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## Fractional strong matching preclusion for two variants of hypercubes

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# Fractional strong matching preclusion for two variants of hypercubes

## **Cover Page Footnote**

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## Abstract

Let  $F$  be a subset of edges and vertices of a graph  $G$ . If  $G - F$  has no fractional perfect matching, then  $F$  is a fractional strong matching preclusion set of  $G$ . The *fractional strong matching preclusion number* is the cardinality of a minimum fractional strong matching preclusion set. In this paper, we mainly study the fractional strong matching preclusion problem for two variants of hypercubes, the multiply twisted cube and the locally twisted cube, which are two of the most popular interconnection networks. In addition, we classify all the optimal fractional strong matching preclusion set of each.

**Keywords:** Fractional perfect matching; Fractional matching preclusion; Fractional strong matching preclusion; Multiply twisted cube; Locally twisted cube

**AMS subject classification 2010:** 05C05, 05C12, 05C76.

## 1 Introduction

All graphs considered in this paper are connected and even order. For a graph  $G$ ,  $V(G)$  and  $E(G)$  are its vertices set and edges set, respectively. A *matching* for a graph is a set of edges such that each vertex is incident with at most one edge in this set. A *perfect matching* for a graph is a set of edges such that each vertex is incident with exactly one edge in this set. An *almost-perfect matching* in a graph is a set of edges such that each vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident with none. Thus, if a graph has a perfect matching, then the graph has even order, if a graph has an almost perfect matching, then the graph has odd order.

Recently there has been an increasing interest in a class of interconnection networks, proposed to serve as the topology of a large-scale parallel and distributed system. Matching preclusion is a measure of robustness when there is a link failure. The matching preclusion problem is a topic researched widely in graph theory. Fractional matching preclusion is a nice generalization for matching preclusion.

### 1.1 Matching preclusion and its generalizations

A set  $F$  of edges is called a *matching preclusion set* (MP set for short) if  $G - F$  has neither a perfect matching nor an almost-perfect matching. The *matching preclusion number* of graph  $G$ , denoted by  $mp(G)$ , is the minimum size of MP sets of  $G$ , that is  $mp(G) = \min\{|F| : F \text{ is a MP set}\}$ . The concept of matching preclusion was introduced by Birgham et al.[2] and investigated in some special graphs. For more information, we refer to [3, 4, 5, 8]. According to the definition of  $mp(G)$ , for a graph  $G$  with the even number of vertices, we have

$$mp(G) \leq \delta(G),$$

where  $\delta(G)$  is the minimum degree of  $G$ . A matching preclusion set is trivial if all the edges in it are incident with a single vertex.

A set  $F$  of edges and vertices is called a *strong matching preclusion set* (SMP set for short) if  $G - F$  has neither a perfect matching nor an almost-perfect matching. The concept of strong

matching preclusion was introduced in [14], for more details see [11]. The *strong matching preclusion number* of a graph  $G$ , denoted by  $smp(G)$ , is given by  $smp(G) = \min\{|F| : F \text{ is a SMP set}\}$ . Then we have

$$smp(G) \leq mp(G).$$

A SMP set  $F$  is *optimal* if  $|F| = smp(G)$ .

Another standard way to see matching in polyhedral combinatorics is as follows. A *matching* is a function  $f$  that assigns to each edge of  $G$  a number in  $\{0, 1\}$  so that  $\sum_{e \sim v} f(e) \leq 1$  for each vertex  $v$  of  $G$ , where the sum taken over all edges  $e$  incident with  $v$ . A matching is *perfect* if  $\sum_{e \sim v} f(e) = 1$  for each vertex  $v$ , that is,

$$\sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \sim v} f(e) = \frac{|V(G)|}{2}.$$

A matching is *almost-perfect* if there exists exactly one vertex  $v'$  such that  $\sum_{e \sim v'} f(e) = 0$  and  $\sum_{e \sim v} f(e) = 1$  for each other vertex  $v$ , that is,

$$\sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \sim v} f(e) = \frac{|V(G)| - 1}{2}.$$

In this paper, we consider a kind of generalization of matching. A *fractional matching* is a function  $f$  that assigns to each edge a number in  $[0, 1]$  so that  $\sum_{e \sim v} f(e) \leq 1$  for each vertex  $v$  of  $G$ , where the sum is taken all edges  $e$  incident with  $v$ . Clearly,

$$\sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \sim v} f(e) \leq \frac{|V(G)|}{2}.$$

As a generalization of matching preclusion, Liu and Liu [10] recently introduced the concept of fractional matching preclusion number. A *fractional perfect matching* is a fractional matching  $f$  satisfying that  $\sum_{e \sim v} f(e) = 1$  for each  $v \in V(G)$ . A fractional matching  $f$  is perfect if and only if  $\sum f(e) = \frac{|V(G)|}{2}$  and a perfect matching is a fractional perfect matching. Many further ideas and results on fractional graph theory can be found in [10, 17].

