Theory and Applications of Graphs

Volume 7 | Issue 1 Article 6

May 2020

Domination number of annulus triangulations

Toshiki Abe
Yokohama National University, abe-toshiki-jg@ynu.jp

Junki Higa
higa-junki-sv@ynu.jp

Shin-ichi Tokunaga
Tokyo Medical and Dental University, tokunaga.las@tmd.ac.jp

Recommended Citation

Abe, Toshiki; Higa, Junki; and Tokunaga, Shin-ichi (2020) "Domination number of annulus triangulations," *Theory and Applications of Graphs*: Vol. 7 : Iss. 1 , Article 6.
DOI: 10.20429/tag.2020.070106
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol7/iss1/6

Abstract

An *annulus triangulation* G is a 2-connected plane graph with two disjoint faces f_1 and f_2 such that every face other than f_1 and f_2 are triangular, and that every vertex of G is contained in the boundary cycle of f_1 or f_2. In this paper, we prove that every annulus triangulation G with t vertices of degree 2 has a dominating set with cardinality at most $\lfloor \frac{|V(G)|+t+1}{4} \rfloor$ if G is not isomorphic to the octahedron. In particular, this bound is best possible.

1 Introduction

In this paper, all graphs are undirected and simple. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v \in V(G)$, let $N(v)$ denote the set of vertices which are adjacent to v. In particular, we call the set $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of v. Moreover, for $S \subset V(G)$, let $N(S)$ denote the *neighborhood* of S, i.e., the set of vertices adjacent to a vertex of S in G. For $S, T \subset V(G)$, we say that S *dominates* T if $T \subset S \cup N(S)$. If $D \subset V(G)$ dominates $V(G)$, then D is called a *dominating set* of G. The *domination number* of G is the minimum cardinality over all dominating sets of G and denoted by $\gamma(G)$.

A *disk triangulation* is a 2-connected plane graph such that every face except for the infinite face is triangular. Matheson and Tarjan proved the following theorem by an elegant coloring method:

Theorem 1 (Matheson and Tarjan [3]) *Let G be a disk triangulation with n vertices. Then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.*

They constructed a disk triangulation with n vertices in which any dominating sets have cardinality at least $\lfloor \frac{n}{3} \rfloor$, and hence the estimation in Theorem 1 is best possible. The examples they constructed are *maximal outerplanar graphs*, (i.e., a 2-connected plane graph such that there is a single face f containing all vertices on the boundary cycle, and that every face other than f is triangular), and so they asked what happens if every face is triangular:

Conjecture 2 (Matheson and Tarjan [3]) *Let G be a planar triangulation with n vertices. If n is sufficiently large, then $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$.*

They constructed a plane triangulation G with n vertices satisfying $\gamma(G) = \lfloor \frac{n}{4} \rfloor$ for any large n. but the conjecture is still open so far. For this conjecture, Plummer, Ye and Zha [4] proved that every 4-connected plane triangulation with $n \geq 26$ vertices satisfies $\gamma(G) \leq \lfloor 5n/16 \rfloor$. In addition, King and Pelsmajer [2] proved that every plane triangulation G of maximum degree 6 with n vertices satisfies that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$. But we do not know of any other progress on the problem.

Let us focus on maximal outerplanar graphs. By Theorem 1, every maximal outerplanar graph G with n vertices has domination number at most $\lfloor \frac{n}{3} \rfloor$. This result is easily obtained by a proper 3-coloring, as follows: A maximal outerplanar graph is known to have a proper 3-coloring $c : V(G) \to \{1, 2, 3\}$. Observe that for $i = 1, 2, 3$, the set $c^{-1}(i)$ dominates G where $c^{-1}(i)$ is the set of vertices colored by i for the coloring c. Hence for some $i \in \{1, 2, 3\}$, we have

$|c^{-1}(i)| \leq \frac{n}{3}$ since $|c^{-1}(1)|+|c^{-1}(2)|+|c^{-1}(3)| = n$, and we are done. Moreover, there exists a maximal outerplanar graph each of whose dominating set requires $\lfloor \frac{n}{3} \rfloor$ vertices [3]. Campos and Wakabayashi [1] pointed out that maximal outerplanar graphs with a large domination number have many vertices of degree 2, and they (and Tokunaga independently)proved the following theorem.

Theorem 3 ([1, 5]) *Let G be a maximal outerplanar graph with n vertices and t vertices of degree 2. Then $\gamma(G) \leq \lfloor \frac{n+t}{4} \rfloor$, where the bound is sharp.*

In this paper, we introduce an "annulus triangulation" and consider its domination number. An *annulus triangulation* is a 2-connected plane graph with two disjoint special faces f_1 and f_2 such that every face of G except for f_1 and f_2 are triangular, and that every vertex of G is contained in the boundary cycle of f_1 or f_2. We call f_1 and f_2 *holed face* and any other faces *facial 3-cycles*. The boundary cycle of f_1 and that of f_2 are called the *boundary* of G. This seems to be a natural extension of maximal outerplanar graphs.

Our main theorem is as follows:

Theorem 4 *Let G be an annulus triangulation with n vertices and t vertices of degree 2. If $n \geq 7$, then $\gamma(G) \leq \lfloor \frac{n+t+1}{4} \rfloor$, where this estimation is sharp.*

A big difference between maximal outerplanar graphs and annulus triangulations is that an annulus triangulation G is not necessarily 3-colorable, and that G might not have vertices of degree 2. In this paper, we elaborate a coloring method in [3, 5] and prove Theorem 4. In Section 2, we will prove lemmas to show the main theorem, and in Section 3, we prove the main theorem.

