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Triangles in Ks-saturated graphs with minimum degree t

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Abstract

For $n \geq 15$, we prove that the minimum number of triangles in an n-vertex K_4-saturated graph with minimum degree 4 is exactly $2n-4$, and that there is a unique extremal graph. This is a triangle version of a result of Alon, Erdős, Holzman, and Krivelevich from 1996. Additionally, we show that for any $s > r \geq 3$ and $t \geq 2(s-2)+1$, there is a K_s-saturated n-vertex graph with minimum degree t that has $\binom{s-2}{r-1}2^{r-1}n + c_{s,r,t}$ copies of K_r. This shows that unlike the number of edges, the number of K_r's $(r>2)$ in a K_s-saturated graph is not forced to grow with the minimum degree, except for possibly in lower order terms.

1 Introduction

Let F be a graph. A graph G is *F-free* if G does not contain F as a subgraph. A graph G is *F-saturated* if G is F-free, and adding a new edge to G creates a copy of F. The minimum number of edges in an F-saturated graph with n vertices is called the *saturation number of F*. Write $\mathrm{sat}(n,F)$ for this minimum, so

$$\mathrm{sat}(n,F) = \min\{|E(G)| : G \text{ has } n \text{ vertices and is } F\text{-saturated}\}.$$

An n-vertex F-saturated graph with $\mathrm{sat}(n,F)$ edges is called an *extremal graph.*

1.1 History and Previous Results

One of the most important results on graph saturation is that for any graph F with at least one edge, there is a constant C_F such that $\mathrm{sat}(n,F) \leq C_F n$. This was proved by Kászonyi and Tuza in 1986 [15], and shows that saturation numbers are linear in n. Since then, the study of saturation has become an established branch of extremal graph theory. Saturation numbers of hypergraphs and of random graphs have been studied as well [4, 17, 21, 20]. The survey paper of Faudree, Faudree, and Schmitt [10] contains many results and references.

For complete graphs, Erdős, Hajnal, and Moon [8] proved that for $s \geq 3$,

$$\mathrm{sat}(n,K_s) = (s-2)(n-s+2) + \binom{s-2}{2}.$$

Furthermore, there is a unique extremal graph which is, up to isomorphism, $K_{s-2} + \overline{K_{n-s+2}}$ (the join of a clique with $s-2$ vertices and an independent set with $n-s+2$ vertices). This graph has minimum degree $s-2$, and the minimum degree of a K_s-saturated graph is at least $s-2$ (since nonadjacent vertices must have a K_{s-2} in their common neighborhood). A natural question is to ask for the minimum number of edges in an F-saturated graph G with $\delta(G) = t$ where $t > s-2$. Given a graph F and an integer t, let

$$\mathrm{sat}_t(n,F) = \min\{|E(G)| : G \text{ has } n \text{ vertices, is } F\text{-saturated, and has } \delta(G) = t\ \}.$$

Duffus and Hanson [7] proved that $\mathrm{sat}_2(n,K_3) = 2n-5$ for $n \geq 5$, and characterized the extremal graphs. They also showed that $\mathrm{sat}_3(n,K_3) = 3n-15$ for $n \geq 10$. For larger t and

s, the bounds are not exact. In 2014, Day [6], resolving a conjecture of Bollobás [14] from 1996, showed that

$$\mathrm{sat}_t(n, K_s) \geq tn - c_t \tag{1}$$

for any $s \geq 3$, $t \geq s-2$. Here c_t is a constant depending only on t. Further discussion on saturated graphs with degree constraints can be found in [10].

In 2014, Alon and Shikhelman [2] introduced a very important generalization of Turán numbers which has since been extensively studied ([9, 11, 13, 18] to name a few). It is connected to the widely studied Turán problem for Berge hypergraphs ([12, 19], for instance). Motivated by this generalization, Kritschgau et al. [16] defined an analogous generalization of saturation numbers. For graphs H and F, let $\mathrm{sat}(n, H, F)$ be the minimum number of copies of H in an F-saturated graph with n vertices. Observe that $\mathrm{sat}(n, K_2, F) = \mathrm{sat}(n, F)$. In this paper, we introduce the function

$$\mathrm{sat}_t(n, H, F)$$

which is defined to be the minimum number of copies of H in an n-vertex F-saturated graph with minimum degree t. This function generalizes both $\mathrm{sat}_t(n, F)$ and $\mathrm{sat}(n, H, F)$. The question we put forth is the following.

Question 1.1. *Let H and F be graphs. What is the minimum number of copies of H in an F-saturated n-vertex graph with minimum degree t?*

Before stating our results, let us recall a result from [16] which we take as a starting point for our work. In [16], the formula

$$\mathrm{sat}(n, K_3, K_4) = n - 2 \tag{2}$$

was proved, and it was shown that $K_2 + \overline{K_{n-2}}$ is the unique extremal graph. Bounds on $\mathrm{sat}(n, K_r, K_s)$ for all $s > r \geq 3$ were also proved in [16], but they did not give an exact result. Recently, resolving a conjecture put forth in [16], Chakraborti and Loh [5] obtained an exact formula for $\mathrm{sat}(n, K_r, K_s)$ for all $s \geq r \geq 2$ provided n is sufficiently large (as a function of s and r).

Motivated by the fact that (2) gives a formula for $\mathrm{sat}(n, K_3, K_4)$, we will study the function $\mathrm{sat}_t(n, K_3, K_4)$ in detail. Our aim is to determine whether or not a statement similar to (1) holds when counting K_3's in a K_4-saturated graph with minimum degree at least t. First let us state some results that answer Question 1.1 in certain cases. The first proposition we give is easy to prove using the fact that in a K_s-saturated graph, any two nonadjacent vertices must have a K_{s-2} in their common neighborhood.

Proposition 1.2. *Let $s > r \geq 2$ be integers. If G is an n-vertex K_s-saturated graph with $\delta(G) = s-2$, then G is isomorphic to $K_{s-2} + \overline{K_{n-s+2}}$ and consequently, has*

$$\binom{s-2}{r} + (n-s+2)\binom{s-2}{r-1}$$

copies of K_r.

In the special case $(r, s) = (3, 4)$ and $\delta(G) = 2$, we have exactly $n - 2$ triangles. A close look at the proof of (2) in [16] shows that if G is not isomorphic to $K_2 + \overline{K_{n-2}}$, then G has at least n triangles. Now if G is K_4-saturated and is not isomorphic to $K_2 + \overline{K_{n-2}}$, then one can prove that G has minimum degree at least 3. Thus, we see a small jump in the number of triangles when G is no longer allowed to have a vertex of degree 2. It turns out that when the minimum degree is 3, we do not just get 2 additional triangles, but we must get and additional $n - 5$ triangles. Before formalizing this as a proposition, we need to introduce a new graph.

Let $a_1a_2a_3a_4a_5a_1$ be a cycle of length five and let b be a vertex joined to each a_i. Call this graph W. Let m_1, m_3, m_4 be positive integers with $m_1 + m_3 + m_4 = n - s + 1$. In W, replace b with a clique of size $s - 3$, and for $i \in \{1, 3, 4\}$, replace a_i with an independent set of size m_i. Vertices in this new graph are adjacent if and only if the vertices they replaced are adjacent vertices in W. Write $W_s(m_1, 1, m_3, m_4, 1)$ for this graph, which has appeared in the literature (see [3]).

Proposition 1.3. *Let $s \geq 3$. If G is an n-vertex K_s-saturated graph with $\delta(G) = s-1$, then G is isomorphic to either $(K_{s-1} - e) + \overline{K_{n-s+1}}$, or $W_s(m_1, 1, m_3, m_4, 1)$ for some m_1, m_3, m_4 with $m_1 + m_3 + m_4 = n - s + 1$.*

It follows from Proposition 1.3 that if G is an n-vertex K_4-saturated graph with minimum degree 3, then G has at least $2n - 7$ triangles, and equality holds only if G is isomorphic to $W_4(m_1, 1, m_3, 1, 1)$ for some $m_1 + m_3 = n - 4$. For a proof of Proposition 1.3, see [16].

1.2 New Results

In light of Day's Theorem (1) and that

- $|V(G)| = n$, G is K_4-saturated, and $\delta(G) = 2 \quad \Rightarrow \quad G$ has at least $n - 2$ triangles,
- $|V(G)| = n$, G is K_4-saturated, and $\delta(G) = 3 \quad \Rightarrow \quad G$ has at least $2n - 7$ triangles,

one may be tempted to conjecture that in general, $\delta(G) = t$ forces at least $(t - 1)n - O(1)$ triangles in any n-vertex K_4-saturated graph. This would then give a version of Day's result for triangles.

It turns out, rather surprisingly, that for any $t \geq 4$ and $n \geq t + 5$, there is an n-vertex K_4-saturated graph that has minimum degree t and only $2n + 2t - 12$ triangles. We call this graph $H_t(n)$ and it is defined in Section 2. Our main theorem determines $\mathrm{sat}_t(n, K_3, K_4)$, and shows $H_4(n)$ is the unique extremal graph.

Theorem 1.4. *Let $n \geq 14$ and G be an n-vertex K_4-saturated graph with $\delta(G) = 4$. Then G contains at least $2n - 4$ triangles. Furthermore, if G contains exactly $2n - 4$ triangles, then G is isomorphic to $H_4(n)$.*

For $t \geq 4$, the graph $H_t(n)$ implies the following upper bound on $\mathrm{sat}_t(n, K_3, K_4)$.

Theorem 1.5. *For integers $t \geq 4$ and $n \geq 2t$,*

$$\mathrm{sat}_t(n, K_3, K_4) \leq 2n + 2t - 12.$$

We conjecture that the upper bound in Theorem 1.5 is best possible, and that $H_t(n)$ is the unique extremal graph for all sufficiently large n.

Conjecture 1.6. *For any integer $t \geq 4$, there is an integer n_t such that for all $n \geq n_t$,*

$$\mathrm{sat}_t(n, K_3, K_4) = 2n + 2t - 12$$

and $H_t(n)$ is the unique extremal graph.

Theorem 1.4 shows that Conjecture 1.6 is correct when $t = 4$. The next theorem gives an upper bound for arbitrary $s > r \geq 3$.

Theorem 1.7. *Let $s > r \geq 3$ and $t \geq 2(s-2)+1$ be integers. For $n \geq 2(s-2)+2t$,*

$$\mathrm{sat}_t(n, K_r, K_s) \leq \binom{s-2}{r-1} 2^{r-1} n + C_{s,r,t}$$

where $C_{s,r,t}$ is a constant depending only on r, s, and t.