An edge subset  $F$  of  $G$  is a *fractional matching preclusion set* (FMP set for short) if  $G - F$  has no fractional perfect matching. The *fractional matching preclusion number* of a graph  $G$ , denoted by  $fmp(G)$ ,  $fmp(G) = \min\{|F| : F \text{ is a FMP set}\}$ . Then we have

$$fmp(G) \leq \delta(G).$$

If the number of vertices in  $G$  is even, we have

$$mp(G) \leq fmp(G).$$

But, if the number of vertices in  $G$  is odd,  $mp(G)$  and  $fmp(G)$  do not satisfy this inequality. Some examples are given in [10].

A set  $F$  of edges and vertices is a *fractional strong matching preclusion set* (FSMP set for short) if  $G - F$  has no fractional perfect matchings. The *fractional strong matching*

preclusion number of a graph  $G$ , denoted by  $fsm_p(G)$ , is given by  $fsm_p(G) = \min\{|F| : F \text{ is a FSMP set}\}$ . Then we have

$$fsm_p(G) \leq fmp(G) \leq \delta(G).$$

A FSMP set  $F$  is optimal if  $|F| = fsm_p(G)$ . A FSMP(FMP) set  $F$  is *trivial* if  $G - F$  has a single vertex.

## 1.2 Two variants of hypercubes

Hypercubes are one of the most basic class of interconnection networks. Based on the graph construction, Vaidya et al. [18] gave a recursive definition of a class of graph, called the *hypercube-like graph* (*HL-graph for short*) and this class of networks are important generalizations of hypercubes and defined recursively as follows.

$$HL_0 = \{K_1\}, HL_n = \{G^0 \oplus_M G^1 | G^0, G^1 \in HL_{n-1}\},$$

where the symbol " $\oplus_M$ " represents the perfect matching operation that connects  $G^0$  and  $G^1$  using some perfect matching, denoted by  $PM(G)$ . For a graph  $G \in HL_n$ ,  $V(G) = V(G^0 \oplus_M G^1) = V(G^0) \cup V(G^1)$  and  $E(G) = E(G^0 \oplus_M G^1) = E(G^0) \cup E(G^1) \cup PM(G)$ , where  $G^0, G^1 \in HL_{n-1}$ . It is clear that for  $G \in HL_n$ ,  $G$  is an  $n$ -regular connected graph of order  $2^n$ , more results are shown in [13, 15, 16].

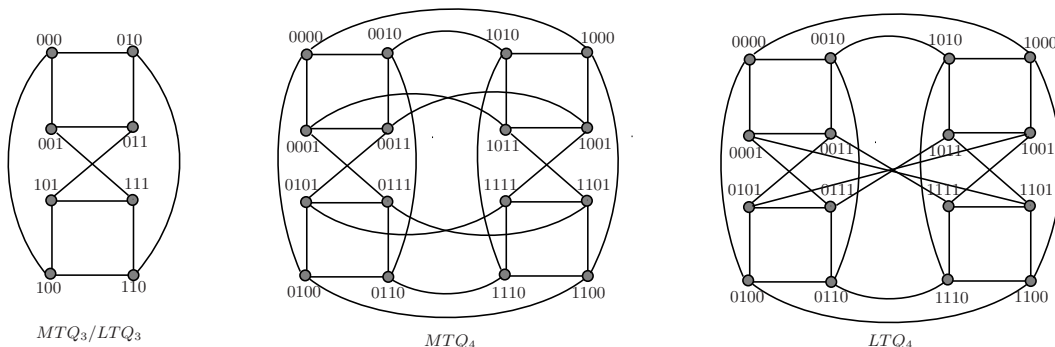


Fig.1 The multiply twisted cube and locally twisted cube when  $n = 3, 4$ .

We first introduce the definition of the multiply twisted cube, which was proposed by Efe in [7]. An  $n$ -dimensional multiply twisted cube denoted by  $MTQ_n = MTQ^0 \oplus_m MTQ^1$  with  $2^n$  vertices, each vertex labeled by an  $n$ -bit binary string  $u_{n-1}u_{n-2} \cdots u_0$  such that  $u_i \in \{0, 1\}$  ( $i = 0, 1, \dots, n - 1$ ).  $MTQ_1$  and  $MTQ_2$  are isomorphic to  $K_2$  and  $Q_2$ , respectively. For  $n \geq 3$ ,  $MTQ_n$  is defined recursively by using two copies of  $(n - 1)$ -dimensional multiply twisted cube with edges between them, the first copy denote by  $MTQ_{n-1}^0$  with vertices  $u = 0u_{n-2}u_{n-3} \cdots u_0$ , another copy is  $MTQ_{n-1}^1$  with vertices  $v = 1v_{n-2}v_{n-3} \cdots v_0$ .  $MTQ_n$  is a  $n$ -regular bipartite graph and  $diam(G) = \lceil \frac{(n+1)}{2} \rceil$ , an edge exists between vertex  $u = u_{n-1}u_{n-2} \cdots u_0$  and  $v = v_{n-1}v_{n-2} \cdots v_0$  provided there is an index  $k$  such that