2 Dominating k-set-assignment

Let G be a graph and k be a positive integer. A *k-coloring* is a map $c : V(G) \to \{1, 2, ..., k\}$, and c is *proper* if $c(x) \neq c(y)$ for any $xy \in E(G)$. A k-coloring c is said to be a *dominating k-coloring* if for any $i \in \{1, \ldots, k\}$, the vertex set $c^{-1}(i)$ is a dominating set of G. By the definition, we have the following:

Proposition 5 *If a graph G admits a dominating k-coloring, then $\gamma(G) \leq \lfloor \frac{|V(G)|}{k} \rfloor$.* ■

Proposition 5 is useful to prove that a maximal outerplanar graph G with n vertices has a dominating set with cardinality at most $\lfloor \frac{n}{3} \rfloor$, since every proper 3-coloring of G is a dominating 3-coloring of G, as is mentioned in the previous section.

Extending the notion of a dominating k-coloring of a graph G, we define a "dominating k-set-assignment", as follows: An assignment $f : V(G) \to 2^{\{1,\ldots,k\}}$ is a *dominating k-set-assignment* if for any $i \in \{1, \ldots, k\}$, the vertex set

$$D_f(i) = \{v \in V(G) \ : \ i \in f(v)\}$$

is a dominating set of G. It is easy to see that f is a dominating k-set-assignment if and only if every vertex v has all k colors in its closed neighborhood. Let

$$d_G(f) = \sum_{i=1}^{k} |D_f(i)|.$$

By the definition, we have:

Proposition 6 *If a graph G admits a dominating k-set-assignment f, then $\gamma(G) \leq \lfloor \frac{d_G(f)}{k} \rfloor$.*

Note that if $|f(v)| = 1$ for every vertex $v \in V(G)$ in Proposition 6, then the statement coincides with Proposition 5. In order to prove our theorem, we give the definition of a property called good. Let G be a graph embedded on the plane. We say a 4-set-assignment f of a graph G is *good* if f satisfies all of the following conditions,

(D1) for each vertex v of degree at least 3 except for at most one vertex u, $|f(v)| = 1$,

(D2) for each vertex w of degree 2 or the vertex u as above (if it exists), $|f(w)| = |f(u)| = 2$, and

(D3) for every facial 3-cycle $C = xyz$ of G, there exist three distinct colors $i_1, i_2, i_3 \in \{1, \dots, 4\}$ such that $i_1 \in f(x), i_2 \in f(y), i_3 \in f(z)$.

Note that if f is good, then we have $d_G(f) \leq n + t + 1$, where n is the number of vertices of G and t is the number of vertices of degree 2 in G. In particular, Tokunaga [5] proved Theorem 3 by constructing, for a maximal outerplanar graph, a good dominating 4-set-assignment with additional properties.

Proposition 7 ([5]) *Let G be a maximal outerplanar graph with n vertices and t vertices of degree 2. Then G has a good dominating 4-set assignment f such that*

(P1) there is no exception in (D1) and hence $d_G(f) = n + t$, and

(P2) for any 4-cycle $xyzw$ in G, the four colors $1, 2, 3, 4$ are contained in the four sets $f(x), f(y), f(z), f(w)$ bijectively.

Let G be an annulus triangulation and let C_1 and C_2 denote boundary components of G. An edge e is a *boundary edge* if e is contained in C_1 or C_2. An edge e is *trivial* if e is not a boundary edge but the endpoints of e are contained in the same boundary component. For example, the edge x_0x_1 in Figure 1 is a boundary edge and y_1y_3 is trivial. We usually represent an annulus triangulation G by a rectangle cutting G along a non-trivial and non-boundary edge x_0y_0, as in Figure 1. By identifying the arrows of both ends, we obtain the annulus triangulation.

Suppose that an annulus triangulation G has a trivial edge $e = xy$ whose endpoints are contained in C_1. Let P and P' be the two paths of G such that $V(P) \cup V(P') = V(C_1)$, that $V(P) \cap V(P') = \{x, y\}$, and that the cycle $P \cup \{e\}$ bounds a maximal outerplane subgraph D of G. We call D the *ear* of G separated by the edge xy. In particular, we say D is *maximal*

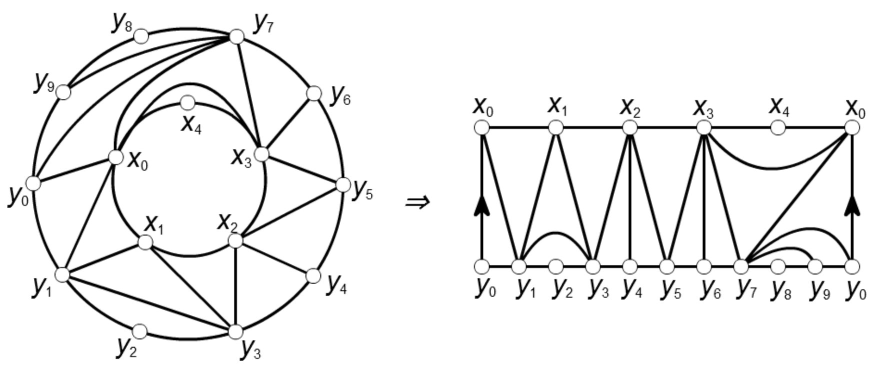

Figure 1: Different representations of an annulus triangulation.

if G has no trivial edge separating an ear including D as a proper subgraph. Removing an ear except for x and y decreases the number of trivial edges. So, repeating this operation, we finally get one with no trivial edges, which is called an *essential* subgraph of G and taken uniquely in G. See Figures 2 and 3. The graph drawn in Figure 3 is the essential subgraph of the graph in Figure 2.

In an essential annulus triangulation G, an edge e is called a *spoke* if an endpoint of e has degree 3. (We note that G has no vertex of degree less than 3 since G is essential.) An edge e is called a *frame edge* if e is neither a spoke nor boundary edge. The *frame* of G is the subgraph of G induced by the frame edges.