It would be interesting to determine if there is a constant $c_{r,s}$, depending only on r and s, such that for all $n \geq n(r,s,t)$ and $t \geq s-2$, there is an n-vertex K_s-saturated graph with minimum degree t having at most $c_{r,s}n + o(n)$ copies of K_r. Theorem 1.5 shows that such a constant exists in the case when $(r,s) = (3,4)$. Theorem 1.7 covers all $s > r \geq 3$, but assumes $t \geq 2(s-2)+1$, and perhaps the coefficient of n could be improved (as in Theorem 1.5 when $(r,s) = (3,4)$). We can improve this upper bound in the case $(r,s) = (3,5)$ as the following result shows.

Theorem 1.8. *For any $t \geq 8$ and $n \geq t + 30$,*

$$\mathrm{sat}_t(n, K_3, K_5) \leq 9n + C_t$$

where C_t is a constant dependent only on t.

Finally, a simple argument in the case $r = 3$ gives a lower bound on $\mathrm{sat}_t(n, K_3, K_s)$.

Proposition 1.9. *Let $s > 3$ and $t \geq 6\binom{s-2}{2}$ be integers. For $n \geq 2s-2$,*

$$\binom{s-2}{2}(n-2) \leq \mathrm{sat}_t(n, K_3, K_s).$$

Proposition 1.9 implies $\mathrm{sat}_t(n, K_3, K_5) \geq 3(n-2)$ for $t \geq 18$. We suspect the upper bound of Theorem 1.8 could give the correct coefficient of n.

In Section 2, we define the graphs that prove Theorems 1.5, 1.7, and Proposition 1.8. In Section 3, we prove Theorem 1.4. In Section 4, we prove Proposition 1.9.

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2 Proof of Theorems 1.5, 1.7, and 1.8

In this section, we construct the graphs that prove Theorems 1.5, 1.7, and 1.8. While the constructions are different, they share a common theme: their vertex set is the union of four sets A, B, X, and Y. Most of the vertices will belong to X, and these vertices will have degree t. The set Y is used to force minimum degree t. In all three constructions, X and Y will be independent sets, every vertex in X will be adjacent to every vertex in $A \cup Y$, and every vertex in Y will be adjacent to every vertex in $B \cup X$. The edges with endpoints in $A \cup B$ must be chosen much more carefully, and this is where the constructions begin to differ.

2.1 Proof of Theorem 1.5

Let $t \geq 4$ and $n > 2t$ be integers. Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_{123}, b_{124}, b_{134}, b_{234}\}$,

$$X = \{x_1, x_2, \ldots, x_{n-t-4}\}, \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_{t-4}\}$$

where $Y = \emptyset$ when $t = 4$. The vertex set of $H_t(n)$ is $A \cup B \cup X \cup Y$. The edges of $H_t(n)$ are defined as follows:

- each a_i is adjacent to b_{jlk} if and only if $i \in \{j, l, k\}$,
- a_1 is adjacent to a_2, a_3 is adjacent to a_4, and each vertex in $\{b_{123}, b_{124}\}$ is adjacent to each vertex in $\{b_{134}, b_{234}\}$,
- every vertex in X is adjacent to every vertex in $A \cup Y$, and every vertex in Y is adjacent to every vertex in $B \cup X$.

This completes the description of $H_t(n)$. We now prove Theorem 1.5 by showing that (i) $H_t(n)$ is K_4-saturated, (ii) the minimum degree of $H_t(n)$ is t, and (iii) $H_t(n)$ has exactly $2n + 2t - 12$ triangles.

Proof of Theorem 1.5. (i) Showing that $H_t(n)$ is K_4-free is equivalent to showing that no vertex in $H_t(n)$ has a triangle in its neighborhood. If $x_i \in X$, then $N(x_i) = Y \cup A$ and the only edges in the subgraph induced by $N(x_i)$ are a_1a_2 and a_3a_4. If $y_i \in Y$, then $N(y_i) = X \cup B$ and the only edges in the subgraph induced by $N(y_i)$ are $b_{123}b_{134}, b_{123}b_{234}, b_{124}b_{134}$, and $b_{124}b_{234}$. These edges form a 4-cycle. Therefore, no K_4 in $H_t(n)$ can contain a vertex in $X \cup Y$. The 8 vertex subgraph induced by $A \cup B$ is K_4-free and thus, $H_t(n)$ must be K_4-free.

Next we show that adding any missing edge to $H_t(n)$ creates a K_4. The possibilities that must be checked are missing edges with one endpoint in Z_1 and the other in Z_2 where (Z_1, Z_2) ranges over (X, X), (X, B), (A, A), (A, Y), (A, B), (Y, Y), and (B, B). A quick remark is that this list is the same set of possibilities that must be checked in the proofs of Theorem 1.7 and Theorem 1.8.

When checking these pairs, we will take advantage of the symmetries within parts X, Y, A, and B.

- If x_ix_j is added, then $\{x_i, x_j, a_1, a_2\}$ is a K_4.

- If $x_i b_{123}$ is added, then $\{x_i, b_{123}, a_1, a_2\}$ is a K_4.
- If $a_1 a_3$ is added, then $\{a_1, a_3, b_{123}, b_{134}\}$ is a K_4.
- If $a_1 y_i$ is added, then $\{a_1, y_i, b_{123}, b_{134}\}$ is a K_4.
- If $a_1 b_{234}$ is added, then $\{a_1, b_{234}, a_2, b_{124}\}$ is a K_4.
- If $y_1 y_2$ is added, then $\{y_1, y_2, b_{123}, b_{134}\}$ is a K_4.
- If $b_{123} b_{124}$ is added, then $\{b_{123}, b_{124}, a_1, a_2\}$ is a K_4.

Thus, $H_t(n)$ is K_4-saturated.

(*ii*) To check that $H_t(n)$ has a minimum degree t, we compute the degree of each vertex. If $x \in X$, then $d(x) = |A|+|Y| = 4+|Y| = t$. If $y \in Y$, then $d(y) = |B|+|X| = 4+|X| = n-t$. If $a \in A$, then $d(a) = 4 + |X| = n - t$. Finally, if $b \in B$, then $d(b) = 5 + |Y| = t + 1$.

(*iii*) The graph $H_t(n)$ contains 12 triangles in the subgraph induced by $A \cup B$, $4(t-4)$ triangles containing a vertex in Y, and $2(n-t-4)$ triangles containing a vertex in X. Thus, there are $2n + 2t - 12$ triangles in $H_t(n)$. □

2.2 Proof of Theorem 1.8

Let $t > 9$ and $n > 2t + 18$ be integers. Let A be the disjoint union of the sets A_1, A_2, and A_3 where $A_1 = \{a_{\{1,4,7\}}, a_{\{2,5,8\}}, a_{\{3,6,9\}}\}$, $A_2 = \{a_{\{1,5,9\}}, a_{\{2,6,7\}}, a_{\{3,4,8\}}\}$, and $A_3 = \{a_{\{1,6,8\}}, a_{\{2,4,9\}}, a_{\{3,5,7\}}\}$. Notice that the subscripts of a_{ℓ_1}, a_{ℓ_2}, $a_{\ell_3} \in A_\ell$ form a class of parallel lines in an affine plane of order 3. Also, the subscripts from two distinct A_ℓ's are different parallel classes. Let B be the disjoint union of sets B_1, B_2, and B_3 where $B_\ell = \{b_{\ell,1}, b_{\ell,2}, \ldots, b_{\ell,9}\}$. Finally, let X and Y be disjoint sets where

$$X = \{x_1, x_2, \ldots, x_{n-t-27}\} \text{ and } Y = \{y_1, y_2, \ldots, y_{t-9}\}.$$

The vertex set of $R_t(n)$ is $A \cup B \cup X \cup Y$. The edges of $R_t(n)$ are defined as follows:

- each A_ℓ induces a K_3 (so $A = A_1 \cup A_2 \cup A_3$ induces three vertex disjoint triangles),
- the set B induces a complete 3-partite graph with parts B_1, B_2, and B_3,
- for $1 \le \ell \le 3$, each $a_{\{j,k,l\}} \in A_\ell$ is adjacent to each vertex in

  $$B_\ell \cup \{b_{m,p} : 1 \le m \le 3, p \in \{j,k,l\}\}.$$

  For instance, $a_{\{1,4,7\}} \in A_1$ is adjacent to all vertices in

  $$B_1 \cup \{b_{2,1}, b_{2,4}, b_{2,7}, b_{3,1}, b_{3,4}, b_{3,7}\}.$$

- Every vertex in X is adjacent to every vertex in $A \cup Y$, and every vertex in Y is adjacent to every vertex in $X \cup B$.

Let us call the graph constructed up to this point $R'_t(n)$. We will need to add some edges to $R'_t(n)$ to obtain $R_t(n)$, but first we will show that $R'_t(n)$ is K_5-free.

Lemma 2.1. *The graph $R_t'(n)$ is K_5-free.*

Proof of Lemma 2.1. We will write $R_t'(n)[Z]$ for the subgraph of $R_t'(n)$ induced by Z. If $x \in X$, then $N(x) = Y \cup A$. This neighborhood induces a K_4-free graph (three disjoint triangles and isolated vertices) and thus, x cannot be in a K_5. A similar argument shows that no vertex in Y is in a K_5. Thus, a possible K_5 in $R_t'(n)$ must only use vertices in $A \cup B$. Suppose K is such a K_5. Since $R_t'(n)[B]$ is K_4-free, K must use at least two vertices in A, say a, a'. These two vertices must be in one of the sets A_1, A_2, or A_3. Suppose $a, a' \in A_\ell$ where $A_\ell = \{a_{\ell_1}, a_{\ell_2}, a_{\ell_3}\}$, and let $a = a_{\ell_1}$, $a' = a_{\ell_2}$. The neighborhoods of a_{ℓ_1} and a_{ℓ_2} in B are

$$N(a_{\ell_1}) = B_\ell \cup \{b_{i,p}, b_{j,p} : p \in \ell_1\} \quad \text{and} \quad N(a_{\ell_2}) = B_\ell \cup \{b_{i,p}, b_{j,p} : p \in \ell_2\}$$

where $\{\ell, i, j\} = \{1, 2, 3\}$. Because $\ell_1 \cap \ell_2 = \emptyset$, the intersection of the two neighborhoods above is B_ℓ which is an independent set. Therefore, K can contain at most one vertex in B and since $R_t'(n)[A]$ is K_4-free, there is no K_5 in $R_t'(n)[A \cup B]$. □

Next we complete the construction of $R_t(n)$ by adding edges between A and B.

- Fix an ordering of the nonedges of $R_t'(n)$ that have one endpoint in A and the other in B. Say this ordering is $e_1, e_2, \ldots, e_z$. We consider these nonedges, in this order, and proceed through this ordering adding edge e to $R_t'(n)$ provided this does not create a K_5.