- (1)  $u_{n-1}u_{n-2} \cdots u_k = v_{n-1}v_{n-2} \cdots v_k$ ,
- (2)  $u_{k-1} \neq v_{k-1}$ ,
- (3)  $u_{k-2} = v_{k-2}$  if  $k$  is even,
- (4) for  $0 \leq i < \lfloor \frac{(k-1)}{2} \rfloor$ ,  $v_{2i+1}v_{2i}$  is pair-related to  $u_{2i+1}u_{2i}$ .

The definition of the locally twisted cube was introduced in [20, 21]. An  $n$ -dimensional locally twisted cube denoted by  $LTQ_n$  with  $2^n$  vertices, and each vertex is an  $n$ -string on  $\{0, 1\}$ .  $LTQ_1$  and  $LTQ_2$  are isomorphic to  $K_2$  and  $Q_2$ , respectively. For  $n \geq 3$ ,  $LTQ_n$  is built from two disjoint copies of  $LTQ_{n-1}$  according to the following steps. Let  $LTQ_{n-1}^0$  and  $LTQ_{n-1}^1$  denote obtained by prefixing labels each vertex of one copy of  $LTQ_{n-1}$  with 0 and 1, respectively. Two vertices  $u = u_1u_2 \dots u_{n-1}u_n$  and  $v = v_1v_2 \dots v_{n-1}v_n$  of  $LTQ_n$  are adjacent if and only if either there is an integer  $1 \leq k \leq n - 2$  such that

- (1)  $u_k \neq v_k$ ,
- (2)  $u_{k+1} = v_{k+1} + v_n$ ,
- (3)  $u_i = v_i$  for all the remaining bits.

where “+” represents the modulo 2 addition, or there is an integer  $k \in \{n - 1, n\}$  such that  $u$  and  $v$  differ only in the  $k$ -th bit.

We illustrate the multiply twisted cube and locally twisted cube in Fig.1 when  $n = 3, 4$ , respectively. More details refer to [1, 9].

In [18, 19] show that the multiply twisted cube and locally twisted cube are members of hypercube-like graphs, which give better performance than hypercube with the same number of edges and vertices.

### 1.3 Related results

In this section, we briefly collect some related results which are vital for our proof. We now present some necessary and sufficient conditions for existence of the fractional perfect matching in a graph, which introduced by Scheinerman and Ullman in [17].

**Lemma 1.1.** [17] *A graph  $G$  has a fractional perfect matching if and only if  $i(G - S) \leq |S|$  for each set  $S \subseteq V(G)$ , where  $i(G - S)$  is the number of isolated vertices of  $G - S$ .*

**Lemma 1.2.** [17] *The graph  $G$  has a fractional perfect matching if and only if there is a partition  $\{V_1, V_2, \dots, V_n\}$  of the vertex set of  $V(G)$  such that, for each  $i$ , the subgraph of  $G$  induced by  $V_i$  is either  $K_2$  or a Hamiltonian graph on odd number of vertices.*

According to the above results, the following observations are immediate.

**Observation 1.3.** (1) *If a graph  $G$  has a perfect matching, then it must have a fractional perfect matching by assigning 1 to each edge of perfect matching and 0 to other edges of  $G$ .*

(2) *If a graph  $G$  is Hamiltonian, we can get a fractional perfect matching by assigning  $\frac{1}{2}$  to each edge of Hamiltonian cycle and 0 to other edges.*

**Lemma 1.4.** [12] *Let  $G$  be a  $n$ -dimensional multiply twisted cube or locally twisted cube for  $n \geq 3$ . Then  $mp(G) = n$ . Moreover, each optimal matching preclusion set is trivial.*

**Lemma 1.5.** [14] *If  $n \geq 3$ , then  $smp(HL_n) = n$ . Furthermore,*

(a) *for  $n \geq 5$ , each of its minimum SMP sets is trivial.*

(b) *for  $n = 4$ , each of its minimum SMP sets is either trivial or a set consisting of a boundary edge  $(v_1, v_{i+1})$  of  $HL^0$ , another boundary edge  $(w_j, w_{j+1})$  of  $HL^1$ , and two white vertices in  $W_i \cup W'_j$  such that  $\bar{W}_i = B'_j$  and  $\bar{B}_i = W'_j$ .*

**Lemma 1.6.** [9, 15] *Let  $G$  be a  $n$ -dimensional multiply twisted cube or locally twisted cube with  $n \geq 3$ , and let  $F$  be a set of fault vertices and/or edges of  $G$ . Then  $G - F$  is Hamiltonian if  $|F| \leq n - 2$ , and Hamiltonian-connected if  $|F| \leq n - 3$ .*

We need to a result in [6] regarding almost perfect matching.

