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Fractional matching preclusion for butterfly derived networks

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Abstract

The *matching preclusion number* of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost perfect matchings. As a generalization, Liu and Liu [17] recently introduced the concept of fractional matching preclusion number. The *fractional matching preclusion number* (FMP number) of G , denoted by $fmp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a fractional perfect matching. The *fractional strong matching preclusion number* (FSMP number) of G , denoted by $fsm(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a fractional perfect matching. In this paper, we study the fractional matching preclusion number and the fractional strong matching preclusion number for butterfly network, augmented butterfly network and enhanced butterfly network.

Keywords: Matching; Fractional matching preclusion; Fractional strong matching preclusion; Butterfly network; Augmented butterfly network; Enhanced butterfly network

AMS subject classification 2010: 05C70, 05C72.

1 Introduction

Let G be a graph. A *matching* M in a graph is a set of pairwise non-adjacent edges. A *perfect matching* in the graph is a set of edges such that every vertex is incident with exactly one edge in this set. An *almost-perfect matching* in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. All graphs considered in this paper are undirected, finite and simple. We refer to the book [2] for graph theoretical notations and terminology not defined here. For a graph G , let $V(G)$, $E(G)$, and $[u, v]$ (uv for short) denote the set of vertices, the set of edges, and the edge whose end vertices are u and v , respectively. We use $G - F$ to denote the subgraph of G obtained by removing all elements of F . A set F of edges in a graph $G = (V, E)$ is called a *matching preclusion set* if $G - F$ has neither a perfect matching nor an almost-perfect matching. The *matching preclusion number* of graph G , denoted by $mp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. The concept of matching preclusion was introduced by Brigham et al. [1]. Matching preclusion has also connections to a number of theoretical topics, including conditional connectivity and extremal graph theory, and was further studied in [3–13, 15, 20], with special attention given to interconnection networks.

In [20], the concept of strong matching preclusion was introduced. The *strong matching preclusion number* of a graph G , denoted by $smp(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. According to the definition of $mp(G)$ and $smp(G)$, we have that $smp(G) \leq mp(G)$.

A *fractional matching* is a function f that assigns to each edge a number in $[0, 1]$ so that $\sum_{e \sim v} f(e) \leq 1$ for each vertex v , where the sum is taken over all edges e incident with v .

Clearly,

$$\sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \sim v} f(e) \leq \frac{|V(G)|}{2}.$$

A *fractional perfect matching* is a fractional matching f satisfying that $\sum_{e \sim v} f(e) = 1$ for every $v \in V(G)$. Clearly, a fractional matching f is perfect if and only if $\sum f(e) = \frac{|V(G)|}{2}$ and a perfect matching is a fractional perfect matching. Many further ideas and results on fractional graph theory can be found in [24].

In organic molecule graphs, *perfect matchings* correspond to *kekulé structures*, playing an important role in analysis of the resonance energy and stability of hydrocarbon compounds. Cyvin and Gutman systematically gave [14] detailed enumeration formulas for kekulé structures of various types of benzenoids. Kardos et al. showed [16] that fullerene graphs have exponentially many kekulé structures. For the details of anti-kekulé number can be found in [22].

Recently, Liu and Liu in [17] introduced some natural and nice generalizations of the above concepts. An edge subset F of G is a *fractional matching preclusion set* (FMP set for short) if $G - F$ has no fractional perfect matchings. The *fractional matching preclusion number* (FMP number for short) of G , denoted by $fmp(G)$, is the minimum size of FMP sets of G , that is, $fmp(G) = \min\{|F| : F \text{ is an FMP set}\}$. We refer the readers to [17] for more details and additional references.

A set F of edges and vertices of G is a *fractional strong matching preclusion set* (FSMP set for short) if $G - F$ has no fractional perfect matchings. The *fractional strong matching preclusion number* (FSMP number for short) of G , denoted by $fsmf(G)$, is the minimum size of FSMP sets of G , that is, $fsmf(G) = \min\{|F| : F \text{ is an FSMP set}\}$.