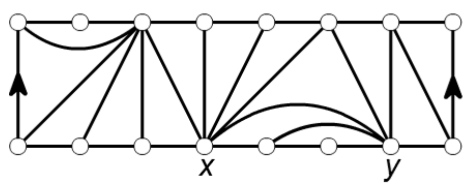

Figure 2: An annulus triangulation

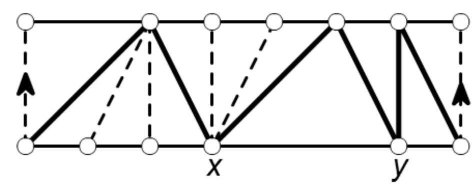

Figure 3: The thick edges are frame and the dotted ones are spoke

We first introduce two propositions for an annulus triangulation.

Proposition 8 *Let G be a non-essential annulus triangulation and let Y be a maximal ear of G separated by a trivial edge $e = xy$. Let G' be the annulus triangulation such that $G' \cup Y = G$ and $V(G') \cap V(Y) = \{x, y\}$ (See Figure 4). Then if G' admits a good dominating 4-set-assignment or if G' is isomorphic to the octahedron, then G has a good dominating 4-set-assignment.*

Proof. Without loss of generality, we may assume $\deg_Y(y) \geq \deg_Y(x)$. First, we will show that the edge xy is incident to a facial 3-cycle in G'. Let f_1 and f_2 be two distinct holed faces such that the vertices x and y are on the boundary of f_1. The edge xy is incident to exactly two faces, say f_1 and f_3 in G. If $f_3 = f_1$, then the edge xy is a cut edge of G, which contradicts 2-connectivity of G. Moreover, if $f_3 = f_2$, e is on the boundary of both f_1 and f_2, which contradicts that f_1 and f_2 are disjoint with each other. Thus e is incident to a

https://digitalcommons.georgiasouthern.edu/tag/vol7/iss1/6
DOI: 10.20429/tag.2020.070106

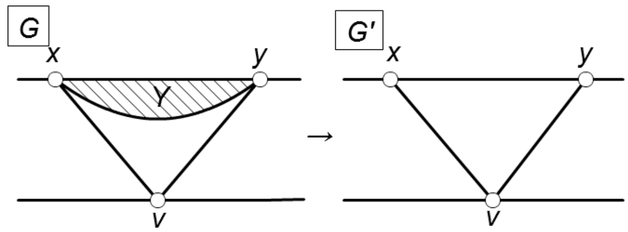

Figure 4: The ear reduction. The shaded area in the left figure is the maximal ear Y of G.

facial 3-cycle $f_3 = xyv$. Moreover, since Y is a maximal ear, the vertex v is on the boundary of f_2.

Next, we show that G has a good dominating 4-set-assignment. If G' has a good dominating 4-set-assignment f', without loss of generality, we may assume $1 \in f'(x)$ and $2 \in f'(y)$. On the other hand, if G' is isomorphic to the octahedron, then we let f' be as shown in Figure 5.

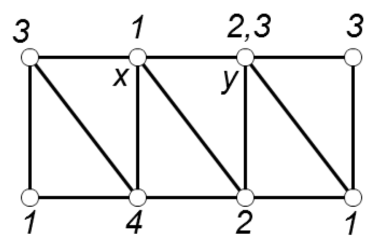

Figure 5: The 4-set-assignment f' of the octahedron.

We divide the proof into two cases depending on $|V(Y)|$.

Case 1 Suppose $|V(Y)| \geq 4$.

Since Y is a maximal outerplane graph and $\deg_Y(y) \geq \deg_Y(x)$, we have $\deg_Y(y) \geq 3$. Thus we may assume that Y has a cycle $C = xyzw$ such that $wy \in E(Y)$. By Proposition 7, Y admits a good dominating 4-set-assignment f_Y such that $1 \in f_Y(x)$, $\{2\} = f_Y(y)$, $3 \in f_Y(z)$ and $4 \in f_Y(w)$.

We define the assignment f as

$$f(u) = \begin{cases} f'(u) & (u \in V(G')), \\ f_Y(u) & (u \in V(Y) - \{x, y\}). \end{cases}$$

By the construction of f, it is sufficient to prove that every vertex which is adjacent to x or y or which is x or y itself has all 4 colors in its closed neighborhood. We see that every vertex in $V(G') - \{y\}$ has all 4 colors in its closed neighborhood by f' in either case. Moreover, since y is contained in the cycle C in Y, y has all 4 colors in its closed neighborhood in Y. Thus every vertex in $V(G')$ has all 4 colors in its closed neighborhood for f. Next, we show that every vertex which is adjacent to x or y in Y has all 4 colors for f. Since $f_Y(y) \subset f(y)$

in either case, the vertices which are adjacent to y also have all 4 colors for f. Moreover, if $\deg_Y(x) = 2$, then $N_Y(x) = \{y, w\}$ and hence $f_Y(x) = \{1, 3\}$ by the assumptions and Proposition 7. Since $3 \in f_Y(z)$, the vertices which are adjacent to x in Y have all 4 colors for f. On the other hand, if $\deg_Y(x) \geq 3$, then we have $f_Y(x) = \{1\}$. In this case, we have $f_Y(x) \subset f(x)$ and hence the vertices which are adjacent to x have all 4 colors for f. Therefore, we see that f is a good dominating 4-set-assignment in G.