Let $R_t(n)$ be the resulting graph (so $R_t(n)$ does depend on the ordering of the nonedges between A and B, but this is not important). By Lemma 2.1 and the manner in which edges are added to obtain $R_t(n)$, the graph $R_t(n)$ is K_5-free.

We now complete the proof of Theorem 1.8 by showing (i) $R_t(n)$ is K_5-saturated, (ii) the minimum degree of $R_t(n)$ is t, and (iii) $R_t(n)$ has $3n + C_t$ triangles.

Proof of Theorem 1.8. (i) By definition, the graph $R_t(n)$ is K_5-free. Next we show that adding a missing edge to $R_t(n)$ creates a K_5. The possibilities that must be checked are the same as those in the proof of Theorem 1.5. Again, we will use the symmetries within parts X, Y, A, and B.

- If $x_i x_j$ is added, then $\{x_i, x_j\} \cup A_1$ is a K_5.
- If $x_i b_{\ell,p}$ is added, then $\{x_i, b_{\ell,p}\} \cup A_\ell$ is a K_5.
- If aa' is added, then we note that a and a' must belong to different A_ℓ's. Let $a \in A_\ell$ and $a' \in A_{\ell'}$, say $a = a_{\ell_1}$ and $a' = a_{\ell_1'}$. Then ℓ_1 and ℓ_1' intersect in exactly one element (they correspond to nonparallel lines in an affine plane), say z. In this case, $\{a_{\ell_1}, a_{\ell_1'}, b_{1,z}, b_{2,z}, b_{3,z}\}$ is a K_5.
- If $a_{\ell_1} y$ is added where $a_{\ell_1} \in A_\ell$ and $y \in Y$, then $\{a_{\ell_1}, y, b_{1,z}, b_{2,z}, b_{3,z}\}$ is a K_5 where z is any element of ℓ_1.
- If ab is added where $a \in A$ and $b \in B$, then this edge ab must lie in a K_5, otherwise, it would have been added to $R_t'(n)$ when constructing $R_t(n)$.

- If $y_i y_j$ is added, then $\{y_i, y_j, b_{1,1}, b_{2,1}, b_{3,1}\}$ is a K_5.
- If $b_{\ell,i} b_{\ell,j}$ is added where $b_{\ell,i}, b_{\ell,j} \in B_\ell$, then $\{b_{\ell,i}, b_{\ell,j}\} \cup A_\ell$ is a K_5.

This covers all cases and since $R_t(n)$ is K_5-free, we have shown that $R_t(n)$ is K_5-saturated.

(ii) To check that $R_t(n)$ has a minimum degree t we compute the degree of each vertex. If $x \in X$, then $d(x) = |A| + |Y| = 9 + |Y| = t$. If $y \in Y$, then $d(y) = |B| + |X| = 27 + |X| = n - t - 9$. If $a \in A$, then $d(a) \geq 17 + |X| = n - t - 10$. If $b \in B$, then $d(b) \geq 23 + |Y| = t + 14$.

(iii) Recall that A has 9 vertices, B has 27 vertices, and Y has $t - 9$ vertices. Thus, the number of triangles that do not use a vertex in X is at most $c = c_t$ triangles where c_t depends only on t. The number of triangles that use a vertex in X is $e(A)|X| = 9(n - t - 27)$. Therefore, the graph $R_t(n)$ has at most $9n + C_t$ triangles where C_t only depends on t. $\square$

Proof of Theorem 1.7

Let $s > r \geq 3$, $t \geq 2(s-2)+1$, and $n \geq 2(s-2) + 2t$. Let A, B, X, and Y be disjoint sets where

$$A = \{a_1, a_2, \ldots, a_{2(s-2)}\}, \quad B = \{b_1, b_2, \ldots, b_{2(s-2)}\},$$
$$X = \{x_1, x_2, \ldots, x_{n-2(s-2)-t}\}, \text{ and } Y = \{y_1, y_2, \ldots, y_{t-2(s-2)}\}.$$

Let $F'_{s,t}(n)$ be the graph with vertex set $A \cup B \cup X \cup Y$, and whose edges are defined as follows:

- the set A induces a complete $(s-2)$ partite graph with parts $\{a_j, a_{s-2+j}\}$ for $j = 1, 2, \ldots, s-2$,
- like A, the set B induces a complete $(s-2)$ partite graph with parts $\{b_j, b_{s-2+j}\}$ for $j = 1, 2, \ldots, s-2$,
- for $1 \leq i \leq 2(s-2)$, a_i is adjacent to each vertex in the set

  $$\{b_i, b_{i+1}, b_{i+2}, \ldots, b_{i+(s-2)-1}\}$$

  where the subscripts of the b_j's are taken modulo $2(s-2)$ using the residues $\{1, 2, \ldots, 2(s-2)\}$,
- each vertex in X is adjacent to each vertex in $A \cup Y$, and each vertex in Y is adjacent to each vertex in $B \cup X$.

Let us call the graph constructed up to this point $F'_{s,t}(n)$. Like $R_t(n)$, we will add edges between A and B to obtain $F_{s,t}(n)$, but first we prove a lemma showing that $F'_{s,t}(n)$ is K_s-free.

Lemma 2.2. *The graph $F'_{s,t}(n)$ is K_s-free.*

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Proof of Lemma 2.2. Aiming for a contradiction, suppose that K is a K_s in $F'_{s,t}(n)$. If $x \in X$, then $N(x) = A \cup Y$ which induces a K_{s-1}-free graph. This shows that no vertex in X can be in K and similarly, no vertex in Y is in K. Without loss of generality, we may assume that a_1 is a vertex in K. Since $F'_{s,t}(n)[A]$ and $F'_{s,t}(n)[B]$ are K_{s-1}-free, K must contain at least two vertices in A and at least two vertices in B. Let us consider the common neighborhood of a_1 and another vertex a_i in $K \cap A$. If $2 \leq i \leq s-2$, then

$$N(a_1) \cap N(a_i) = \{b_i, b_{i+1}, \ldots, b_{s-2}\}. \tag{3}$$

If $i = j + (s-2)$ where $2 \leq j \leq s-2$, then

$$N(a_1) \cap N(a_{(s-2)+j}) = \{b_1, b_2, \ldots, b_{j-1}\}. \tag{4}$$

Suppose K contains $a_{i_1}, a_{i_2}, \ldots, a_{i_\alpha}$ and $a_{j_1+(s-2)}, a_{j_2+(s-2)}, \ldots, a_{j_\beta+(s-2)}$ where

$$2 \leq i_1 < i_2 < \cdots < i_\alpha \leq s-2 \tag{5}$$

and

$$2 \leq j_1 < j_2 < \cdots < j_\beta \leq s-2. \tag{6}$$

First consider the case when $\beta = 0$, which forces $\alpha \geq 1$ since K must contain at least two vertices in A. If b_ℓ is a vertex in $K \cap B$, then by (3), $i_\alpha \leq \ell \leq s-2$, and there are $s-2-i_\alpha+1 = s-1-i_\alpha$ integers satisfying this inequality. Therefore, the number of vertices in K is at most

$$s - 1 - i_\alpha + \alpha + 1 = s - i_\alpha + \alpha \leq s - (\alpha+1) + \alpha = s-1.$$

For the last inequality, we have used (5). This shows that K has at most $s-1$ vertices and so cannot be a K_{s-1}. The case when $\beta = 0$ can be dealt with using a very similar argument.

Now suppose both α and β are not zero. If b_ℓ is in $K \cap B$, then (3) implies $\ell \geq i_\alpha$, and (4) implies $\ell \leq j_1 - 1$. Since ℓ, j_1, and i_α are integers, the number of possible ℓ satisfying $i_\alpha \leq \ell \leq j_1 - 1$ is $(j_1 - 1) - i_\alpha + 1 = j_1 - i_\alpha$. By (5) and (6),

$$j_1 - i_\alpha \leq (s-2) - \beta + 1 - (\alpha + 1) = s - 2 - \alpha - \beta.$$

Therefore, the number of vertices in $K \cap B$ is at most $s - 2 - \alpha - \beta$, and the number of vertices in $K \cap A$ is $1 + \alpha + \beta$. We conclude that K has at most $s-1$ vertices, a contradiction. This completes the proof of Lemma 2.2. □

We now complete the construction of $F_{s,t}(n)$ by adding edges between A and B.

- Fix an ordering of the nonedges of $F'_{s,t}(n)$ that have one endpoint in A and the other in B. Say this ordering is $e_1, e_2, \ldots, e_z$. We consider these nonedges, in this order, and proceed through this ordering adding edge e to $F'_{s,t}(n)$ provided this does not create a K_s.

Let $F_{s,t}(n)$ be the resulting graph. By Lemma 2.2 and the manner in which edges were added to obtain $F_{s,t}(n)$, the graph $F_{s,t}(n)$ is K_s-free.

We now prove Theorem 1.7 by showing that (i) $F_{s,t}(n)$ is K_s-saturated, (ii) the minimum degree of $F_{s,t}(n)$ is t, and (iii) $F_{s,t}(n)$ has $\binom{s-2}{r-1}2^{r-1}n + C_{r,s,t}$ triangles.

Proof of Theorem 1.7. (i) The graph $F_{s,t}(n)$ is K_s-free by definition. Now we show that adding any missing edge to $F_{s,t}(n)$ creates a K_s. We will use the symmetry within the parts A, B, X, and Y. The possibilities that must be checked are the same as in the proof of Theorems 1.5 and 1.7.

- If x_ix_j is added, then $\{x_i, x_j, a_1, a_2, \ldots, a_{s-2}\}$ is a K_s.
- If x_ib_{s-2} is added, then $\{x_i, b_{s-2}, a_1, a_2, \ldots, a_{s-2}\}$ is a K_s.
- If $a_1a_{(s-2)+1}$ is added, then $\{a_1, a_2, a_3, \ldots, a_{(s-2)+1}\} \cup \{x\}$ is a K_s where x is an arbitrary vertex in X.
- If ab is added where $a \in A$ and $b \in B$, then a K_s is created otherwise the edge ab would have been added when constructing $F_{s,t}(n)$ from $F'_{s,t}(n)$.
- If a_1y_j is added, then $\{a_1, y_j, b_1, b_2, \ldots, b_{s-2}\}$ is a K_s.
- If y_iy_j is added, then $\{y_i, y_j, b_1, b_2, \ldots, b_{s-2}\}$ is a K_s.
- If $b_1b_{(s-2)+1}$ is added, then $\{b_1, b_2, \ldots, b_{(s-2)+1}\} \cup \{y\}$ is a K_s where y is an arbitrary vertex in Y.