An FMP (FSMP) set of minimum cardinality is called *optimal*. An FMP set F is *trivial* if there is a single vertex of G incident to every edge in F , and an FSMP set F is *trivial* if there is a vertex v such that every vertex in F is a neighbour of v and every edge in F is incident to v . A graph G is *fractional super matched* (*fractional strongly super matched*) if every optimal FMP (FSMP) set is trivial.

We summarize some knowledge which will be needed later.

If a graph G is Hamiltonian, we can assign to each edge which is in the Hamiltonian cycle of G a number $\frac{1}{2}$, and assign to other edges a number 0. It is obvious that we obtain a fractional perfect matching of G . Thus, the following proposition is immediate.

Proposition 1.1. *If a graph G is Hamiltonian, then G has a fractional perfect matching.*

Proposition 1.2. *Let G be a graph. Then $fmp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G . If the number of vertices in G is even, then $mp(G) \leq fmp(G)$.*

However, if the number of vertices in G is odd, $mp(G)$ and $fmp(G)$ do not have the same inequality relation. Some examples are given in [17].

Proposition 1.3. *Let G be a graph. Then $fsmf(G) \leq fmp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G .*

Proposition 1.4. [24] *A graph G has a fractional perfect matching if and only if $i(G - S) \leq |S|$ for every set $S \subseteq V(G)$, where $i(G - S)$ is the number of isolated vertices of $G - S$.*

The vertex set of the hypercube Q_n is the set of all binary strings of length n . Let $r \geq 1$ be an integer. The r -dimensional butterfly network denoted by $BF(r)$ has vertex set $V = (i, x) : x \in V(Q_r)$, where i is a non-negative integer and $0 \leq i \leq r$. Where any two vertices (i, x) and (j, y) of $BF(r)$ are adjacent if and only if $j = i + 1$ and either:

- (1) $x = y$ or
- (2) x differs from y in precisely the j th bit.

For $x = y$, the edge is said to be a straight edge. Otherwise, the edge is a cross edge. For fixed i , the vertex (i, x) is a vertex on layer i . The vertices at layer 0 and r are of degree 2 and all other vertices are of degree 4. Clearly, the number of vertices and edges in the butterfly network $BF(r)$ is equal to $(r + 1)2^r$ and $r2^{r+1}$, respectively. Let E_S denote the set of all straight edges in $BF(r)$, and E_C denote the set of all cross edges in $BF(r)$ such that $E_S \cup E_C = E(BF(r))$. In [18] Manuel et al. proved that perfect matchings do not exist for a butterfly network of even dimension. $BF(r) - F'$ is an induced subgraph obtained by deleting F' , where F' is the set of vertices or edges.

For convenience, we give some symbols as follows. When r is odd, $BF(r)$ has perfect matchings M_S and M_C , where M_S is given by all the straight edges namely $[(i, x_1x_2 \cdots x_r), (i + 1, x_1x_2 \cdots x_r)]$ with i is even and $0 \leq i \leq r - 1$, and M_C is given by all the cross edges namely $[(i, x_1x_2 \cdots x_r), (i + 1, x_1x_2 \cdots x_i\bar{x}_{i+1} \cdots x_r)]$ with i is even and $0 \leq i \leq r - 1$.

Butterfly networks have many weaknesses. It is non-Hamiltonian, not pancyclic and its toughness is less than one. But augmented butterfly network retains most of the favorable properties.