**Lemma 1.7.** [6] *Suppose  $G$  has an almost perfect matching  $M$  missing  $v$ . If  $v$  is not an isolated vertex in  $G$ , then  $G$  has an almost perfect matching missing a vertex other than  $v$ .*

## 2 Initial case

Since  $HL_n$  has usually an even number of vertices, and we know from lemma 1.4 that  $mp(HL_n) = \delta(HL_n)$  for  $n \geq 3$ , it is easy to determine the fractional matching preclusion number of the multiply twisted cube and locally twisted cube.

Now we study the fractional strong matching preclusion problem of multiply twisted cube and locally twisted cube. Without loss of generality, we can assume that any edge in a FSMP set  $F$  is not incident with vertices in  $F$  in the following.

**Theorem 2.1.** *Let  $n \geq 3$ . Then  $fmp(MTQ_n) = fmp(LTQ_n) = n$ .*

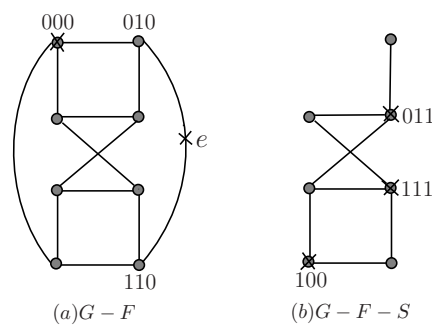


Fig.2 Illustration about the proof of Lemma 2.2.

**Lemma 2.2.** *Let  $G$  be a 3-dimensional multiply twisted cube or locally twisted cube, then  $fsmpp(G) = 2$ . Moreover, the optimal fractional strong matching preclusion set must contain exactly one vertex and one edge.*

*Proof.* It is clear that the 3-dimensional multiply twisted cube and locally twisted cube are isomorphic. By Lemma 1.6, we have  $fsm_p(G) \geq 2$ . Let  $e = (010, 110)$  and  $F = \{000, e\}$ , see Fig.2(a). Choose  $S = \{011, 111, 100\}$ . Then  $i((G - F) - S) = 4 > |S|$ , see Fig.2(b). By Lemma 1.1,  $G - F$  has no fractional perfect matching, and hence  $F$  is a FSMP set of  $G$ . Thus,  $fsm_p(G) = 2$ . Since  $sm_p(G) = 3$  by Lemma 1.5, it follows that any optional FSMP set must contain one vertex and one edge.  $\square$

For simplicity, we briefly introduce some notations throughout this paper. If  $G$  is a  $n$ -dimensional multiply twisted cube or locally twisted cube, we denote by  $G^0$  and  $G^1$  two copies of  $G$ . Given  $F \subseteq V(G) \cup E(G)$ , let  $F^i = F \cap (V(G^i) \cup E(G^i))$  for  $i = 0, 1$ .

**Lemma 2.3.** *Let  $G$  be a 4-dimensional multiply twisted cube or locally twisted cube. Then  $fsm_p(G) = 3$ . Moreover, the optimal fractional strong matching preclusion set of  $G$  must contain exactly one vertex and two edges.*

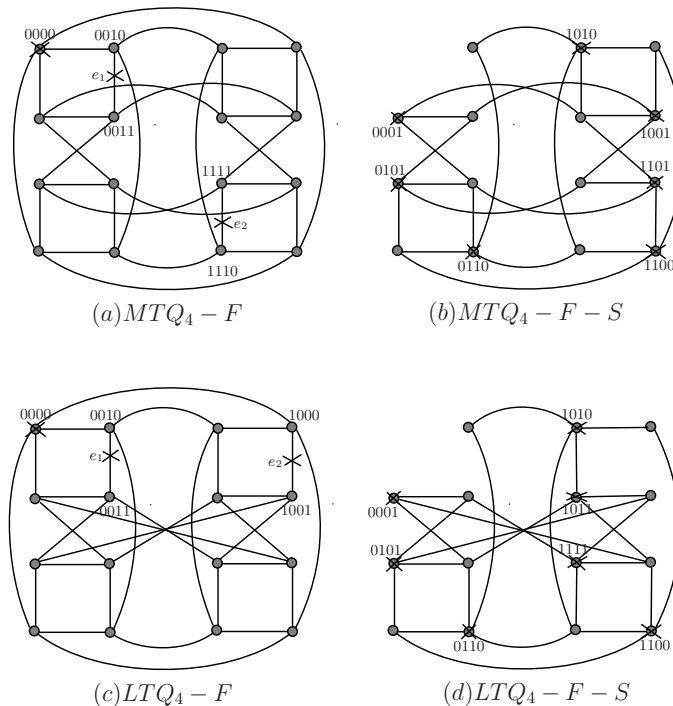


Fig.3. Illustration about the proof of Lemma 2.3.