The above shows that $F_{s,t}$ is K_s-saturated.

(ii) If $x \in X$, then $d(x) = |A| + |Y| = t$. If $a \in A$, then

$$d(a) \geq |X| + (2(s-2) - 1) + (s-2) = n - t + s - 3.$$

If $y \in Y$, then $d(y) = |X| + |B| = n - t$. Finally, if $b \in B$, then

$$d(b) \geq |Y| + (2(s-2) - 1) + (s-2) = t + s - 3 \geq t.$$

(iii) The number of copies of K_r that use a vertex in X is $\beta|X|$ where β is the number of copies of K_{r-1} in the subgraph induced by A. Since A induces a complete $(s-2)$-partite graph with 2 vertices in each part, $\beta = \binom{s-2}{r-1}2^{r-1}$. Thus, the number of K_r's using a vertex in X is

$$(n - 2(s-2) - t)\binom{s-2}{r-1}2^{r-1}.$$

The number of K_r's not using a vertex in X is at most $C_{s,t,r}$ where $C_{s,t,r}$ is a constant depending only on s, t, and r (the part sizes of A, B, and Y only depend on s and t). The conclusion is that the number of K_r's in $F_{s,t}(n)$ is at most

$$n\binom{s-2}{r-1}2^{r-1} + C_{s,t,r}$$

where $C_{s,t,r}$ depends on s, t, and r. $\square$

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3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The proof will be broken up into several lemmas. We write $k_3(G)$ for the number of triangles in G. For $x \in V(G)$, $N(x)$ is the neighborhood of x. Let us set up some specialized notation that will be used in the rest of this section. This same notation was used in [1].

Setup and Notation: For the remainder of Section 3, let G be an n-vertex graph that is K_4-saturated, and has $\delta(G) = 4$. Let $x \in V(G)$ be a vertex of degree 4, and let

$$N(x) = \{x_1, x_2, x_3, x_4\} \quad \text{and} \quad N[x] = \{x\} \cup N(x).$$

Let $Y = V(G) \backslash N[x]$. If $y \in Y$, then $N(y) \cap N(x)$ must contain an edge so that every vertex in Y is adjacent to at least two vertices in $N(x)$. For $S \subseteq \{1, 2, 3, 4\}$, let V_S be the vertices $y \in Y$ such that y is a neighbor of x_i if and only if $i \in S$. For notational convenience, we will omit braces and commas so if, say $S = \{1, 3, 4\}$, then we write V_{134} rather than $V_{\{1,3,4\}}$. Given sets $S, T \subseteq \{1, 2, 3, 4\}$, write

$$V_S \sim V_T$$

if all vertices in V_S are adjacent to all vertices in V_T, and $V_S \nsim V_T$ if no vertex in V_S is adjacent to a vertex in V_T.

An important observation is that if $V_S \neq \emptyset$, then there must be a pair $\{i, j\} \subseteq S$ such that x_i is adjacent to x_j. Also, if $\{x_i : i \in S\}$ forms an independent set in $N(x)$, then $V_S = \emptyset$. Because of this, we ignore all V_S for which $\{x_i : i \in S\}$ is an independent set in $N(x)$.

Given a subset $S \subset \{1, 2, 3, 4\}$ with $i \notin S$ for some $i \in \{1, 2, 3, 4\}$, let $\mathcal{T}_{S,i}$ be the set of all $T \subseteq \{1, 2, 3, 4\}$ for which $i \in T$, and for any $j, k \in S$ for which x_j and x_k are adjacent vertices in $N(x)$, we have $|T \cap \{j, k\}| \leq 1$. Equivalently, $\mathcal{T}_{S,i}$ is the set of all $T \subseteq \{1, 2, 3, 4\}$ for which $i \in T$ and $\{x_i : i \in T \cap S\}$ is an independent set.

Lemma 3.1 (Rules Lemma). *Let $S \subset \{1, 2, 3, 4\}$ with $i \notin S$ for some $i \in \{1, 2, 3, 4\}$. If $V_S \neq \emptyset$, then y is adjacent to some vertex in*

$$\bigcup_{T \in \mathcal{T}_{S,i}} V_T$$

and in particular, this union is not empty.

Proof. Suppose $y \in V_S$. Since $i \notin S$, the vertex y is not adjacent to x_i where $x_i \in N(x)$. There must be an edge $\alpha\beta$ with both α and β in the intersection $N(y) \cap N(x_i)$. If $\alpha, \beta \in N(x)$, then $\{x, x_i, \alpha, \beta\}$ is a K_4 which is a contradiction. Thus, we may assume that $\alpha \in Y$, say $\alpha \in V_T$. Because α is adjacent to x_i, we have $i \in T$. If there is a $j, k \in S$ for which $x_j x_k$ is an edge in G and $j, k \in T$, then $\{y, \alpha, x_j, x_k\}$ is a K_4. We conclude that T must be some member of $\mathcal{T}_{S,i}$. $\square$

It will be convenient to represent Lemma 3.1 with an arrow diagram and we refer to these as Rules. Assuming the set up and conclusion of Lemma 3.1 where $\mathcal{T}_{S,i} = \{T_1, T_2, \ldots, T_\ell\}$, then we illustrate using the figure below.

$$\begin{tikzcd} & V_{T_1} \\ V_S \arrow[ur] \arrow[r] \arrow[ddr] & V_{T_2} \\ & \vdots \\ & V_{T_\ell} \end{tikzcd}$$

Rule for V_S and choice of $i \notin S$

We also use the notation $V_S \to V_{T_1} \cup V_{T_2} \cup \cdots \cup V_{T_\ell}$. Note that the assertion of Lemma 3.1 is not that $V_{T_j} \neq \emptyset$ for every $T_j \in \mathcal{T}_{S,i}$, but only that at least one of these sets is not empty.

The first application of Lemma 3.1 will be in the proof of Lemma 3.4. First we prove two easier lemmas.

Lemma 3.2. *The number of edges in $N(x)$ is at least 2.*

Proof of Lemma 3.2. See Lemma 3.2 of [16] (the proof is not difficult). □

Lemma 3.3. *If $n \geq 14$ and the number of edges in $N(x)$ is 2, then*

$$k_3(G) \geq 2n - 4.$$

Furthermore, equality holds if and only if G is isomorphic to $H_4(n)$.

Proof of Lemma 3.3. There are two possibilities for the edges in $N(x)$.

Case 1: The two edges in $N(x)$ form a path.

Suppose the edges in $N(x)$ are x_1x_2 and x_2x_3. Every vertex in Y must be adjacent to x_2 since $N(x) \cap N(y)$ must contain an edge for all $y \in Y$, and every edge in $N(x)$ has x_2 as an endpoint. Therefore, x_2 is adjacent to all vertices in G except for x_4. The intersection $N(x_3) \cap N(x_4)$ must contain an edge, say y_1y_2. The edge y_1y_2 must have both endpoints in Y since x is the only neighbor of x_4 in $N[x]$, but then $\{y_1, y_2, x_2, x_3\}$ is a K_4 in G (y_1 and y_2 are both adjacent to x_2). This is a contradiction.

Case 2: The two edges in $N(x)$ share no endpoints.

Assume the edges in $N(x)$ are x_1x_2 and x_3x_4. Then V_{12} and V_{34} must be empty (see [1]). Since $N(y) \cap N(x)$ must contain an edge for each $y \in Y$ and $V_{12} = V_{34} = \emptyset$, we can partition Y as follows:

$$Y = V_{123} \cup V_{124} \cup V_{134} \cup V_{234} \cup V_{1234}.$$

Note that every vertex in Y is adjacent to either both x_1 and x_2, or to both x_3 and x_4.

Suppose $y \in V_{1234}$. We claim that the degree of y is exactly 4. If y is adjacent to some $z \in Y \backslash \{y\}$, then we may assume, without loss of generality, that z is adjacent to both x_1 and x_2. However, this implies $\{y, z, x_1, x_2\}$ is a K_4 in G which is a contradiction. Consequently, $d(y) = 4$ whenever $y \in V_{1234}$, and y lies in exactly two triangles: yx_1x_2 and yx_3x_4.

We now focus on

$$Y' := V_{123} \cup V_{124} \cup V_{134} \cup V_{234}.$$

The intersection $N(x_1) \cap N(x_3)$ must contain an edge $\alpha\beta$. This edge has both endpoints in Y since the only common neighbor of x_1 and x_3 in $N[x]$ is x. Because α and β are adjacent

vertices in Y, neither can be in V_{1234}. Thus, $\alpha, \beta \in V_{123} \cup V_{134}$. If α and β are both in V_{123}, then $\{\alpha, \beta, x_1, x_2\}$ is a K_4. A similar argument shows α and β cannot both be in V_{134}. We may assume that $\alpha \in V_{123}$ and $\beta \in V_{134}$, which shows $V_{123} \neq \emptyset$ and $V_{134} \neq \emptyset$. By symmetry, V_{124} and V_{234} are not empty.

It is proved in [1] that $G[Y']$ is a complete bipartite graph with parts

$$V_{123} \cup V_{124} \text{ and } V_{134} \cup V_{234}.$$

This determines the graph G up to the number of vertices in the parts V_{123}, V_{124}, V_{134}, and V_{234} (which are all not empty), and V_{1234} (which may be empty).

Let $y_{123} \in V_{123}$, and define y_{124}, y_{134}, and y_{234} similarly. The subgraph G_1' of G induced by $N[x] \cup \{y_{123}, y_{124}, y_{134}, y_{234}\}$ has 9 vertices and 14 triangles; it is exactly one of the graphs appearing in the proof of Theorem 8 in [1]. These triangles are

$$xx_1x_2,\ xx_3x_4,\ x_1x_2y_{123},\ x_1x_2y_{124},\ x_3x_4y_{134},\ x_3x_4y_{234}\ y_{123}y_{134}x_1,\ y_{123}y_{134}x_3,$$
$$y_{134}y_{124}x_1,\ y_{134}y_{124}x_4,\ y_{124}y_{234}x_2,\ y_{124}y_{234}x_4,\ y_{123}y_{234}x_2,\ y_{123}y_{234}x_3.$$

We now estimate the number of triangles in G by adding back the remaining $n-9$ vertices in $V(G)\backslash V(G_1')$ one by one. At each step, we count the number of triangles that contain the new added vertex and vertices in $V(G_1')$. By counting triangles in this way, we never count the same triangle more than once.