Let $n \geq 1$ be an integer. The vertices of the n -dimensional augmented butterfly network denoted by $ABF(n)$ are the pairs (r, x) where r is a non-negative integer $0 \leq r \leq n$ called the layer, and $x \in V(Q_n)$. In $ABF(n)$ the vertex (r, x) , $1 \leq r \leq n - 1$, is adjacent to the vertices $(r + 1, x)$, $(r + 1, x_1x_2 \cdots x_r\bar{x}_{r+1}x_{r+2} \cdots x_n)$, $(r, x_1x_2 \cdots x_{r-1}\bar{x}_rx_{r+1} \cdots x_n)$ and $(r, x_1x_2 \cdots x_r\bar{x}_{r+1}x_{r+2} \cdots x_n)$. In particular, when $r = 0$, the vertex $(0, x_1x_2 \cdots x_n)$ is adjacent to the vertices $(1, x_1x_2 \cdots x_n)$, $(1, \bar{x}_1x_2 \cdots x_n)$, and $(0, \bar{x}_1x_2 \cdots x_n)$. Also when $r = n$, the vertex $(n, x_1x_2 \cdots x_n)$ is adjacent to the vertices $(n, x_1x_2 \cdots x_{n-1}\bar{x}_n)$, $(n - 1, x_1x_2 \cdots x_n)$, and $(n - 1, x_1x_2 \cdots x_{n-1}\bar{x}_n)$. Clearly, the number of vertices and edges in the augmented butterfly network $ABF(n)$ is equal to $(n + 1)2^n$ and $3n \times 2^n$, respectively [18]. The vertices at layer 0 and n are of degree 3 and all other vertices are of degree 6.

The edges between (r, x) and $(r, x_1x_2 \cdots x_{r-1}\bar{x}_rx_{r+1} \cdots x_n)$, where $0 < r \leq n$ and between (r, x) and $(r, x_1x_2 \cdots x_r\bar{x}_{r+1}x_{r+2} \cdots x_n)$, where $0 \leq r < n$ are called level edges. The edges between (r, x) and $(r + 1, x)$ are called straight edges. While the edges between (r, x) and $(r + 1, x_1x_2 \cdots x_r\bar{x}_{r+1}x_{r+2} \cdots x_n)$, where $0 \leq r \leq n - 1$ are called cross edges.

Let E_S denote the set of all straight edges in $ABF(n)$, E_C denote the set of all cross edges in $ABF(n)$, and E_L denote the set of all level edges in $ABF(n)$ such that $E_S \cup E_C \cup E_L = E(ABF(n))$. Thus $E(ABF(n))$ is partitioned into three categories of edges namely straight, cross and level edges.

Consider the r -dimensional butterfly network $BF(r)$. Place a new vertex in each 4-cycle of $BF(r)$ and join this vertex to the four vertices of the 4-cycle. The resulting graph is called an enhanced butterfly network $EBF(r)$. This network has $2^{r-1}(3r + 2)$ vertices and $r \times 2^{r+2}$ edges. Additional vertices are given by the labels $(x_1x_2 \cdots x_r, x_1x_2 \cdots \bar{x}_{i+1} \cdots x_r)$ with the labels $(i, x_1x_2 \cdots x_r)$, $(i, x_1x_2 \cdots \bar{x}_{i+1} \cdots x_r)$, $(i + 1, x_1x_2 \cdots x_r)$, $(i + 1, x_1x_2 \cdots \bar{x}_{i+1} \cdots x_r)$, $0 \leq i \leq r - 1$. Let L_i denote the set of vertices $\{(i, x_1x_2 \cdots x_r) | x_1x_2 \cdots x_r \in V(Q_r)\}$, and L'_j

denote the set of vertices $\{(x_1x_2 \cdots x_r, x_1x_2 \cdots \bar{x}_j \cdots x_r) \mid x_1x_2 \cdots x_r \in V(Q_r)\}$.

In this paper, we investigate the fractional matching preclusion (FMP) number and fractional strong matching preclusion (FSMP) number of butterfly network, augmented butterfly network and enhanced butterfly network.

2 Butterfly Network

Theorem 2.1. *Let $r \geq 1$ be an integer. Then*

$$fmp(BF(r)) = \begin{cases} 2 & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

Proof. We first consider the case that r is odd. Since $\delta(BF(r)) = 2$, it follows from Proposition 1.2 that $fmp(BF(r)) \leq 2$. In order to obtain our result, we only show that $fmp(BF(r)) \geq 2$, that is, $BF(r) - \{e\}$ has a fractional perfect matching for any edge e of $BF(r)$. If we delete any straight edge from M_S , then there exists a perfect matching M_C in $BF(r)$. If we delete any cross edge from M_C , then there exists a perfect matching M_S in $BF(r)$. Then $fmp(BF(r)) > 1$, and have $fmp(BF(r)) = 2$. Next, we consider the case that r is even. Let $S = \{(i, x)\}$, where i is odd and $1 \leq i \leq r - 1$. Since $i(BF(r) - S) = r2^{r-1} + 2^r > |S| = r2^{r-1}$, it follows from Proposition 1.4 that $BF(r)$ has no fractional perfect matchings, and have $fmp(BF(r)) = 0$. The proof is now complete. \square