*Proof.* By Lemma 1.6, we have  $fsm_p(G) \geq 3$ . If  $G$  is a 4-dimensional multiply twisted cube, let  $e_1 = (0010, 0011)$ ,  $e_2 = (1111, 1110)$  and  $F = \{0000, e_1, e_2\}$ , see Fig.3(a). Choose  $S = \{0001, 0101, 0110, 1010, 1001, 1101, 1100\}$ . Clearly,  $i((G - F) - S) = 8 > |S|$ , see Fig.3(b).

If  $G$  is a 4-dimensional locally twisted cube, let  $e_1 = (0010, 0011)$ ,  $e_2 = (1000, 1001)$  and  $F = \{0000, e_1, e_2\}$ , see Fig.3(c). Choose  $S = \{0001, 0101, 0110, 1010, 1011, 1111, 1100\}$ . Clearly,  $i((G - F) - S) = 8 > |S|$ , see Fig.3(d). From Lemma 1.1, we know that  $G - F$  has no fractional perfect matching, which means that  $F$  is a FSMP set of  $G$ . Hence we have  $fsm_p(G) = 3$ .



Since  $smp(G) = 4$ , it follows that  $G - F$  has perfect matching if  $F$  consists of three edges or consists of two vertices and one edge. Thus we only consider the case that  $F$  consists of three vertices. By deleting any three vertices of  $G$ , there are at least two vertices belong to one copy, say  $G^0$ . If there are two vertices in  $G^0$  and one vertex in  $G^1$ , then we can find a perfect matching with six vertices in  $G^0$  and a Hamiltonian cycle with seven vertices in  $G^1$ . If there are three vertices in  $G^0$ , then we have an almost-perfect matching with five vertices in  $G^0$ , say vertex  $v$  is incident with none edge, then there exists a vertex  $v' \in V(G^1)$  such that edge  $vv' \in E(G)$ , then we have a perfect matching with six vertices,  $G^1 - \{v'\}$  has a Hamiltonian cycle with seven vertices, then we can find a fractional perfect matching of  $G$ . So any optional FSMP set must contain one vertex and two edges.  $\square$

**Lemma 2.4.** *Let  $G$  be a 5-dimensional multiply twisted cube or locally twisted cube. Then  $fsmf(G) = 5$ .*

*Proof.* Since  $fsmf(G) \leq \delta(G) = 5$ , we need to prove  $fsmf(G) \geq 5$ . Let  $F \subseteq V(G) \cup E(G)$  for  $|F| = 4$ ,  $F_V \subseteq V(G)$ . It suffices to prove that  $G - F$  has a fractional perfect matching. If  $|F_V|$  is even, then  $G - F$  has a perfect matching by Lemma 1.5, so we only consider the case that  $|F_V|$  is odd. Thus, there is at least one vertex in  $F$ . Without loss of generality, suppose that this vertex belongs to  $F^0$  and  $|F^0| \geq |F^1|$ .

**Case 1.**  $|F^0| = 4$ .

Since  $F$  contains at least one vertex, say  $v$ , we let  $F^{0'} = F^0 - \{v\}$ . By Lemma 1.5,  $G^0 - F^{0'}$  has a perfect matching  $M$ . Let  $(u, v) \in M$ , we assume that  $u'$  is a neighbor of  $u$  in  $G^1$ . By Lemma 2.3,  $G^1 - \{u'\}$  has a fractional perfect matching  $f_1$ . Thus  $M - (u, v) \cup (u, u') \cup f_1$  induces a fractional perfect matching of  $G - F$ .

**Case 2.**  $|F^0| = 3$ .

If  $F^0$  contains two vertices and one edge or  $F^0$  contains three vertices, from Lemma 2.3,  $G^0 - F^0$  and  $G^1 - F^1$  have a fractional perfect matching  $f_0$  and  $f_1$ , respectively. Then  $f_0 \cup f_1$  induces a fractional perfect matching of  $G - F$ . So, we discuss that  $F^0$  contains one vertex and two edges.

Clearly,  $G^0 - F^0$  has an almost-perfect matching with missing vertex  $u$ . Since each vertex of  $G^0$  is incident with exactly one vertex of  $G^1$ , it follows that there exist a vertex  $u' \in G^1$  such that edge  $(u, u') \in G$ , and hence  $G^0 - F^0 - \{u\}$  has a perfect matching  $M$ . On the other hand, combining  $|F^1| \leq 1$  and Lemma 2.3,  $G^1 - F^1 - u'$  has a fractional perfect matching  $f_1$ . Clearly,  $M \cup (u, u') \cup f_1$  induces a fractional perfect matching of  $G - F$ .