If a vertex y is added to V_{1234}, then we obtain exactly two new triangles: yx_1x_2 and yx_3x_4. We can add any number of vertices to V_{1234} and the resulting graph is K_4-saturated. Now suppose a vertex y is added to V_{123} (the other cases are the same by symmetry). This creates at least five new triangles: yx_1x_2, $yy_{134}x_1$, $yy_{134}x_3$, $yy_{234}x_2$, and $yy_{234}x_3$. The conclusion is that

$$k_3(G) \geq k_3(G_1') + 2(n-9) = 14 + 2(n-9) = 2n - 4.$$

We have $k_3(G) = 2n - 4$ if and only if all of the $n-9$ vertices $V(G)\backslash V(G_1')$ are contained in V_{1234}. This is precisely the graph $H_4(n)$. □

Method and Notation For Triangle Counting: The counting method used in the last three paragraphs of the proof of Lemma 3.3 will be used multiple times in the proof of Lemmas 3.4 and 3.5. We find a small subgraph G_i' of G, count the number of triangles in G_i', and then count the number of triangles created when a vertex y is added to G_i'. The crucial point is that when a vertex y is added, we are only counting triangles that contain y and vertices in G_i'. This means that we never count the same triangle more than once. In an effort to be concise, we will always write y for the added vertex, and then list the triangles in the following way. If y is added to V_S and this creates triangles $y\alpha_1\beta_1$, $y\alpha_2\beta_2, \ldots, y\alpha_k\beta_k$, we will write

$$V_S \ :\ y\alpha_1\beta_1, y\alpha_2\beta_2, \ldots, y\alpha_k\beta_k.$$

We preface this with the statement "When y is added to V_S," and then list the triangles containing y using the notation shown above.

Lemma 3.4. *Suppose $n \geq 12$. If the number of edges in $N(x)$ is 3, then*

$$k_3(G) \geq 3n - 18.$$

Proof of Lemma 3.4. Since $N(x)$ must be triangle free, the three edges in $N(x)$ either form a star or a path of length three.

Suppose first the edges in $N(x)$ are x_1x_2, x_1x_3, and x_1x_4. Since every edge in $N(x)$ has x_1 as an endpoint and every vertex $y \in Y$ must be joined to an edge in $N(x)$, vertex x_1 is adjacent to all vertices in G. Let H be the subgraph of G induced by $N(x_1)$. Then H is a K_3-saturated $(n-1)$-vertex graph with $\delta(H) = 3$. Since $n \geq 11$, a result of Duffus and Hanson [7] implies that H has at least $3(n-1) - 15 = 3n - 18$ edges. Each of these edges forms a triangle with x and so $k_3(G) \geq 3n - 18$.

Assume now the edges in $N(x)$ are x_1x_2, x_2x_3, and x_3x_4. First we show that V_{12} and V_{34} are empty. We prove this in full, but then we will make use of the results proved in [1] concerning adjacencies between a V_S and a V_T. If $y \in V_{12}$, then y is not adjacent to x_4, and so there must be an edge $\alpha\beta$ in the intersection $N(y) \cap N(x_4)$. Since y and x_4 have no common neighbors in $N[x]$, the endpoints of $\alpha\beta$ lie in Y. If α or β is adjacent to both x_1 and x_2, then either $\{x_1, x_2, y, \alpha\}$ or $\{x_1, x_2, y, \beta\}$ is a K_4. The sets $N(\alpha) \cap N(x)$ and $N(\beta) \cap N(x)$ must contain at least one of the edges in $N(x)$, other than x_1x_2. All of the edges in $N(x)$, other than x_1x_2, have x_3 as an endpoint and so α and β are both adjacent to x_3, but then $\{\alpha, \beta, x_3, x_4\}$ is a K_4. The conclusion is that $V_{12} = \emptyset$. By symmetry, $V_{34} = \emptyset$.

As in the proof of Lemma 3.3, if $y \in V_{1234}$, then $d(y) = 4$ and $N(y) = \{x_1, x_2, x_3, x_4\}$. Let $Y' = Y \backslash V_{1234}$. Because $V_{12} = V_{34} = \emptyset$, we have

$$Y' := V_{23} \cup V_{123} \cup V_{124} \cup V_{134} \cup V_{234}. \tag{7}$$

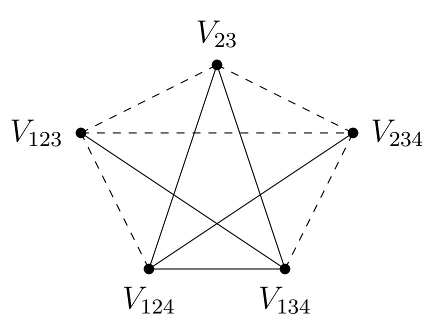
If y and z are adjacent vertices with $y, z \in V_{23} \cup V_{123} \cup V_{234}$, then $\{y, z, x_2, x_3\}$ is a K_4. Thus,

$$V_{23} \nsim V_{123} \quad \text{and} \quad V_{23} \nsim V_{234} \quad \text{and} \quad V_{123} \nsim V_{234}. \tag{8}$$

Similar arguments show that

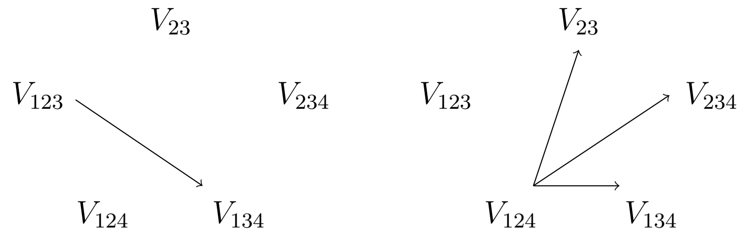
$$V_{123} \nsim V_{124}, V_{134} \nsim V_{234}, V_{23} \sim V_{124}, V_{23} \sim V_{134}, V_{123} \sim V_{134}, V_{124} \sim V_{134} \text{ and } V_{124} \sim V_{234}$$

(see Case 2 of Theorem 8 in [1]). Summarizing, $V_S \sim V_T$ if the intersection $S \cap T$ does not contain one of the pairs $\{1, 2\}$, $\{2, 3\}$, or $\{3, 4\}$. Also, $V_S \nsim V_T$ if $S \cap T$ contains at least one of the pairs $\{1, 2\}$, $\{2, 3\}$, or $\{3, 4\}$. We represent this using a graph. A solid edge between V_S and V_T indicates $V_S \sim V_T$, a dashed edge indicates $V_S \nsim V_T$, and no edge between V_S and V_T indicates that it is possible for a vertex in V_S to be adjacent or not adjacent to vertex in V_T (this last possibility does not occur here, but will occur in the proof of Lemma 3.5).



Adjacencies among the V_S and V_T

From Lemma 3.1, we have that two Rules for this graph are



Rule 1 (on the left) and Rule 2 (on the right)

By symmetry (which is highlighted by the way we have chosen to show Rules 1 and 2), we have $V_{234} \to V_{124}$ and $V_{134} \to V_{23} \cup V_{123} \cup V_{124}$. We also refer to the assertion $V_{234} \to V_{124}$ as Rule 1, and the assertion $V_{134} \to V_{23} \cup V_{123} \cup V_{124}$ as Rule 2.

We take a moment to carefully show how Lemma 3.1 gives Rule 2. Choosing $S = \{1, 2, 4\}$ and $i = 3$, we find all T for which $3 \in T$, and $|T \cap \{1, 2\}| \leq 1$. These two conditions fail for $\{1, 2, 3\}$, but hold for $\{2, 3\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$. Thus, $V_{124} \to V_{23} \cup V_{134} \cup V_{234}$.

The structure of G is now determined up to the sizes of the parts V_{1234}, V_{23}, V_{123}, V_{124}, V_{134}, and V_{234}.

If all of the vertices in Y belong to V_{1234}, then $k_3(G) \geq 3 + 3(n-5) = 3n - 12$ and we are done. Let $Y' := Y \backslash V_{1234}$ and assume $Y' \neq \emptyset$.

Case 1: $V_{23} \neq \emptyset$.

Let $y_{23} \in V_{23}$. Since $\delta(G) \geq 4$, y_{23} has at least two other neighbors aside from x_2 and x_3. Let z_1, z_2 be two other neighbors of y_{23}. These two neighbors must be in the union $V_{124} \cup V_{134}$.

First suppose $z_1 := z_{124} \in V_{124}$ and $z_2 := z_{134} \in V_{134}$. Let G_2' be the subgraph of G induced by $N[x] \cup \{y_{23}, z_{124}, z_{134}\}$. This subgraph has 8 vertices and 11 triangles (see the Appendix for the list of triangles). When y is added to V_S,

$$V_{23} \; : \; yx_2x_3, yz_{124}x_2, yz_{134}x_3, \; yz_{124}z_{134}$$
$$V_{123} \; : \; yx_1x_2, yx_2x_3, yx_1z_{134}, yx_3z_{134} \text{ } (V_{234} \text{ is the same by symmetry})$$
$$V_{124} \; : \; yx_1x_2, yx_2y_{23}, yz_{134}y_{23} \text{ } (V_{134} \text{ is the same by symmetry})$$

The conclusion is that

$$k_3(G) \geq k_3(G_2') + 3(n-8) = 11 + 3(n-8) = 3n - 13.$$

Now suppose that $z_1 := z_{124}$ and $z_2 := z_{124}'$ are in V_{124}, and that $V_{134} = \emptyset$. If $V_{123} \neq \emptyset$, then by Rule 2, $V_{123} \to V_{134}$ implying $V_{134} \neq \emptyset$, a contradiction. Therefore, $V_{123} = \emptyset$.

We consider two subcases.

Subcase 1: $V_{234} \neq \emptyset$.

Let $z_{234} \in V_{234}$ and G_3' be the subgraph of G induced by $N[x] \cup \{y_{23}, z_{124}, z_{124}', z_{234}\}$. This subgraph has 9 vertices and 14 triangles. When y is added to V_S,

$$V_{23} \; : \; yx_2x_3, yz_{124}x_2, yz_{124}'x_2 \qquad V_{124} \; : \; yx_1x_2, yy_{23}x_2, yz_{234}x_2, yz_{234}x_4$$
$$V_{234} \; : \; yx_2x_3, yx_3x_4, yz_{124}x_2, yz_{124}'x_2$$

We conclude that $k_3(G) \geq k_3(G'_3) + 3(n-9) = 14 + 3(n-9) = 3n - 13$.

Subcase 2: $V_{234} = \emptyset$.

The neighborhood x_1x_4 must contain an edge, say $\alpha\beta$. This edge cannot use the vertex x since x_1 is not adjacent to x_3, and x_4 is not adjacent to x_2. Furthermore, neither endpoint is in V_{23} because x_1 is not adjacent to any vertex in V_{23}. We conclude that the endpoints of α and β must both be in V_{124}, but then $\{\alpha, \beta, x_1, x_2\}$ is a K_4.