Remark 1. Let $F = \{(0, 00 \cdots 0), (1, 00 \cdots 0)\}, [(1, 00 \cdots 0), (0, 10 \cdots 0)]\}$. Then $i(BF(r) - F - \{(1, 10 \cdots 0)\}) = 2 > 1$, which implies $BF(r) - F$ does not have a fractional perfect matching from Proposition 1.4. Thus, we have $BF(r)$ is not fractional super matched.

Theorem 2.2. *Let $r \geq 1$ be an integer. Then*

$$fsmf(BF(r)) = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

Proof. We first consider the case that r is odd. Let $F = \{(1, 00 \cdots 0)\}, S = \{(1, 10 \cdots 0)\}$. Since $i(BF(r) - F - S) = 2 > |S| = 1$, it follows from Proposition 1.4 that $BF(r) - F$ has no fractional perfect matchings. It remains for us to show $fsmf(BF(r)) \geq 1$, that is, $BF(r)$ has a fractional perfect matching. Since there are perfect matchings in $BF(r)$, it follows that $fsmf(BF(r)) > 0$, and have $fsmf(BF(r)) = 1$. Next, we consider the case that r is even. By Theorem 2.1, $fmp(BF(r)) = 0$ if r is even, which implies $BF(r)$ has no fractional perfect matchings. So $fsmf(BF(r)) = 0$. The proof is now complete. \square

3 Augmented Butterfly Network

Lemma 3.1. [21] *Let G be the augmented butterfly network $ABF(n)$. Then $mp_1(G) = 3$.*

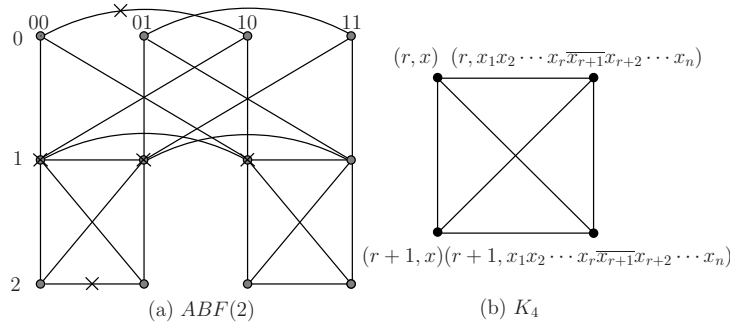


Figure 1: (a) $ABF(2)$, (b) The induced subgraph of $ABF(n)$

Theorem 3.2. *Let $n \geq 2$ be an integer. Then*

$$fmp(ABF(n)) = \begin{cases} 2 & \text{if } n = 2, \\ 3 & \text{if } n \geq 3. \end{cases}$$

Proof. We first consider the case that $n = 2$. Let $F = \{[(0, 00), (0, 10)], [(2, 00), (2, 01)]\}$, $S = \{(1, 00), (1, 01), (1, 10)\}$. Then $i(ABF(2) - F - S) = 4 > |S| = 3$ (see Figure 1 (a)), it follows from Proposition 1.4 that $ABF(2) - F$ has no fractional perfect matchings, and have $fmp(ABF(2)) \leq 2$. In order to obtain our result, we only show that $fmp(ABF(2)) \geq 2$, that is, $ABF(2) - \{e\}$ has a fractional perfect matching for any edge e of $ABF(2)$. Let M_{SL} be a perfect matching consisting of all the edges of the form $\{[(1, x_1x_2), (2, x_1x_2)], [(0, x_1x_2), (0, \bar{x}_1x_2)]\}$. Let M_{LC} be a perfect matching consisting of all the edges of the form $\{[(0, x_1x_2), (1, \bar{x}_1x_2)], [(2, x_1x_2), (2, x_1\bar{x}_2)]\}$. Let M_L be a perfect matching consisting of all the edges of the form $\{[(0, x_1x_2), (0, \bar{x}_1x_2)], [(1, x_1x_2), (1, \bar{x}_1x_2)], [(2, x_1x_2), (2, x_1\bar{x}_2)]\}$. If $e \in E_S$ or $e \in E_C$, then there exists a perfect matching M_L in $ABF(n)$. If $e \in E_L$, then there exists a perfect matching M_{SL} or M_{LC} in $ABF(n)$. Thus, we have $fmp(ABF(2)) = 2$. Next, we consider the case that $n \geq 3$. Since $\delta(ABF(n)) = 3$, it follows from Proposition 1.2 that $fmp(ABF(n)) \leq 3$. It suffices to prove that $fmp(ABF(n)) \geq 3$. Since $mp_1(ABF(n)) = 3$ from Lemma 3.1, the resulting graph by deleting any two edges of $ABF(n)$ still has perfect matchings. So $fmp(ABF(n)) > 2$. Hence $fmp(ABF(n)) = 3$. The proof is now complete. \square