Case 2 Suppose $|V(Y)| = 3$.
In this case, Y is isomorphic to the complete graph K_3. Let $w \in V(Y)$ be the vertex which is neither x nor y. In this case, we get a good dominating 4-set-assignment f of G from f' such that

$$f(u) = \begin{cases} f'(u) & (u \in V(G')), \\ \{3, 4\} & (u = w). \end{cases} \qquad \blacksquare$$

Proposition 9 *Let G be an essential annulus triangulation and v be a vertex to which at least three consective spokes av, bv, cv are incident. Moreover, let G' be the graph obtained from G by removing the three edges av, bv, cv and smoothing the vertices a, b, c of degree 2, as shown in Figure 6. If G' is simple and admits a good dominating 4-set-assignment or if G' is isomorphic to the octahedron, then G admits a good dominating 4-set-assignment.*

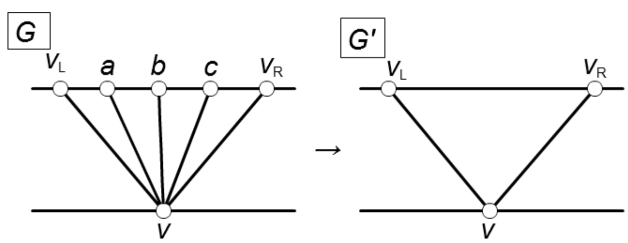

Figure 6: the spoke reduction

Proof. We devide the proof into two cases whether G' has a dominating 4-set-assignment f' or G is isomorphic to the octahedron.

Case 1 Suppose G' has a good dominating 4-set-assignment f'.

Clearly, G' has no vertices of degree 2. Let v_L (v_R respectively) be the vertex which is adjacent to v and a (v and c respectively) with $v_L \neq b$ ($v_R \neq b$ respectively) as in Figure 6. Without loss of generality, we may assume $1 \in f'(v_L), 2 \in f'(v)$ and $3 \in f'(v_R)$.

We define $f : V(G) \to 2^{\{1,2,3,4\}}$ as

$$f(z) = \begin{cases} f'(z) & (z \in V(G')), \\ \{3\} & (z = a), \\ \{4\} & (z = b), \\ \{1\} & (z = c). \end{cases}$$

DOI: 10.20429/tag.2020.070106

We can easily check that all vertices except for v_L and v_R in G have all 4 colors in their closed neighborhoods and all facial cycles have distinct three colors. Suppose f is not a good dominating 4-set-assignment. By symmetry, we may assume v_L does not have four colors in its closed neighborhood. Since f' is a good dominating 4-set-assignment in G', we have $f(v_R) = \{3, 4\}$ and $N_G[v_L] \cap D_f(4) = \emptyset$. On the other hand, G has a facial cycle uv_Lv such that $u \neq v_R$. Since f' is good, we have $f(u) = \{3\}$. In this case, by exchanging the color of the vertices a and b in f, we obtain a good dominating 4-set-assignment in G.

Case 2 Suppose G' is isomorphic to the octahedron.
By symmetric, G is isomorphic the graph as shown in Figure 7 and we assign a 4-set assignment f to G as follows.

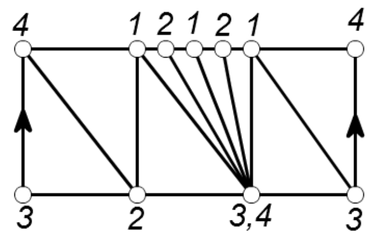

Figure 7: The graph G obtained from the octahedron by adding three spokes.

It is easy to see that f is a good dominating 4-set-assignment in G. ■

3 Domination number of annulus triangulations

For a graph G, a proper 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$ is an *admissible 4-coloring* of G if every four vertices of G contained in a 4-cycle have four distinct colors. We can easily check that a 4-set-assignment f which includes an admissible coloring c (i.e. $c(v) \in f(v)$ for every $v \in V(G)$) satisfies the conditions $(D3)$ and $(P2)$. It is easy to see following.

Lemma 10 *Every maximal outerplane graph with $n \geq 4$ vertices has an admissible 4-coloring.*

Let G be a maximal outerplane graph and $c : V(G) \to \{1, 2, 3, 4\}$ be an admissible 4-coloring. Since every 4-cycle has all 4 colors, each color class $c^{-1}(i)$ dominates all vertices of degree at least 3. On the other hand, every vertex v of degree 2 has exactly one color i such that the set $c^{-1}(i)$ does not dominate v. In this case, the color i is the *missing color* for v.

The following is a key claim for the proof.

Theorem 11 *Let G be an annulus triangulation with n vertices which is not isomorphic to the octahedron. If G has no vertex of degree 2 or at least 7, then G has a good dominating 4-set-assignment.*

Proof. Let G be a minimum counterexample of Theorem 11.

Claim 12 *G does not have a vertex of degree* 6.

Proof. Suppose not. Let v be a vertex of degree 6 and let $v_1, v_2, v_3, v_4, v_5, v_6$ be the neighbors of v in this order with respect to the rotation of v, as in Figure 8. Let G' be the maximal outerplane graph obtained from G by removing v, v_3 and v_4, where we note that exactly one of v_1 and v_2, say x, has degree 2 in G' and so does exactly one of v_5 and v_6, say y. Since G is essential, G' has no vertex of degree 2 except for x and y, and hence by Lemma 10, G' has an admissible 4-coloring c such that each of x and y has a missing color. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of x is 4. Let $c(v_6) = a_1, c(v_5) = a_2$ and let the missing color of y be a_3. By Lemma 10, it is easy to see that a_1, a_2 and a_3 are distinct. Now we construct a good dominating 4-set-assignment f in G as follows.

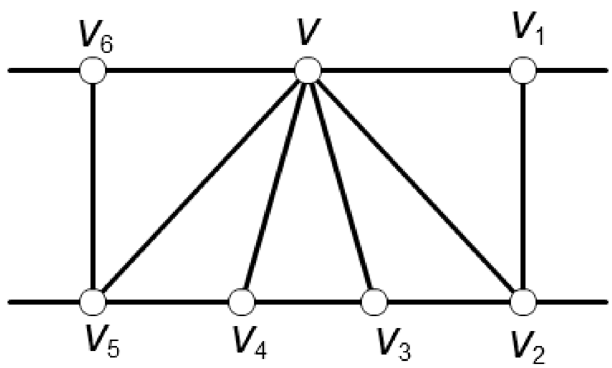

Figure 8: The vetex v and the neighbor of v

Case 12.1 Suppose $a_3 \in \{1, 2, 3\}$.