Case 2: $V_{23} = \emptyset$.

Here we will consider two subcases.

Subcase 1: $V_{123} \neq \emptyset$

Let $y_{123} \in V_{123}$. By Rule 1, $V_{123} \to V_{134}$ so there is a $y_{134} \in V_{134}$, and y_{123} is adjacent to y_{134}.

If $V_{234} \neq \emptyset$, say $y_{234} \in V_{234}$, then by Rule 1, $V_{234} \to V_{124}$. Let $y_{124} \in V_{124}$ be a neighbor of y_{234}. Let G'_4 be the subgraph of G induced by $N[x] \cup \{y_{123}, y_{134}, y_{234}, y_{124}\}$. This graph has 9 vertices and 15 triangles. When y is added to V_S,

$$\begin{aligned} V_{123} &: yx_1x_2, yx_2x_3, yy_{134}x_1, yy_{134}x_3 \text{ } (V_{234} \text{ is the same by symmetry}) \\ V_{134} &: yx_3x_4, yy_{123}x_1, yy_{123}x_3, yy_{124}x_4 \text{ } (V_{124} \text{ is the same by symmetry}) \end{aligned}$$

Thus,

$$k_3(G) \geq k_3(G'_4) + 4(n-9) = 15 + 4(n-9) = 4n - 21 \geq 3n - 14.$$

Now suppose $V_{234} = \emptyset$. Let G'_5 be the subgraph induced by $N[x] \cup \{y_{123}, y_{134}\}$. The graph G'_5 has 7 vertices and 8 triangles. When y is added to V_S,

$$V_{123} : yx_1x_2, yx_2x_3, yy_{134}x_1, yy_{134}x_3 \qquad V_{134} : yx_3x_4, yy_{123}x_1, yy_{123}x_3$$
$$V_{124} : yx_1x_2, yy_{134}x_1, yy_{134}x_4$$

Thus, $k_3(G) \geq k_3(G') + 3(n-7) = 8 + 3(n-7) = 3n - 13$.

Subcase 2: $V_{123} = \emptyset$.

By symmetry, we may assume $V_{234} = \emptyset$ (otherwise we are back in Subcase 1 with V_{234} replacing V_{123}). Then all of the vertices of G not in $N[x]$ must be in $V_{124} \cup V_{134} \cup V_{1234}$. Let $y_{124} \in V_{124}$. By Rule 1, $V_{124} \to V_{23} \cup V_{234} \cup V_{134}$, but $V_{23} = V_{234} = \emptyset$. Thus, there is a $y_{134} \in V_{134}$. If we take G'_6 to be the subgraph of G induced by $N[x] \cup \{y_{124}, y_{134}\}$, then G'_6 has 6 vertices and 7 triangles. When y is added to V_S,

$$V_{124} : yx_1x_2, yy_{134}x_1, yy_{134}x_4 \text{ } (V_{134} \text{ is the same by symmetry})$$

Therefore, $k_3(G) \geq 3(n-7) + 7 = 3n - 14$. □

Lemma 3.5. *If $n \geq 15$ and the number of edges in $N(x)$ is 4, then*

$$k_3(G) \geq 2n - 3.$$

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Proof of Lemma 3.5. Since G is K_4-free, $N(x)$ must be triangle-free and so the four edges in $N(x)$ form a C_4. Assume x_1x_2, x_2x_3, x_3x_4, and x_4x_1 are the four edges in $N(x)$. The set Y can be partitioned into the disjoint union

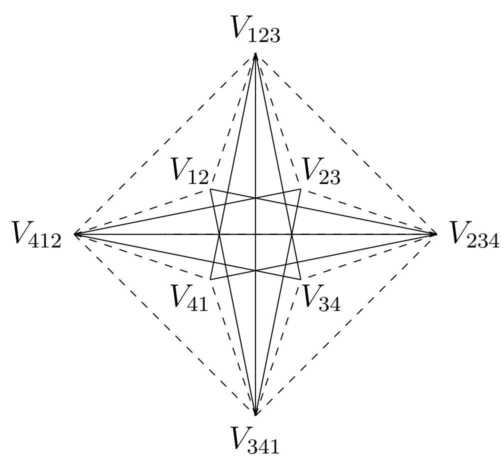
$$Y = V_{12} \cup V_{23} \cup V_{34} \cup V_{14} \cup V_{123} \cup V_{124} \cup V_{134} \cup V_{234} \cup V_{1234}.$$

The first step is to deal with vertices in V_{1234}. If $y \in V_{1234}$, then y cannot have a neighbor in Y, otherwise we obtain a K_4. Thus, $N(y) = \{x_1, x_2, x_3, x_4\}$ for all $y \in V_{1234}$. Such a vertex lies in four triangles: yx_1x_2, yx_2x_3, yx_3x_4, and yx_4x_1. We will therefore assume that $V_{1234} = \emptyset$ (if G' is the subgraph of G obtained by removing the vertices in Y_{1234} and we can prove the result for G', then the result easily follows for G).

For the rest of the proof, we focus on

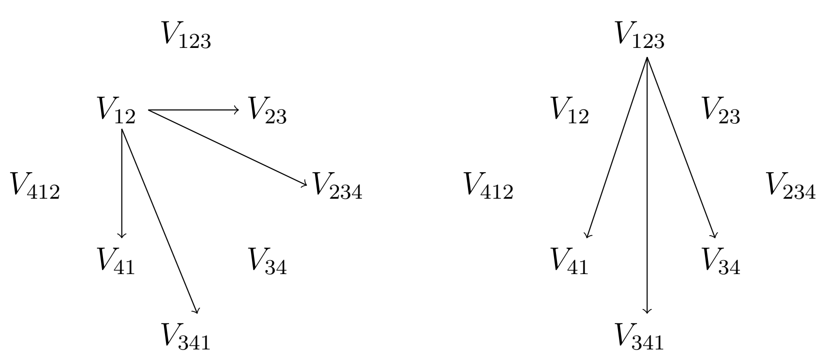
$$Y' := V_{12} \cup V_{23} \cup V_{34} \cup V_{14} \cup V_{123} \cup V_{124} \cup V_{134} \cup V_{234}.$$

As stated in [1], the following relationships hold among these 8 sets. Recall a solid/dashed edge indicates that all vertices of V_S are adjacent/not adjacent to all vertices of V_T.



The only missing edges in this figure are those with endpoints in $\{V_{12}, V_{23}, V_{34}, V_{41}\}$. At this stage, we do not have enough information to determine adjacencies between these sets.

Next, we apply Lemma 3.1 to obtain Rules 3 and 4.



Rule 3 (on the left) and Rule 4 (on the right)

Again, the pictures are drawn to highlight the symmetry, and we use Rule 3 and Rule 4 accordingly (for example, the assertion $V_{412} \to V_{23} \cup V_{234} \cup V_{34}$ is also called Rule 4). Rule 3 is obtained by applying Lemma 3.1 with $S = \{1, 2\}$ and $i = 3$, which gives $V_{12} \to V_{23} \cup V_{234}$, and then again with $S = \{1, 2\}$ with $i = 4$, which gives $V_{12} \to V_{41} \cup V_{341}$.

The proof of Lemma 3.5 from this point forward will be divided into cases.

Case 1: $V_{12} \cup V_{23} \cup V_{34} \cup V_{14} = \emptyset$.

Since G has more than 5 vertices, we can assume there is some $y_{123} \in V_{123}$. By Rule 4, $V_{123} \to V_{41} \cup V_{341} \cup V_{34}$, but $V_{41} = V_{34} = \emptyset$, so y_{123} has a neighbor $y_{134} \in V_{134}$. Hence, we know that $V_{123} \neq \emptyset$ and $V_{134} \neq \emptyset$.

Subcase 1: $V_{234} \cup V_{124} = \emptyset$.

Let G'_7 be the subgraph of G induced by $N[x] \cup \{y_{123}, y_{134}\}$. Then G'_7 has 7 vertices and 10 triangles. When y is added to V_S,

$$V_{123} \ : \ yx_1x_2, yx_2x_3, yy_{134}x_1, yy_{134}x_3 \qquad (V_{341} \text{ is the same by symmetry})$$

Thus, $k_3(G) \geq 10 + 4(n-7) = 4n - 18 \geq 2n - 2$.

Subcase 2: All of V_{123}, V_{134}, V_{234}, and V_{124} are not empty.

Before dealing with Subcase 2, we note that Subcases 1 and 2 do cover all possibilities. This is because if $V_{234} \neq \emptyset$ (or $V_{124} \neq \emptyset$), then Rule 4 and the fact that $V_{12} \cup V_{23} \cup V_{34} \cup V_{41} = \emptyset$ implies $V_{124} \neq \emptyset$ (or $V_{234} \neq \emptyset$). Therefore, it cannot be the case that exactly one of V_{123}, V_{134}, V_{234}, V_{124} is empty.

Assume $y_{234} \in V_{234}$ and $y_{412} \in V_{412}$. Let G'_8 be the subgraph of G induced by $N[x] \cup \{y_{123}, y_{234}, y_{134}, y_{412}\}$. This graph has 9 vertices and 16 triangles. When y is added to V_S,

$$V_{123} \ : \ yx_1x_2, yx_2x_3, yy_{134}x_1, yy_{134}x_3 \qquad (V_{234},\ V_{134},\ V_{412} \text{ are the same})$$

Thus, $k_3(G) \geq 16 + 4(n-9) = 4n - 20 \geq 2n - 2$.

Case 2: $V_{12} \cup V_{23} \cup V_{34} \cup V_{14} \neq \emptyset$.

For Case 2 we need an additional lemma which is proved in [1].

Lemma 3.6. *If F is the subgraph of G induced by $V_{12} \cup V_{23} \cup V_{34} \cup V_{14}$, then F is a 4-partite graph that is K_4-saturated with respect to the parts.*

In particular, Lemma 3.6 implies that if one of the four parts $V_{12}, V_{23}, V_{34}, V_{14}$ is empty, then the remaining parts induce a complete 1-partite, 2-partite, or 3-partite graph depending on the number of nonempty parts.

Subcase 1: $V_{12} \neq \emptyset$ and $V_{23} = V_{34} = V_{14} = \emptyset$.