Remark 2. Let $F = \{[(0, 00 \cdots 0), (0, 10 \cdots 0)], [(0, 00 \cdots 0), (1, 00 \cdots 0)], [(1, 00 \cdots 0), (0, 10 \cdots 0)]\}$. Then $i(BF(r) - F - \{(1, 10 \cdots 0)\}) = 2 > 1$, which implies $ABF(n) - F$ does not have a fractional perfect matching from Proposition 1.4. So $ABF(n)$ is not fractional super matched.

Theorem 3.3. *Let $n \geq 1$ be an integer. Then $fsmf(ABF(n)) = 2$.*

Proof. Let $F = \{(1, 00 \cdots 0), [(0, 00 \cdots 0), (0, 10 \cdots 0)]\}$, $S = \{(1, 10 \cdots 0)\}$. Since $i(ABF(n) - F - S) = 2 > |S| = 1$, it follows from Proposition 1.4 that $ABF(n) - F$ has no fractional perfect matchings, and have $fsmf(ABF(n)) \leq 2$. Next, we need prove that $fsmf(ABF(n)) \geq 2$, that is, the resulting graph deleting any vertex or any edge of $ABF(n)$ still has fractional perfect matchings. By deleting any edge e of $ABF(n)$, the resulting graph $ABF(n) - \{e\}$ still has a perfect matching by Theorem 3.2. So we only consider the case that

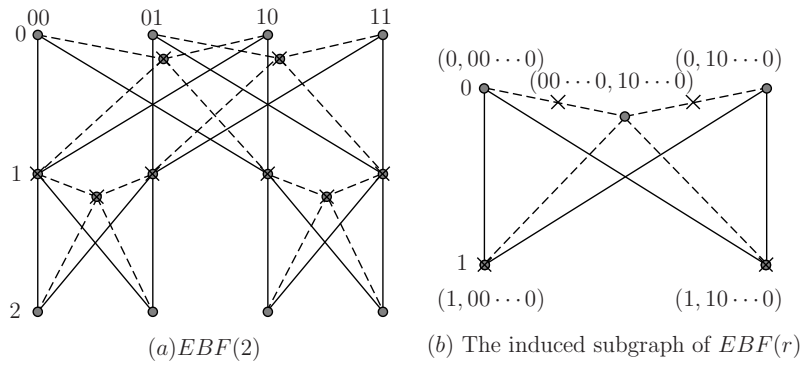


Figure 2: (a) $EBF(2)$, (b) The induced subgraph of $EBF(r)$

we delete a vertex (r, x) from $ABF(n)$. Thus there exists three neighbors of (r, x) such that they induces a complete graph say K_4 (see Figure 1 (b)). It is obvious that $ABF(n) - K_4$ has a perfect matching induced by level edges. As we can see, deleting any vertex of $ABF(n)$ can be regarded as deleting any vertex of K_4 . Clearly, after deleting any vertex of K_4 , the resulting graph still has a fractional perfect matching. After deleting any vertex of $ABF(n)$, the resulting graph still has fractional perfect matchings. Hence $fsmf(ABF(n)) > 1$. So $fsmf(ABF(n)) = 2$. The proof is now complete. \square