We let $b_3, b_4 \in \{1, 2, 3, 4\}$ as follows.

$$b_4 \in \begin{cases} \{2\} & (a_2 \neq 2, a_3 \neq 2), \\ \{1, 3\} - \{a_2\} & (otherwise). \end{cases}$$

$$b_3 \in \begin{cases} \{1, 3\} - \{a_3\} & (a_2 \neq 2, a_3 \neq 2), \\ \{1, 3\} - \{b_4\} & (otherwise). \end{cases}$$

Then we define an assignment f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{a_3, 4\} & (z = v), \\ \{b_4\} & (z = v_4), \\ \{b_3\} & (z = v_3). \end{cases}$$

If $a_2 \neq 2$ and $a_3 \neq 2$, then we have $\{b_3, b_4, a_3\} = \{1, 2, 3\}$. Otherwise, we have $\{b_3, b_4\} = \{1, 3\}$. In either case, we can easily check that every vertex has distinct four colors in its closed neighborhood and that f also satisfies good in G, which contradicts the assumption.

Case 12.2 Suppose $a_3 = 4$.
We assign

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{4\} & (z = v), \\ \{1,2\} & (z = v_4), \\ \{3\} & (z = v_3). \end{cases}$$

In this case, it is easy to see this assignment f is also a good dominating 4-set-assignment. It contradicts the assumption. $\square$

Suppose G has a vertex of degree 5, say v. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in this order with respect to the rotation of v, as in Figure 9. Let G' be the maximal outerplane graph obtained from G by removing v and v_3, where we note that exactly one of v_1 and v_2, say x, has degree 2 in G' and so does exactly one of v_4 and v_5, say y. By the assumption, G' has no vertex of degree 2 except for x and y, and hence by Lemma 10, G' has an admissible 4-coloring c such that each of x and y has a missing color. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of x is 4. Let $c(v_5) = a_1, c(v_4) = a_2$ and the missing color of y be a_3.

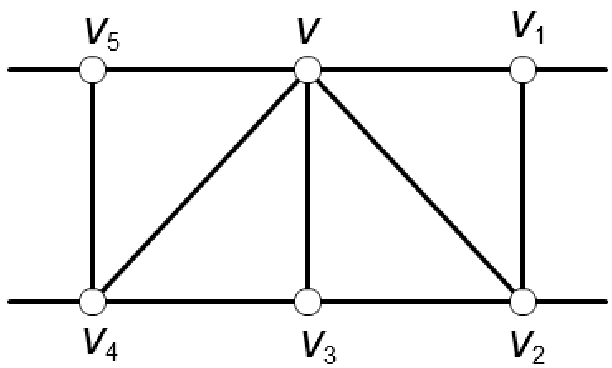

Figure 9: a vertex of degree 5

Claim 13 *If G has a vertex v of degree 5, then $a_2 = 4$ and $a_3 = 2$ where a_2 and a_3 are defined as above.*

Proof. Suppose not.

Case 13.1 Suppose $a_3 \in \{1, 2, 3\}$.
We let b_3 as follows.

$$b_3 \in \begin{cases} \{1,3\} - \{a_2\} & (a_2 \notin \{2,4\}), \\ \{1,3\} - \{a_3\} & (otherwise). \end{cases}$$

Then we define f as

$$f(z)=\begin{cases}\{c(z)\} & (z\in V(G')),\\ \{a_3,4\} & (z=v),\\ \{b_3\} & (z=v_3),\end{cases}$$

If $a_2\notin\{2,4\}$, then b_3 is uniquely obtained and it is easy to see that f is a good dominating 4-set-assignment in G. If $a_2=2$, then we have $\{a_3,b_3\}=\{1,3\}$ and it is easy to see that f is a good dominating 4-set-assignment in G. Moreover, if $a_2=4$ and $a_3\neq 2$, then we have $\{a_3,b_3\}=\{1,3\}$ and hence f is a good dominating 4-set-assignment in G.

Case 13.2 Suppose $a_3=4$
We define f as

$$f(z)=\begin{cases}\{c(z)\} & (z\in V(G')),\\ \{4\} & (z=v),\\ \{1,3\} & (z=v_3),\end{cases}$$

In this case, we have f is a good dominating 4-set-assignment in G. $\square$

Suppose G does not have a vertex of degree 4. In this case, every vertex in G has degree 3 or 5. Now we will show that G is uniquely obtained in this case. Since G is essential, we see that every vertex $v\in V(G)$ is an endpoint of a frame edge if and only if $\deg_G(v)=5$. Let $C=x_0x_1...x_{k-1}$ and $C'=y_0y_1...y_{m-1}$ be two distinct boundary components in G. First, suppose that G has a boundary edge x_ix_{i+1} such that $\deg_G(x_i)=\deg_G(x_{i+1})=5$, where the subscript is taken modulo k. Since x_i and x_{i+1} are endpoints of the frame edges and $x_ix_{i+1}\in E(G)$, G has a vertex $y_j\in V(C')$ such that x_iy_j, y_jx_{i+1} are frame edges of G. Moreover, since y_j is endpoint of the frame edges and $x_ix_{i+1}\in E(G)$, we have $deg_G(y_j)=4$. This contradicts the assumption. Next, suppose that G has a boundary edge x_ix_{i+1} such that $\deg_G(x_i)=\deg_G(x_{i+1})=3$. In this case, neither x_i nor x_{i+1} are endpoints of frame edges. Since they are endpoints of the spokes, they are adjacent to a common vertex $y\in V(C')$. This indicates that y has degree at least 6 and this fact contradicts the assumption. Thus, the vertices of degree 3 and ones of degree 5 appear alternatively in C and C'. Moreover, by counting the number of non-boundary edges, we have $2k=2m$. This indicates that $|V(C)|=|V(C')|$. Without loss of generality, we may assume $\deg_G(x_0)=3$ and $x_0y_0\in E(G)$. Since G is an annulus triangulation, we have $\deg_G(y_0)=5$ and hence $y_0x_1\in E(G)$. Moreover, we see that $x_1y_1, x_1y_2\in E(G)$ and that $y_2x_2\in E(G)$ by the same reason as above. By repeating these argument, G is uniquely obtained as shown in Figure 10.