Let $y_{12} \in V_{12}$. By Rule 3, $V_{12} \to V_{41} \cup V_{341}$ and $V_{12} \to V_{23} \cup V_{234}$, but $V_{41} = V_{23} = \emptyset$ so there must be vertices $y_{234} \in V_{234}$ and $y_{134} \in V_{134}$ that are both adjacent to y_{12}. Let G'_9 be the subgraph of G induced by $N[x] \cup \{y_{12}, y_{234}, y_{134}\}$. Then G' has 11 triangles and 8 vertices. When y is added to V_S,

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V_{12} : $yx_1x_2, yy_{234}x_2, yy_{134}x_1$ V_{123} : $yx_1x_2, yx_2x_3, yy_{134}x_1$ (V_{412} is the same)
V_{234} : $yx_2x_3, yx_3x_4, yy_{12}x_2$ (V_{341} is the same)

We conclude that

$$k_3(G) \geq k_3(G'_9) + 3(n-8) = 11 + 3(n-8) = 3n - 13 \geq 2n - 2$$

where we have used the assumption $n \geq 15$ for the last inequality.

Subcase 2: $V_{12} \neq \emptyset$ and $V_{23} \neq \emptyset$, $V_{34} = V_{14} = \emptyset$.

Let $y_{12} \in V_{12}$ and $y_{23} \in V_{23}$. By Lemma 3.6, any vertex in V_{12} will be adjacent to all vertices in V_{23}. By Rule 3, $V_{12} \to V_{41} \cup V_{341}$ so, since $V_{41} = \emptyset$, there is a vertex $y_{134} \in V_{134}$. This vertex must be adjacent to both y_{12} and y_{23}. Let G'_{10} be the subgraph of G induced by $N[x] \cup \{y_{12}, y_{23}, y_{134}\}$. Then G'_{10} has 8 vertices and 12 triangles. When y is added to V_S,

V_{12} : yx_1x_2 $yy_{23}x_2$, $yy_{134}x_1$ (V_{23} is the same),
V_{123}: yx_1x_2, yx_2x_3, $yy_{134}x_1$,$yy_{134}x_4$ V_{134} :yx_1x_4, yx_3x_4, $yy_{12}x_1$,
V_{234}: yx_2x_3, yx_3x_4, $yy_{12}x_2$ (V_{412} is the same).

Therefore, $k_3(G) \geq 12 + 3(n-8) = 3n - 12 \geq 2n - 2$.

Subcase 3: $V_{12} \neq \emptyset$ and $V_{34} \neq \emptyset$, $V_{23} = V_{14} = \emptyset$.

Let $y_{12} \in V_{12}$ and $y_{34} \in V_{34}$. By Rule 3, $V_{12} \to V_{23} \cup V_{234}$, $V_{12} \to V_{41} \cup V_{341}$, but $V_{23} = V_{14} = \emptyset$. Thus, there are $y_{134} \in V_{134}$ and $y_{234} \in V_{234}$ with both y_{134} and y_{234} adjacent to y_{12}. Also by Rule 3, $V_{34} \to V_{23} \cup V_{123}$ and $V_{34} \to \cup V_{41} \cup V_{412}$. This implies there are vertices $y_{123} \in V_{123}$ and $y_{412} \in V_{412}$ where both of these vertices are adjacent to y_{34}. Let G'_{11} be the subgraph of G induced by $N[x] \cup \{y_{12}, y_{34}, y_{134}, y_{234}, y_{123}, y_{412}\}$. Then G' has 11 vertices and has 22 triangles. When y is added to V_S,

V_{12}: yx_1x_2 $yy_{134}x_1$, $yy_{234}x_2$ (V_{34} is the same),
V_{123}: yx_1x_2, yx_2x_3, $yy_{34}x_3$, $yy_{134}x_1$, $yy_{134}x_3$ (V_{234}, V_{134}, and V_{124} are the same).

Thus, $k_3(G) \geq k_3(G'_{11}) + 3(n-11) = 3n - 11 \geq 2n - 2$.

Subcase 4: $V_{12} \neq \emptyset$, $V_{23} \neq \emptyset$, $V_{34} \neq \emptyset$, $V_{14} = \emptyset$.

Let $y_{12} \in V_{12}$, $y_{23} \in V_{23}$, and $y_{34} \in V_{34}$. Note that $V_{12} \cup V_{23} \cup V_{34}$ is a complete 3-partite graph by Lemma 3.6. By Rule 3, $y_{34} \to V_{14} \cup V_{124}$, but $V_{14} = \emptyset$. Therefore, there is a $y_{124} \in V_{124}$ and this vertex is adjacent to both y_{34} and y_{23}. Let G'_{12} be the subgraph of G induced by $N[x] \cup \{y_{12}, y_{23}, y_{34}, y_{124}\}$. Then G'_{12} has 9 vertices and has 15 triangles. When y is added to V_S,

V_{12} : yx_1x_2 $yy_{23}x_2$, $yy_{23}y_{34}$ V_{23} : yx_2x_3, $yy_{34}x_3$, $yy_{124}x_2$, $yy_{34}y_{12}$, $yy_{34}y_{124}$
V_{34} : yx_3x_4, $yy_{23}x_3$, $yy_{124}x_4$, $yy_{23}y_{12}, yy_{23}y_{124}$ V_{123} : yx_1x_2 yx_2x_3, $yy_{34}x_3$
V_{234} : yx_2x_3, yx_3x_4, $yy_{12}y_2$, V_{124} : yx_1x_2, yx_1x_4, $yy_{23}x_2$, $yy_{34}x_4$
V_{341}: yx_3x_4, yx_1x_4, $yy_{12}x_1$.

Hence, $k_3(G) \geq k_3(G'_{12}) + 3(n-9) = 3n - 12$.

Subcase 5: Each of V_{12}, V_{23}, V_{34}, V_{14} is not empty.

For this subcase, we will count triangles in G in a different way than the previous subcases. Let

$$X = \{x_1, x_2, x_3, x_4\},\ S = V_{12} \cup V_{23} \cup V_{34} \cup V_{41}, \text{ and } T = V_{123} \cup V_{234} \cup V_{341} \cup V_{412}.$$

Claim 3.7. *The number of triangles that contain at least one vertex in X is at least $2n-6$.*

Proof of Claim 3.7. There are four triangles that contain x, $2|T|$ triangles that contain two vertices in X and one in T, and $|S|$ triangles that contain two vertices in X and one in S. By Rule 3, a vertex $y_{12} \in V_{12}$ lies in a triangle of the form $y_{12}zx_2$ where z is some vertex in $V_{23} \cup V_{234}$. Similar statements hold for vertices in V_{23}, V_{34}, and V_{41}. Altogether, we have $4 + 2|T| + 2|S| = 4 + 2(n-5) = 2n-6$ triangles. □

If $|S| = 4$, then we let $V_{ij} = \{y_{ij}\}$ for $(i,j) \in \{(1,2),(2,3),(3,4),(4,1)\}$. Since G is K_4-free, Lemma 3.6 implies that S induces a $K_4 - e$. The two distinct cases, up to symmetry, are the missing edge e is $y_{12}y_{23}$ or $y_{12}y_{34}$.

Suppose first $e = y_{12}y_{34}$. For any $y_{123} \in V_{123}$, we have that $y_{123}y_{34}y_{14}$ is a triangle. Likewise, $y_{234} \in V_{234}$ implies $y_{234}y_{12}y_{14}$ is a triangle, $y_{341} \in V_{341}$ implies $y_{341}y_{23}y_{12}$ is a triangle, and $y_{412} \in V_{234}$ implies $y_{412}y_{34}y_{12}$ is a triangle. The two triangles $y_{12}y_{23}y_{34}$ and $y_{12}y_{41}y_{34}$ have no vertex in X. The number of triangles containing no vertex in X is at least

$$2 + |T| = 2 + n - 9 = n - 7.$$

Thus, by Claim 3.7, G contains at least $3n - 13 \geq 2n - 2$ triangles.

Now suppose $e = y_{12}y_{23}$. By Rule 3, y_{12} has a neighbor in $V_{23} \cup V_{234} = \{y_{23}\} \cup V_{234}$, but y_{12} is not adjacent to y_{23} so $V_{234} \neq \emptyset$. Similarly, y_{23} has a neighbor in $V_{12} \cup V_{412}$ but y_{23} is not adjacent to y_{12} so $V_{412} \neq \emptyset$. For any $y_{234} \in V_{234}$, $y_{234}y_{12}y_{41}$ is a triangle. Likewise, $y_{412} \in V_{412}$ implies $y_{412}y_{23}y_{34}$ is a triangle, and $y_{123} \in V_{123}$ implies $y_{123}y_{34}y_{41}$ is a triangle. Therefore, there are at least

$$\gamma := 2 + |V_{234}| + |V_{412}| + |V_{123}| = 2 + |T| - |V_{314}|$$

triangles in G with no vertex in X. Since V_{234} and V_{412} are not empty,

$$|V_{314}| \leq n - |X \cup \{x\}| - |S| - |V_{234}| - |V_{412}| \leq n - 5 - 4 - 1 - 1 = n - 11.$$

Hence, $\gamma \geq 2 + (n-9) - (n-11) = 4$. Combining this with Claim 3.7 gives

$$k_3(G) \geq 2n - 2.$$

The final possibility is if at least one of the parts V_{12}, V_{23}, V_{34}, V_{41} contains 2 or more vertices. Assume $|V_{12}| > 1$ and let y_{12}^1, y_{12}^2 be distinct vertices in V_{12}. Let $y_{23} \in V_{23}$, $y_{34} \in V_{34}$, and $y_{41} \in V_{41}$. The set $\{y_{12}^1, y_{23}, y_{34}, y_{41}\}$ cannot form a K_4 and so by Lemma 3.6, this set of four vertices induces a $K_4 - e$. There are two triangles using these vertices. Similarly, there are two triangles using the vertices $\{y_{12}^2, y_{23}, y_{34}, y_{41}\}$ and regardless of which pair of vertices

is not adjacent in this set, one of the triangles must contain y_{12}^2. Thus, we have 3 distinct triangles contained in S. By Claim 3.7,

$$k_3(G) \geq 2n - 3.$$

This completes the proof of Lemma 3.5. □

Combining Lemmas 3.2, 3.3, 3.4, and 3.5 implies Theorem 1.4.

4 Proof of Proposition 1.9

Let $s > 3$ and $t \geq 6\binom{s-2}{2}$ be integers. Suppose G is an n-vertex K_s-saturated graph with minimum degree t where $n \geq 2s-2$. We must show that G has at least $\binom{s-2}{2}(n-2)$ triangles.