4 Enhanced Butterfly Network

Lemma 4.1. [21] *Let G be the enhanced butterfly network $EBF(r)$, $r \geq 2$. Then G has a perfect matching.*

Lemma 4.2. [21] *Let G be the enhanced butterfly network $EBF(r)$, $r \geq 2$. Then $mp_1(G) = 2$.*

Theorem 4.3. *Let $r \geq 2$ be an integer. Then $fsmf(EBF(r)) = 2$.*

Proof. When $r \geq 2$, any enhanced butterfly network $EBF(r)$ has a induced subgraph; see Figure 2 (b). Let $F = \{[(0, 00 \dots 0), (00 \dots 0, 10 \dots 0)], [(00 \dots 0, 10 \dots 0), (0, 10 \dots 0)]\}$, $S = \{(1, 00 \dots 0), (1, 10 \dots 0)\}$. Since $i(EBF(r) - F - S) = 3 > |S| = 2$, it follows from Proposition 1.4 that $EBF(r) - F$ has no fractional perfect matchings, and have $fsmf(EBF(r)) \leq 2$. Next, we only prove that $fsmf(EBF(r)) \geq 2$. From Lemma 4.2, we have $mp_1(EBF(r)) = 2$, and have the resulting graph by deleting any an edge of $EBF(r)$ still has perfect matchings. Hence $fsmf(EBF(r)) > 1$. So $fsmf(EBF(r)) = 2$. The proof is now complete. \square

Theorem 4.4. *Let $r \geq 2$ be an integer. Then*

$$fsmf(EBF(r)) = \begin{cases} 1 & \text{if } r = 2, \\ 2 & \text{if } r \geq 3. \end{cases}$$

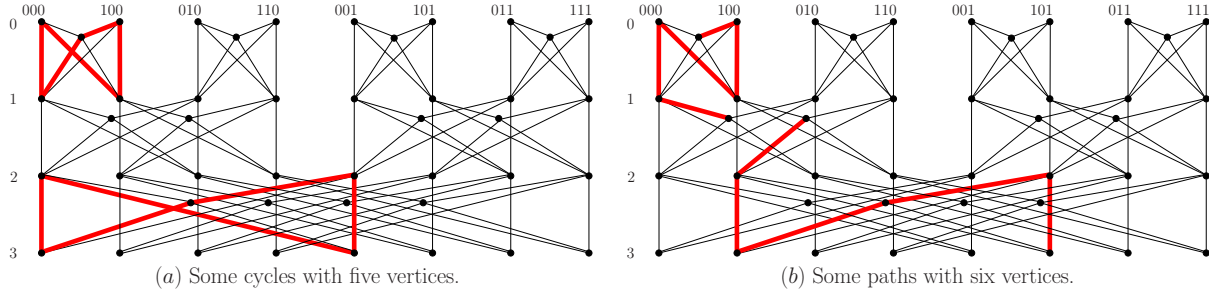


Figure 3: (a) Some cycles with five vertices, (b) Some paths with six vertices.