Let $G'=G-\{x_1,y_1\}$. By lemma 10, G' has an admissible coloring c so that $c(x_2)=1, c(y_2)=2$ and missing color of x_2 is 4. We can easily check this coloring c satisfies that if $c(y_0)=4$, then the missing color of x_0 is 1 and hence c does not satisfy Claim 13. Thus we may assume G has a vertex of degree 4. Next, we prove that G must be a 4-regular graph.

Claim 14 *G does not have a vertex of degree* 5.

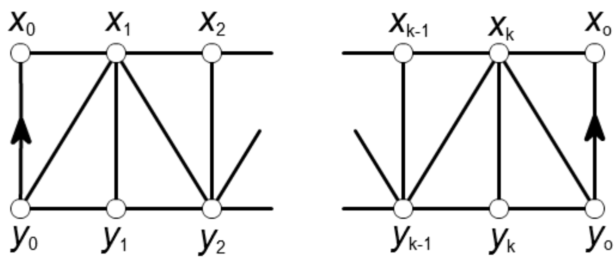

Figure 10: A situation without vertices of degree 4

Proof. Suppose not. Since G has a vertex of degree 4, G has a frame edge connecting a vertex of degree 5 and one of degree 4. Let v be a vertex of degree 5 and let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in this order with respect to the rotation of v, as in Figure 9. We may assume that v_2 is a vertex of degree 4. Let u be the endpoint of the frame edge which is incident to v_1 with $u \neq v_2$. Since $G' = G - \{v, v_3\}$ is a maximal outerplanar graph, G' has an admissible coloring c by Lemma 10. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of v_2 is 4. By Claim 13, $c(v_4) = 4$ and hence $u \neq v_4$. Next, we construct a good dominating 4-set-assignment as follows.

Case 14.1 Suppose $\deg_G(v_1) = 4$. We define an assignment f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G') - \{v_2\}), \\ \{2\} & (z = v), \\ \{4\} & (z = v_2), \\ \{1, 3\} & (z = v_3). \end{cases}$$

In this case, it is easy to see that every vertex which is not adjacent to u and whose degree is at least 3 in G' has all 4 colors in its closed neighborhood by f. Moreover, since $u \neq v_4$ and any 4-cycles in G' except for the cycle bounded by v_2v_1wu have all 4 colors by Lemma 10, where w is a vertex which is adjacent to v_1 and u with $w \neq v_2$, we have $|N[u] \cap c^{-1}(2)| = 2$. Thus the vertex u also has all 4 colors in its closed neighborhood by f. Moreover, it is easy to see that every vertex $v_i (i \in \{1, 2, ..., 5\})$ has all 4 colors in its closed neighborhood by f. Thus f is a good dominating 4-set-assignment in G.

Case 14.2 Suppose $\deg_G(v_1) = 5$.

In this case, v_1 has one spoke v_1w'. If $u = v_4$, then it is easy to see that $N_{G'}[v_4] \cap N_{G'}[v_5] \cap c^{-1}(2) \neq \emptyset$, which contradicts Claim 13. Thus we conclude $u \neq v_4$. We define an assignment

f as

$$f(z)=\begin{cases}\{c(z)\} & (z\in V(G')-\{v_2,w'\}),\\ \{2,4\} & (z=v),\\ \{3\} & (z=v_2),\\ \{1\} & (z=v_3),\\ \{2\} & (z=w').\end{cases}$$

By the similar argument as before, we conclude f is a good dominating 4-set-assignment. □

By Claim 12 and 14, the degree of every vertex in G is at most 4. Suppose that G has a vertex of degree 3, say v. Since G does not have a vertex of degree 2, v must be an endpoint of a spoke vw. This implies that the degree of w is at least 5, which is a contradiction. Thus G is a 4-regular graph. Let $C=x_0x_1...x_k$ and $C'=y_0y_1...y_m$ be two distinct boundary components of G. Without loss of generality, we may assume $x_0y_0, x_0y_1\in E(G)$. We can easily check $|V(C)|=|V(C')|$ and G is uniquely obtained, as shown in Figure 11.

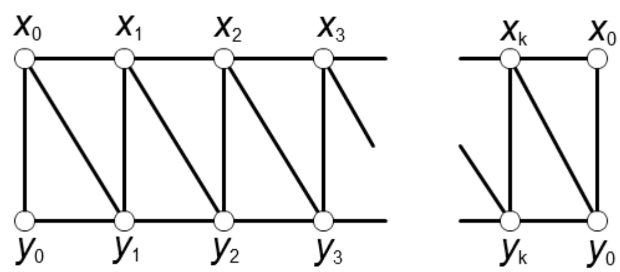

Figure 11: A 4-regular graph

Claim 15 $|V(G)|$ *is at most* 6.