**Case 3.**  $|F^0| = 2$ .

By Lemma 2.3,  $G^0 - F^0$  and  $G^1 - F^1$  have a fractional perfect matching  $f_0$  and  $f_1$ , respectively. Thus  $f_0 \cup f_1$  induces a fractional perfect matching of  $G - F$ .  $\square$

We use the similar notation as above in following results.

**Lemma 2.5.** *Let  $G$  be a 5-dimensional multiply twisted cube or locally twisted cube. Then each optimal FSMP sets of  $G$  is trivial.*

*Proof.* Let  $F \subseteq V(G) \cup E(G)$  with  $|F| = 5$ , where  $F = F^0 \cup F^1$ . To the contrary, we assume  $G - F$  has a fractional perfect matching. If  $|F_V|$  is even, then  $G - F$  has a perfect matching by Lemma 1.5. So we only consider the case that  $|F_V|$  is odd. For notational convenience, we assume that  $|F^0| \geq |F^1|$ .

**Case 1.**  $|F^0| = 5$ .

*Subcase 1.1.*  $F^0$  contains at least three vertices  $u, v, w$ .

Let  $F^{0'} = F^0 - \{u, v, w\}$ . From Lemma 1.5,  $G^0 - F^{0'}$  has a perfect matching  $M$ . Let  $\{(u, x), (v, y), (w, z)\} \subset M$ , note that  $|F - F^0| = 0$ , so we can assume that  $x', y', z'$  are neighbours of  $x, y$  and  $z$ , respectively, where  $x', y', z' \in G^1$ . From Lemma 2.3,  $G^1 - \{x', y', z'\}$  has a fractional perfect matching  $f_1$ . Clearly,  $(M - \{u, v, w\}) \cup \{(x, x'), (y, y'), (z, z')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ .

*Subcase 1.2.*  $F^0$  contains one vertex  $u$ .

Let  $F^{0'} = F^0 - \{u\}$ . From Lemma 1.5, either  $G^0 - F^{0'}$  has a perfect matching  $M$ , or  $F^{0'}$  is a SMP set of  $G^0$ . If  $G^0 - F^{0'}$  has a perfect matching  $M$ , then we can assume  $(u, x) \in M$ . Note that  $|F - F^0| = 0$ , we can assume that  $x' \in G^1$  is neighbour of  $x$ . From Lemma 2.3,  $G^1 - \{x'\}$  has a fractional perfect matching  $f_1$ . Clearly,  $(M - \{u\}) \cup \{(x, x')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ . If  $F^{0'}$  is a SMP set of  $G^0$ , then we assume that it is induced by  $u$ . Similarly, we can assume that  $x', y'$  are neighbours of  $x$  and  $y$ , respectively, where  $x', y' \in G^1$ .  $G^1 - \{x', y'\}$  has a fractional perfect matching  $f_1$ . Thus we can easily obtain a desired fractional perfect matching of  $G - F$ .

**Case 2.**  $|F^0| = 4$ .

By Lemma 1.5, there are at most two vertices  $u$  and  $v$  such that they are missed by a perfect matching  $M$ . Since  $F$  is not a FSMP set of  $G$ , we assume that  $u', v'$  are neighbours of  $u$  and  $v$ , respectively, where  $u', v' \in G^1$ . From Lemma 2.3,  $G^1 - F^1 - \{u', v'\}$  has a fractional perfect matching  $f_1$ . Hence  $(M - \{u, v\}) \cup \{(u, u'), (v, v')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ .

**Case 3.**  $|F^0| = 3$ .

In fact, we only consider that  $F^0$  contains one vertex. Otherwise, by Lemma 2.3,  $G^0 - F^0$  and  $G^1 - F^1$  have fractional perfect matchings  $f_0$  and  $f_1$ , respectively. Thus we can easily obtain a desired fractional perfect matching of  $G - F$ . From Lemma 1.7,  $G^0 - F^0$  has an almost perfect matching  $M$ . By Lemma 1.7 and Lemma 2.3, there exists a vertex  $v$  missed by  $M$  such that  $G^1 - F^1 - \{v'\}$  has a fractional perfect matching  $f_1$ , where  $v' \in G^1$  is a neighbour of  $v$ . Clearly,  $(M - \{v\}) \cup \{(v, v')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ .  $\square$

### 3 Main results

Now is the time to present the main results of this paper.

