H_{k-1}$ . This is the longest  $u, v$ -path in  $G$ , which shows that Lemma 1 is sharp. The same example, with the inclusion of the edge  $uv$  to complete a cycle, shows that Theorem 3 is sharp.

For Theorem 4, choose  $v_1, \dots, v_s$  from  $C$  to obtain the desired bound. In this situation, because these vertices must be leaves of the constructed tree, we may use the vertices of at most  $k - s + 1$  components  $H_i$  in building  $T$ . Note also that if  $s > k$ , a similar result cannot hold because, if we choose all of  $C$  and at least one vertex of  $G \setminus C$ , at least one vertex of  $C$  must not be a leaf of a tree including these vertices.

The authors hope that the results contained in this work may be applied in other works. Like Theorems 3 and 4 we believe that many results will follow from this work and perhaps other proofs may be simplified through use of Lemma 1.

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