Proof. Suppose not. Since $G'=G-\{x_1,y_1\}$ is a maximal outerplanar graph, G' has the admissible coloring c by Lemma 10. Without loss of generality, we may assume $c(x_2)=1, c(y_2)=2$ and $c(y_3)=3$. If k is odd, then we see that $(c(x_0),c(y_0))=(1,2)$ and hence we can get a good dominating 4-set-assignment f in G naturally as follows.

$$f(z)=\begin{cases}\{c(z)\} & (z\in V(G')),\\ \{4\} & (z=x_1),\\ \{3\} & (z=y_1).\end{cases}$$

Otherwise, we see that $(c(x_0),c(y_0))=(4,3)$. In this case, we define f as

$$f(z)=\begin{cases}\{c(z)\} & (z\in V(G')-\{y_2\}),\\ \{2\} & (z=x_1),\\ \{1,3\} & (z=y_1),\\ \{4\} & (z=y_2).\end{cases}$$

DOI: 10.20429/tag.2020.070106

Since G is not isomorphic to the octahedron, we see that $k \geq 4$. Every vertex in $V(G') - \{x_0, x_2, y_2, y_3\}$ has all 4 colors in its closed neighborhood by f. Moreover, since $c(y_4) = 2$, the vertex y_3 also has all 4 colors in its closed neighborhood. It is easy to see that any other vertices have all 4 colors in their closed neighborhood. Thus f is a good dominating 4-set-assignment in G. Therefore, by Claim 12, 14 and 15, if G is not isomorphic to the octahedron, then G has a good dominating 4-set-assignment. ■

4 Proof of Theorem 4

Proof. By Proposition 6, it is sufficient to prove that G has a good domminating 4-set-assignment unless G is the octahedron. Let G be a counterexample as above with minimum cardinality. Suppose G is non-essential (i.e. G has a trivial edge xy). Let G' be a graph obtained by removing a maximal ear Y of G except for xy. It is easy to see that G' is an annulus triangulation. By the minimality of G, G' has a good dominating 4-set-assignment or that G' is isomorphic to the octahedron. On the other hand, by Proposition 8, we conclude that G has a good dominating 4-set-assignment, which contradicts the assumption. Thus we may assume G is essential.

Suppose G has a vertex of degree at least 7. Then G has a vertex v such that v is an endpoint of at least three spokes av, bv, cv. Let G' be the graph obtained from G by removing the three edges av, bv, cv and smoothing the vertices a, b, c of degree 2. Let v_L (v_R respectively) be the vertex which is adjacent to v and a (v and c respectively) with $v_L \neq b$ ($v_R \neq b$ respectively) as in Figure 6. It is easy to see that G' is not simple if and only if $v_L v_R \in E(G)$ and $\deg_G(v) = 7$. If G' is simple, then G' has a good dominating 4-set-assignment or G' is isomorphic to the octahedron. By Proposition 9, G has a good dominating 4-set-assignment for either case, which contradicts the assumption. Therefore, we may assume that G is not simple, then G has the edge $v_L v_R$ and $\deg_G(v) = 7$. Since $v_L v_R \in E(G)$ and since G is a simple annulus triangulation, the structure of G is restricted as shown in Figure 12. The ? areas in Figure 12 may have some spokes.

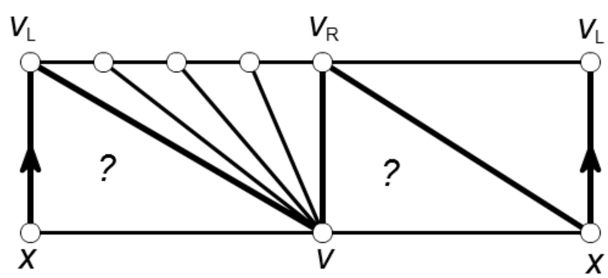

Figure 12: A situation of G such that G' is not simple.

By symmetry, we may assume $\deg_G(v_L) \leq \deg_G(v_R)$. If G satisfies that $\deg_G(v_R) \geq 8$ or that $\deg_G(v_R) = 7$ and $xv \notin E(G)$, then we obtain the simple annulus triangulation G''

by focusing on v_R instead of v. Thus G has a good dominating 4-set-assignment by the induction hypothesis, which contradicts the assumption. Moreover, if $deg_G(v_R) = 4$, then $deg_G(v_L) = 4$ and hence G has a multiple edge. Thus we have $5 \leq \deg_G(v_R) \leq 7$ We construct a good dominating 4-set-assignment in G depending on $\deg_G(v_R)$ as follows.

Case 1 Suppose that $\deg_G(v_R) = 7$ and $xv \in E(G)$.

We assign the 4-set-assignment as shown in Figure 13.

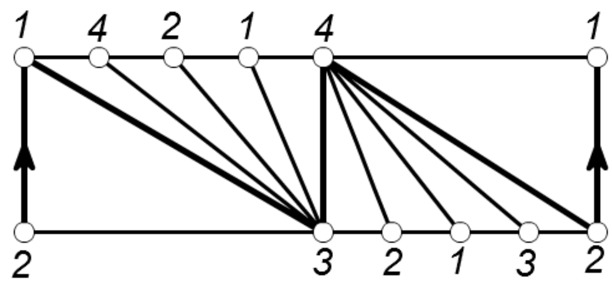

Figure 13: The degree of v_R is 7 and $xv \in E(G)$.

Case 2 Suppose that $deg_G(v_R) = 6$.

In this case, we see that $4 \leq \deg_G(v_L) \leq 6$. We assign the 4-set-assignment of G as in Figures 14 to 16.

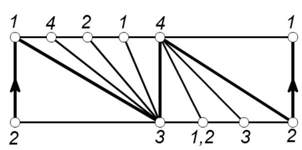

Figure 14: $\deg_G(v_L) = 4$

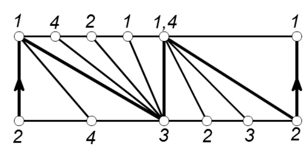

Figure 15: $\deg_G(v_L) = 5$

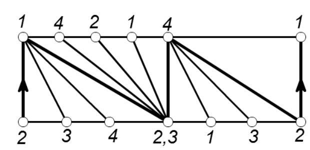

Figure 16: $\deg_G(v_L) = 6$

Case 3 Suppose that $\deg_G(v_R) = 5$.

In this case, we see that $4 \leq \deg_G(v_L) \leq 5$. We assign the 4-set-assignment as in Figures 17 and 18.