**Theorem 3.1.** *Let  $G$  be a  $n$ -dimensional multiply twisted cube or locally twisted cube.*

$$f_{smp}(G) = \begin{cases} n - 1, & \text{if } 3 \leq n \leq 4; \\ n, & \text{if } n \geq 5. \end{cases}$$

*Moreover, for  $n \geq 5$ , each optimal FSMP set of  $G$  is trivial.*

*Proof.* By Lemmas 2.2 and 2.3, the results holds for  $n = 3, 4$ . Now we only to prove  $f_{smp}(G) = n$  for  $n \geq 5$  by induction on  $n$ . The statement is true for  $n = 5$  by Lemma 2.4. For  $n \geq 6$ , we assume that the result holds for  $n - 1$ . Now we show that  $f_{smp}(G) = n$ . Let  $F \subseteq V(G) \cup E(G)$ ,  $|F| = n - 1$  and  $F = F^0 \cup F^1$ . It suffices to prove that  $G - F$  has a

fractional perfect matching. If  $|F_V|$  is even, then  $G - F$  has a perfect matching by Lemma 1.5. So we only consider the case that  $|F_V|$  is odd. We suppose that  $|F^0| \geq |F^1|$ .

**Case a.**  $|F^0| = n - 1$ .

Since  $F$  contains at least one vertex, we can assume that  $F^{0'} = F^0 - \{v\}$  and  $v \in F$ . By Lemma 1.5,  $G^0 - F^{0'}$  has a perfect matching  $M$ . Let  $(u, v) \in M$ , we assume that  $u'$  is a neighbor of  $u$  in  $G^1$ . From the induction hypothesis,  $G^0 - \{u'\}$  has a fractional perfect matching  $f$ . Thus  $(M - (u, v)) \cup (u, u')$  and  $f$  induce a fractional perfect matching of  $G - F$ .

**Case b.**  $|F^0| = n - 2$ .

By the induction hypothesis, we can know that the  $G^0 - F^0$  and  $G^1 - F^1$  have fractional perfect matching  $f_0$  and  $f_1$ , respectively. Thus,  $f_0$  and  $f_1$  induce a fractional perfect matching of  $G - F$  and  $f_{smp}(MTQ_n) \geq n$ .

Since  $f_{smp}(G) \leq \delta(G)$ , there are  $f_{smp}(G) \leq n$ , and hence  $f_{smp}(G) = n$ .

Next, we classify the optimal solutions, for  $F \subseteq V(G) \cup E(G)$  with  $|F| = n$ . We only show that one of the two cases holds: (i)  $F$  is a trivial FSMP set, (ii)  $G - F$  has a fractional perfect matching. If  $F$  contains even number of vertices, then  $G - F$  has either perfect matching or isolated vertex by Lemma 1.5. Thus we only consider the case that  $F$  contains odd number of vertices. For notational convenience, assume that  $|F^0| \geq |F^1|$ . We distinguish the following the following three cases.

**Case 1.**  $|F^0| = n$ .

Clearly,  $F$  contains odd number of vertices. Let  $u \in F$ . We consider the following two subcases.

*Subcase 1.1.*  $F^0$  contains at least an edge  $(w, s)$ .

Let  $F^{0'} = F^0 - \{u, (w, s)\}$ , then  $|F^{0'}| = n - 2$ . Since  $G^0 - F^{0'}$  has an even number of vertices and  $|F^{0'}| = n - 2$ , it follows from Lemma 1.5 that  $G^0 - F^{0'}$  has a perfect matching  $M$ . Let  $(u, v) \in M$ , since  $|F^1| = 0$ , it follows that we may assume that  $v', w', s'$  are neighbours of  $v, w$  and  $s$ , respectively, where  $v', w', s' \in G^0$ . We consider the case that  $(w, s) \in M$ , otherwise, it is easy. Since  $n \geq 6$  and  $|\{v', w', s'\}| = 3$ , it follows from the induction hypothesis that  $G^1 - \{v', w', s'\}$  has a fractional perfect matching  $f_1$ . Clearly,  $(M - \{u, (w, s)\}) \cup \{(v, v'), (w, w'), (s, s')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ .

*Subcase 1.2.*  $F^0$  contains all vertices.

Since  $n - 1 \geq 6 - 1 > 3$ , we can pick two additional vertices  $v$  and  $w$  in  $F_0$ . Let  $F^{0'} = F^0 - \{u, v, w\}$ . Since  $G^0 - F^{0'}$  has an even number of vertices and  $|F^{0'}| = n - 3$ , it follows from Lemma 1.5 that there exists a perfect matching  $M$ . Let  $\{(u, x), (v, y), (w, z)\} \subseteq M$ . For  $|F - F^0| = 0$ , we may assume that  $x', y', z'$  are neighbours of  $x, y$  and  $z$ , respectively, where  $x', y', z' \in G^1$ . Since  $n \geq 6$  and  $|\{x', y', z'\}| = 3$ , it follows from the induction hypothesis that  $G^1 - F^1$  has a fractional perfect matching  $f_1$ . Clearly,  $(M - \{u, v, w\}) \cup \{(x, x'), (y, y'), (z, z')\} \cup f_1$  induces a fractional perfect matching of  $G - F$ .