First assume every edge of G lies in a triangle. If $t(e)$ is the number of triangles that contain the edge e, then the number of triangles in G is

$$\frac{1}{3}\sum_{e \in E(G)} t(e) \geq \frac{e(G)}{3} \geq \frac{tn}{6} \geq \binom{s-2}{2}n.$$

Now assume there is an edge xy in G that does not lie in any triangle. Let $A = N(x)$ and $B = N(y)$. Because xy is not in any triangle, $A \cap B = \emptyset$. If $a \in A$, then $N(y) \cap N(a)$ must contain a copy of K_{s-2}. The number of triangles that contain a and an edge from this K_{s-2} is $\binom{s-2}{2}$. The same argument applies to a vertex $b \in B$. Let $C = V(G)\backslash(\{x, y\} \cup A \cup B)$. If $c \in C$, then both $N(c) \cap N(x)$ and $N(c) \cap N(y)$ must contain a copy of K_{s-2}. These two copies of K_{s-2} cannot share any edges because $A \cap B = \emptyset$. Thus, G must contain at least

$$|A|\binom{s-2}{2} + |B|\binom{s-2}{2} + |C| \cdot 2\binom{s-2}{2} \geq \binom{s-2}{2}(n-2)$$

triangles. This completes the proof of Proposition 4.

References

[1] N. Alon, P. Erdős, R. Holzman, M. Krivelevich, On k-saturated graphs with restrictions on the degrees, *J. Graph Theory* 23 (1996), no. 1, 1–20.

[2] N. Alon, C. Shikhelman, Many T copies in H-free graphs, *J. Combin. Theory Ser.* B **121** (2016), 146–172.

[3] K. Amin, J. Faudree, R. Gould, The edge spectrum of K_4-saturated graphs, *J. Combin. Math. Combin. Comput.* **81** (2012), 233–242.

[4] B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** 1965 447–452.

[5] D. Chakraborti, P. Loh, Minimizing the numbers of cliques and cycles of fixed size in an F-saturated graph, arXiv:1907.01603 Sep 2019.

[6] A. N. Day, Saturated graphs of prescribed minimum degree, *Combin. Probab. Comput.* 26 (2017), no. 2, 201–207.

[7] D. Duffus, D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, *J. Graph Theory* 10 (1986), no. 1, 55–67.

[8] P. Erdős, A. Hajnal, J. W. Moon, A problem in graph theory, *Amer. Math. Monthly* **71** 1964 1107–1110.

[9] B. Ergemlidze, E. Györi, A. Methuku, N. Salia, E. A note on the maximum number of triangles in a C_5-free graph, J. Graph Theory 90 (2019), no. 3, 227–230.

[10] J. R. Faudree, R. J. Faudree, and J. R. Schmitt, A survey of minimum saturated graphs, *Electron. J. Combin.*, DS19, (2011).

[11] D. Gerbner, E. Györi, A. Methuku, M. Vizer, Generalized Turán problems for even cycles, arXiv:1712.07079 Dec 2018.

[12] D. Gerbner, A. Methuku, M. Vizer, Asymptotics for the Turán number of Berge-$K_{2,t}$, *J. Combin. Theory Ser. B*, to appear.

[13] L. Gishboliner, A. Shapira, A generalized Turán problems and its applications, Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, 760–772, ACM, New York, 2018. Results and open problems on minimum saturated hypergraphs, *Ars Combin.* 72 (2004), 111–127.

[14] R. L. Graham, M. Grötschel, L. Lovász, editors, Handbook of Combinatorics, Vol. 2 Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995.

[15] L. Kászonyi, Zs. Tuza, Saturated graphs with minimal number of edges, *J. Graph Theory* 10 (1986), no. 2, 203–210.

[16] J. Kritschgau, A. Methuku, M. Tait, C. Timmons, Few T-copies in H-saturated graphs, arXiv:1810.00939v1 Oct 2018.

[17] D. Korándi, B. Sudakov, Saturation in random graphs, *Random Structures and Algorithms* 51 (2017), no. 1, 169–181.

[18] J. Ma, Y. Qiu, Some sharp results on the generalized Turán numbers, arXiv:1802.01091 Feb 2018.

[19] C. Palmer, M. Tait, C. Timmons, A. Zsolt Wagner, Turán problems for Berge-hypergraphs and related extremal problems, *Discrete Math.* 342 (2019), no. 6, 1553–1563.

[20] O. Pikhurko, The minimum size of saturated hypergraphs, *Combin. Probab. Comput.* 8 (1999), no. 5, 483–492.

[21] O. Pikhurko, Results and open problems on minimum saturated hypergraphs, *Ars Combin.* 72 (2004), 111–127.

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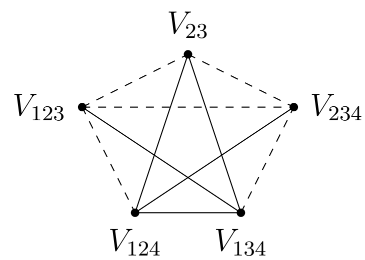
5 Appendix

For G_1', the edges in $N(x)$ are x_1x_2 and x_3x_4.

14 Triangles in G_1': xx_1x_2, xx_3x_4, $y_{123}x_1x_2$, $y_{124}x_1x_2$, $y_{134}x_3x_4$, $y_{234}x_3x_4$, $y_{123}y_{134}x_1$, $y_{123}y_{134}x_3$, $y_{123}y_{234}x_2$, $y_{123}y_{234}x_3$, $y_{124}y_{134}x_1$, $y_{124}y_{134}x_4$, $y_{124}y_{234}x_2$, $y_{124}y_{234}x_4$

For G_2' through G_6', the edges in $N(x)$ are x_1x_2, x_2x_3, and x_3x_4.



11 Triangles in G_2': xx_1x_2, xx_2x_3, xx_3x_4, $y_{23}x_2x_3$, $z_{124}x_1x_2$, $z_{134}x_3x_4$, $y_{23}z_{124}x_2$, $y_{23}z_{134}x_3$, $z_{124}z_{134}x_1$, $z_{124}z_{134}x_4$, $y_{23}z_{124}z_{134}$

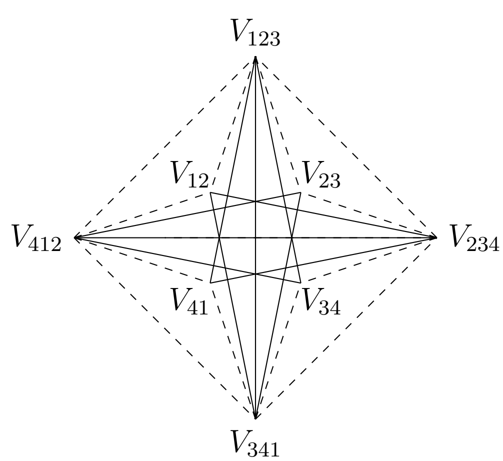
14 Triangles in G_3': xx_1x_2, xx_2x_3, xx_3x_4, $y_{23}x_2x_3$, $z_{124}x_1x_2$, $z'_{124}x_1x_2$, $z_{234}x_2x_3$, $z_{234}x_3x_4$, $y_{23}z_{124}x_2$, $y_{23}z'_{124}x_2$, $z_{124}z_{234}x_2$, $z_{124}z_{234}x_4$, $z'_{124}z_{234}x_2$, $z'_{124}z_{234}x_4$

15 Triangles in G_4': xx_1x_2, xx_2x_3, xx_3x_4, $y_{123}x_1x_2$, $y_{123}x_2x_3$, $y_{124}x_1x_2$, $y_{234}x_2x_3$, $y_{234}x_3x_4$, $y_{134}x_3x_4$, $y_{123}x_1y_{134}$, $y_{123}x_3y_{134}$, $y_{124}x_1y_{134}$, $y_{124}x_4y_{134}$, $y_{124}x_2y_{234}$, $y_{124}x_4y_{234}$

8 Triangles in G_5': xx_1x_2, xx_2x_3, xx_3x_4, $y_{123}x_1x_2$, $y_{123}x_2x_3$, $y_{134}x_3x_4$, $y_{123}y_{134}x_1$, $y_{123}y_{134}x_3$

7 Triangles in G_6': xx_1x_2, xx_2x_3, xx_3x_4, $y_{124}x_1x_2$, $y_{134}x_3x_4$, $y_{124}y_{134}x_1$, $y_{124}y_{134}x_4$

For G_7' through G_{12}', the edges in $N(x)$ are x_1x_2, x_2x_3, x_3x_4, and x_4x_1.



10 Triangles in G_7': xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{123}x_1x_2$, $y_{123}x_2x_3$, $y_{134}x_3x_4$, $y_{134}x_4x_1$, $y_{123}y_{134}x_1$, $y_{123}y_{134}x_3$

16 Triangles in G'_8: xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{123}x_1x_2$, $y_{123}x_2x_3$, $y_{412}x_1x_2$, $y_{412}x_4x_1$, $y_{234}x_2x_3$, $y_{234}x_3x_4$, $y_{134}x_3x_4$, $y_{134}x_4x_1$, $y_{123}y_{134}x_1$, $y_{123}y_{134}x_3$, $y_{412}y_{234}x_2$, $y_{412}y_{234}x_4$

11 Triangles in G'_9: xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{12}x_1x_2$, $y_{134}x_3x_4$, $y_{134}x_4x_1$, $y_{234}x_2x_3$, $y_{234}x_3x_4$, $y_{12}y_{134}x_1$, $y_{12}y_{234}x_2$

12 Triangles in G'_{10}: xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{12}x_1x_2$, $y_{23}x_2x_3$, $y_{134}x_3x_4$, $y_{134}x_4x_1$, $y_{12}y_{134}x_1$, $y_{23}y_{134}x_1$, $y_{12}y_{23}x_2$, $y_{12}y_{23}y_{134}$

22 Triangles in G'_{11}: xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{12}x_1x_2$, $y_{34}x_3x_4$, $y_{123}x_1x_2$, $y_{123}x_2x_3$, $y_{412}x_4x_1$, $y_{412}x_1x_2$, $y_{234}x_2x_3$, $y_{234}x_3x_4$, $y_{341}x_3x_4$, $y_{341}x_4x_1$, $y_{12}y_{134}x_1$, $y_{12}y_{234}x_2$, $y_{34}y_{124}x_4$, $y_{34}y_{123}x_3$, $y_{134}y_{123}x_1$, $y_{134}y_{123}x_3$, $y_{234}y_{412}x_2$, $y_{234}y_{412}x_4$

15 Triangles in G'_{12}: xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_1, $y_{12}x_1x_2$, $y_{23}x_2x_3$, $y_{34}x_3x_4$, $y_{124}x_1x_2$, $y_{124}x_4x_1$, $y_{12}y_{23}x_2$, $y_{23}y_{34}x_3$, $y_{124}y_{23}x_2$, $y_{124}y_{34}x_4$, $y_{12}y_{23}y_{34}$, $y_{124}y_{23}y_{34}$

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