Proof. We first consider the case that $r = 2$. Let $F = \{(1, 00)\}$ and $S = \{(00, 10), (01, 11), (00, 01), (10, 11), (1, 01), (1, 10), (1, 11)\}$. Since $i(EBF(2) - F - S) = 8 > |S| = 7$ (see Figure 2(a)), it follows from Proposition 1.4 that $EBF(2) - F$ has no fractional perfect matchings, and have $fsmpt(EBF(2)) \leq 1$. In order to obtain our result, we have only to prove that $fsmpt(EBF(2)) \geq 1$, that is, $EBF(2)$ has a fractional perfect matching. From Lemma 4.1, any enhanced butterfly network $EBF(r)$ has a perfect matching, and have $fsmpt(EBF(2)) \geq 1$. So $fsmpt(EBF(2)) = 1$. Next, we consider the case that $r \geq 3$. Let $F = \{(1, 00 \cdots 0), (00 \cdots 0, 10 \cdots 0)\}$, $S = \{(1, 10 \cdots 0)\}$. Then $i(EBF(r) - F - S) = 2 > |S| = 1$. So $EBF(r) - F$ has no fractional perfect matchings by Proposition 1.4, and we have $fsmpt(EBF(r)) \leq 2$. It suffices to prove that $fsmpt(EBF(r)) \geq 2$, that is, after deleting any vertex or any edge of $EBF(r)$, the resulting graph still has a fractional perfect matching. By deleting any edge e of $EBF(r)$, the resulting graph $EBF(r) - \{e\}$ still has a perfect matching by Theorem 4.3. So we only need to consider the case where we delete some vertex v of $EBF(r)$. Let us prove the result by induction on r . We begin with $r = 3$. If $v \in (L_i \cup L'_1 \cup L'_3)$ for $i = 0, 1, 2, 3$, then there exist 8 vertex-disjoint cycles with five vertices such that v is a vertex of a cycle with five vertices from Figure 3 (a). Note that the additional vertex $(x_1x_2x_3, x_1\bar{x}_2x_3)$ of L'_2 is adjacent to either the vertex $(1, x_1x_2x_3)$ or the vertex $(2, x_1x_2x_3)$, so it is easy to find two different paths with six vertices whose end-vertex is $(x_1x_2x_3, x_1\bar{x}_2x_3)$; see Figure 3 (b). So $EBF(3) - \{v\}$ can be decomposed into a path with four vertex, 4 paths with six vertices and 3 cycles with five vertices. Thus, $EBF(3) - \{v\}$ has a fractional perfect matching. If $v \in L'_2$, then $EBF(3) - \{v\}$ can be decomposed into 3 paths with six vertices, and 5 cycles with five vertices. Thus, $EBF(3) - \{v\}$ has a fractional perfect matching. Next, we consider the case that $r = 4$. If $v \in (L_i \cup L'_1 \cup L'_4)$ for $i = 0, 1, 3, 4$, then there exist 16 vertex-disjoint cycles with five vertices such that v is a vertex of a cycle with five vertices; see Figure 4. Note that, there exists a perfect matching M of vertices in L_2, L'_2 and L'_3 ; see Figure 4. It follows that $EBF(4) - \{v\}$ has a fractional perfect matching. If $v \in (L_2 \cup L'_2 \cup L'_3)$, then v' is adjacent to a vertex of a cycle with five vertices, where $vv' \in M$. It is easy to find a path with six vertices whose end-vertex is v' ; see Figure 5. Thus $EBF(4) - \{v\}$ has a fractional perfect matching. Assume that the argument is true for $EBF(r - 2)$. We need to show the argument is true for $EBF(r)$. Let H be a subgraph of $EBF(r)$ induced by $L_r \cup L'_r \cup L_{r-1} \cup L'_{r-1}$. Note that the vertex $(x_1x_2 \dots x_r, x_1x_2 \dots x_{r-2}\bar{x}_{r-1}x_r)$ of L'_{r-1} is adjacent to either the vertex $(r - 2, x_1x_2 \dots x_r)$ or the vertex $(r - 1, x_1x_2 \dots x_r)$. Since the subgraph induced by $L_r \cup L'_r \cup L_{r-1}$ can be decomposed into 2^{r-1} vertex-disjoint cycles with five vertices, H can be decomposed into 2^{r-1}