**Case 2.**  $|F^0| = n - 1$ .

By the induction hypothesis, either  $G^0 - F^0$  has a fractional perfect matching  $f_0$  or  $F^0$  is a trivial FSMP set of  $G^0$ . In the first case, since  $|F - F^0| = 1$ , it follows from the induction hypothesis that  $G^1 - F^1$  has a fractional perfect matching  $f_1$ . Clearly,  $f_0 \cup f_1$  induces a fractional perfect matching of  $G - F$ . Now we consider the case that  $F^0$  is a trivial FSMP set of  $G^0$  and it is induced by the vertex  $u$ . If  $u$  has no neighbour in  $G^1 \cup (G - F)$ , then  $u$  is an isolated vertex in  $G - F$ , so we are done. Thus we may assume that  $u$  has a neighbour  $u' \in G^1$  in  $G - F$ . We distinguish the following two subcases to show this case.

*Subcase 2.1.*  $F^0$  contains only edges.

Since  $|F^0| \geq 6$  and  $|F - F^0| = 1$ , there exists an edge  $(u, v)$  in  $F^0$  such that  $v$  has a neighbour  $v' \in G^1$  in  $G - F$ . Let  $F^{0'} = F^0 - \{(u, v)\}$  and  $F^{1'} = F^1 \cup (V(G^1) \cap \{u', v'\})$ . Clearly,  $|F^{0'}| \leq n - 2$  and  $|F^{1'}| \leq 3$ . From the induction hypothesis,  $G^1 - F^{0'}$  has a fractional perfect matching. Note that  $u$  has only one neighbour  $v$  in  $G^0 - F^{0'}$ . From Lemma 1.2, the subgraph of  $G^0 - F^{0'}$  induced  $\{u, v\}$  is  $K_2$  and  $G^0 - F^{0'} - \{u, v\}$  can be partitioned into disjoint union of some  $K_2$  and Hamiltonian graphs on odd number of vertices. It is obvious that  $G^0 - F^{0'} - \{u, v\}$  has a fractional perfect matching  $f_0$ . Furthermore, note that  $n - 1 \geq 5$  and  $|F^{1'}| \leq 3$ . By the induction hypothesis,  $G^1 - F^{1'}$  has a fractional perfect matchings  $f_1$ . Clearly,  $f_0 \cup f_1 \cup \{(u, u'), (v, v')\}$  induces a fractional perfect matching of  $G - F$ .

*Subcase 2.2.*  $F^0$  contains at least one vertex  $v$ .

Let  $F^{0'} = F^0 - \{v\}$  and  $F^1 \cap \{u'\}$  for  $2 \leq i \leq n$ . Clearly,  $|F^{0'}| \leq n - 2$  and  $|F^1 \cap \{u'\}| \leq 2$ . By the induction hypothesis,  $G^0 - F^{0'}$  has a fractional perfect matching. Note that  $u$  has only one neighbour  $v$  in  $G^0 - F^{0'}$ . From Lemma 1.2, the subgraph of  $G^0 - F^{0'}$  induced by  $\{u, v\}$  is  $K_2$  and  $G^0 - F^{0'} - \{u, v\}$  can be partitioned into disjoint union of some  $K_2$  and Hamiltonian graphs on odd number of vertices. It is obvious that  $G^0 - F^{0'} - \{u, v\}$  has a fractional perfect matching  $f_0$  by Lemma 1.2. Furthermore, noting that  $n - 1 \geq 5$  and  $|F^1 \cap \{u'\}| \leq 2$ , by the induction hypothesis,  $G^1 - F^1 \cap \{u'\}$  has a fractional perfect matchings  $f_1$ . Thus  $f_0 \cup f_1$  and  $\{(u, u')\}$  induces a fractional perfect matching of  $G - F$ .

**Case 3.**  $|F^0| \leq n - 2$ .

In this case,  $|F^1| \leq |F^0| \leq n - 2$ . By the induction hypothesis,  $G^i - F^i$  has a fractional perfect matching  $f_i$  for  $0 \leq i \leq 1$ . It is clear that  $f_0 \cup f_1$  induces a fractional perfect matching of  $G - F$ .  $\square$

## 4 Conclusion

We believe that the concept of fractional matching preclusion introduced in [10] is very interesting and it will gain more attention in the future. In this paper, we studied this topic of strong version for multiply twisted cube and locally twisted cube, two special members of HL graphs. It is not difficult to see that the proofs for them are similar. One may ask why not prove the more general HL graphs. That's our early vision, but we didn't achieve it. Clearly, the fractional strong preclusion number for 4-dimensional multiply twisted cube or locally twisted cube is 3, but not 4. For some of the other HL graphs, such as Mcube, which was proposed in [22], however we found the fractional strong preclusion number for 4-dimensional Mcube is 4, but not 3. Therefore, the induction can't be used always. We believe that the fractional strong preclusion number for  $n$ -dimensional HL graphs is  $n$  if  $n \geq 5$ . We are looking forward to more results for this topic.

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