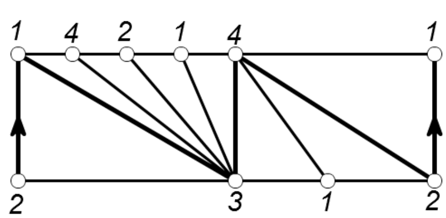

Figure 17: $\deg_G(v_L) = 4$

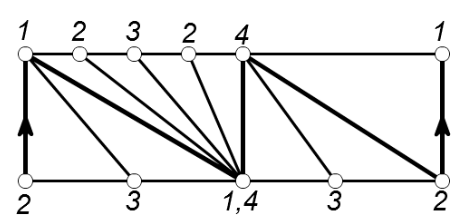

Figure 18: $\deg_G(v_L) = 5$

In either case, we see that each assignment as above is a good dominating 4-set-assignment in G, which contradicts the assumption. Thus we may assume that G does not have a vertex of degree at least 7.

By Theorem 11, G has a good dominating 4-set-assignment. Thus in any cases except for the octahedron, we constructed a good dominating 4-set-assignment with $d_G(f) \leq n+t+1$. ■

In order to prove the sharpness of the theorem, we construct an annulus triangulation satisfying the equality of the estimation. See Figure 19. We show $\gamma(G) = 7$. Let A_i be the closed neighborhood of a_i, for $i = 1, 2, 3, 4, 5, 6$. Then observe that $A_1, \ldots, A_6$ are pairwise disjoint. Thus we must have $\gamma(G) \geq 6$, since we have to choose at least one vertex from A_i for $i = 1, 2, 3, 4, 5, 6$, in order to dominate a_i. Hence we suppose that G has a dominating set S with $|S| = 6$. It is trivial $|S \cap A_i| = 1$ for any i. Observe that b_1 is the only vertex in $\bigcup A_i$ adjacent to x and so $S \cap A_1 = \{b_1\}$. Next, in order to dominate the vertex c_1, we have $S \cap A_3 = \{b_3\}$. By the same reason, we have $b_5 \in S$ to dominate c_3. By any choice of three vertices in A_2, A_4, A_6, S does not dominate y. Hence $\gamma(G) > 6$.

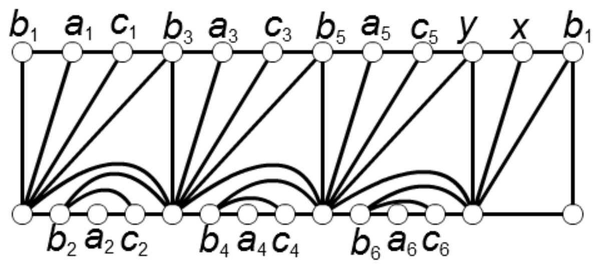

Figure 19: $n = 24, t = 3, \gamma(G) = 7$

By the similar discussion, we have an annulus triangulation with $\gamma(G) = \lfloor \frac{n+t+1}{4} \rfloor$ for some $n \geq N$, where N is a large constant.

5 Domination number of k-holed triangulations

As a natural extension of maximal outerplanar graph, we concern a graph called k-holed triangulation. A *k-holed plane triangulation* is a 2-connected plane graph with k disjoint special faces $f_1, \ldots, f_k$ such that every face of G except for f_i are triangular, and that every vertex of G is contained in the boundary cycle of boundary cycles. If $k = 1$, then G is a maximal outerplane graph. Moreover if $k = 2$, then G is an annulus triangulation.

It is not hard to see the following.

Proposition 16 *Let G be a k-holed triangulation with n vertices and t vertices of degree 2. Then $\gamma(G) \leq \lfloor \frac{n+t+2(k-1)}{4} \rfloor$.*

The upper bound is tight since the it holds with equality for the octahedron. However, there seems to be no other graphs for which the bounds hold equality, hence we conjecture as follows.

Conjecture 17 *Let G be a k-holed triangulation with n vertices and t vertices of degree 2. If G is not isomorphic to the octahedron, then $\gamma(G) \leq \lfloor \frac{n+t+(k-1)}{4} \rfloor$.*

The graph shown in Figure 20 holds with equality for $k = 4$. In a similar way, we can construct k-holed triangulations with $\gamma(G) = \lfloor \frac{n+t+(k-1)}{4} \rfloor$ for any k.

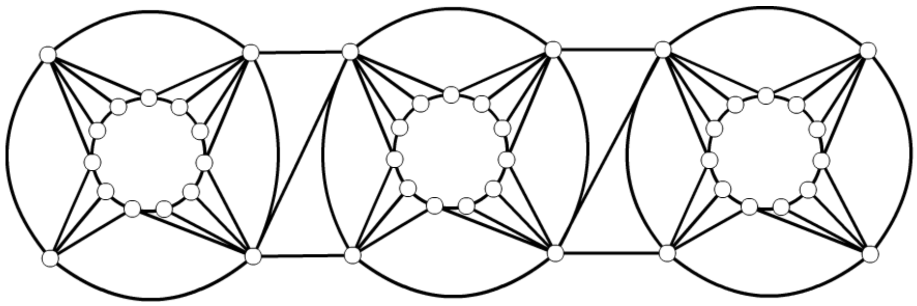

Figure 20: A 4-holed triangulation with $n = 45, t = 0, k = 4$ and $\gamma(G) = 12$

References

[1] C.N. Campos, Y. Wakabayasi, On dominating sets of maximal outerplanar graphs, *Discrete Appl. Math.* **161** (2013), 330–335.

[2] E.L.C. King and M. J. Pelsmajer, Dominating sets in plane triangulation, *Discrete Math.* **310** (2010), 2221-2230.

[3] L. Matheson, R. Tarjan, Dominating sets in planar graphs, *European J. Combin.* **17** (1996), 565–568.

[4] M.D. Plummer, D. Ye, X. Zha, Dominating plane triangulation, *Discrete Appl. Math.* **211** (2016), 175–182.

[5] S. Tokunaga, Dominating sets of maximal outerplanar graphs, *Discrete Appl. Math.* **161** (2013), 3097–3099.

DOI: 10.20429/tag.2020.070106