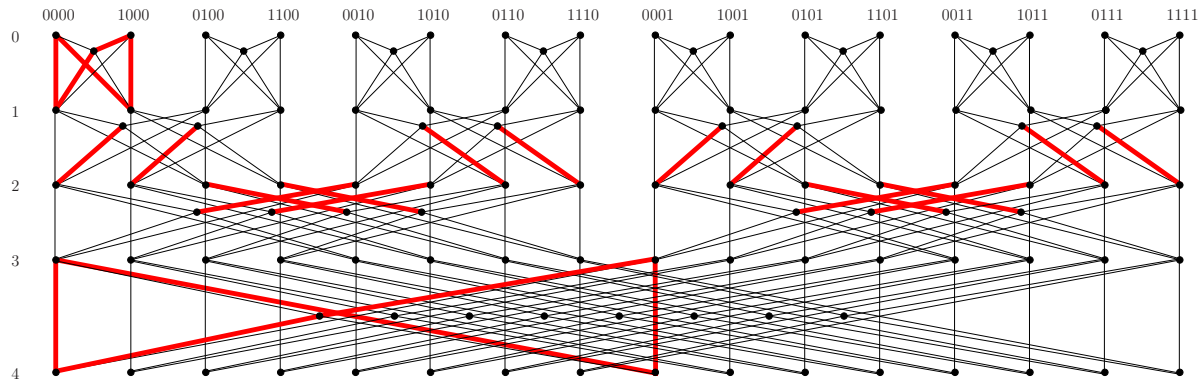


Figure 4: Some cycles with five vertices and a perfect matching M .

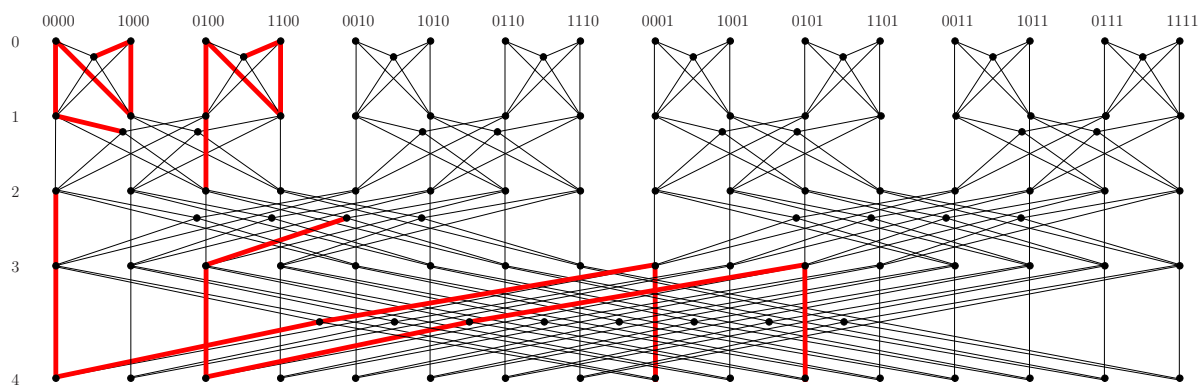


Figure 5: Some paths with six vertices.

vertex-disjoint paths with six vertices whose end-vertex is $(x_1x_2 \dots x_r, x_1x_2 \dots x_{r-2}\overline{x_{r-1}}x_r)$. Clearly, $EBF(r)$ can be decomposed into a subgraph H and 4 subgraphs that is isomorphic to $EBF(r - 2)$, say G_i for $i = 1, 2, 3, 4$. If v is a vertex of G_i , then $G_i - \{v\}$ has fractional perfect matchings by induction hypothesis, where $i \in \{1, 2, 3, 4\}$. Moreover, G_j has fractional perfect matchings for $j \neq i$. Hence $EBF(r) - \{v\}$ has a fractional perfect matching. We now consider the case that v is a vertex of H . If $v \in L'_{r-1}$, then $H - \{v\}$ can be decomposed into a cycle with five vertices and $2^{r-1} - 1$ paths with six vertices. Hence $EBF(r) - \{v\}$ has a fractional perfect matching. If $v \in (L_r \cup L'_r \cup L_{r-1})$, then there exists a path six vertices such that v is a vertex of this path. Let u be a end-vertex of this path such that $u \in L'_{r-1}$. Note that u is adjacent to a vertex in G_i , say u' , where $i \in \{1, 2, 3, 4\}$. It is not difficult to see that $H - \{v, u\}$ can be decomposed into a path with four vertices and $2^{r-1} - 1$ paths with six vertices. Since $G_i - \{u'\}$ has a fractional perfect matching by induction hypothesis, where $i \in \{1, 2, 3, 4\}$, $EBF(r) - \{v\}$ has a fractional perfect matching. Hence $fsm_p(EBF(r)) > 1$. So $fsm_p(EBF(r)) = 2$. The proof is now complete. \square

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