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Gallai-Ramsey and vertex proper connection numbers

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ABSTRACT

Given a complete graph G, we consider two separate scenarios. First, we consider the minimum number N such that every coloring of G using exactly k colors contains either a rainbow triangle or a monochromatic star on t vertices. This number is known for small cases and generalized for larger cases for a fixed k. Second, we introduce the vertex proper connection number of a graph and provide a relationship to the chromatic number of minimally connected subgraphs. Also a notion of total proper connection is introduced and a question is asked about a possible relationship between the total proper connection number and the vertex and edge proper connection numbers.

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Chapter 1

Basic Graph Theory

1.1 Introduction

The origin of graph theory can be traced back to the early 18th century and the famous Königsberg bridge problem. In 1736, Leonard Euler presented the problem of the seven bridges of Königsberg. There were seven bridges connecting two islands of Königsberg Germany, and the goal was to find a path traversing all seven bridges exactly once. He very quickly realized that there was no such path, a proposition that has many different proofs today. Another famous question in graph theory is the four color question. Suppose one wanted to color a map such that no two countries of the same color share an edge. It was found that four colors was enough to accomplish this task, but it took centuries for a valid proof to emerge. Since this birth of graph theory, the study of graphs has been expanded upon by many famous mathematicians such as Leibniz, Listing, and Cayley and though its early discoveries had many pratical applications in the field of geography, graph theory has since grown into its own unique discipline.

Graph theory now has many different branches with diverse applications. A prominent applicaiton of graph theory would be in the field of cyber security and internet connections. The connections between computers can be thought of as a large graph, where the "wires" connecting each computer are the edges and the computers themselves the vertices. The reliability of the network can be studied using connectivity of the graph. By coloring the edges or vertices of the graph, one can model its network security, or lack thereof. Many other aspects of computer networks can also be modeled by placing restrictions upon the graph's colors, connection numbers, and chromatic properties which may reflect firewalls, cookies, or something similar.

1.2 Some Defintions

A *graph* G is a collection of *vertices* v and *edges* e such that the edges connect pairs of distinct vertices. If not all vertices are associated with an edge, these are refered to as *singletons*. The collection of vertices of a graph G is denoted by $V(G)$, and the collection of edges of a graph G is denoted by $E(G)$. A vertex v is said to be *adjacent* to a vertex u if they are connected by some edge e, denoted as vu. A graph G is said to be *connected* if between all pairs of vertices $v, u \in V(G)$ there exists an edge e. The *degree* of a graph can be associated to both the number of vertices or the number of edges, denoted $|V(G)|$ and $|E(G)|$ respectively, where it defines the number of edges extending from each vertex.

Some graphs may have only one edge associated to each pair of vertices, while others may have multiple edges. A graph is said to be a *multigraph* if it has multiple edges between vertices. A *path* P is defined as a sequence of edges e_i for $i \in I$ connecting a sequence of vertices v_j such that all $v_j's$ for $i \in I$ are distinct.

If each vertex or edge of a graph G can be associated with a certain label or coloring, G is called *colored*. If a graph can be divided into two subsets X, Y such that each edge has one vertex in X and the other in Y, then it is called *bipartite*.

The *diameter* of a graph is the longest length (number of edges) of a shortest $u - v$

paths over all pairs of vertices u and v. A coloring of the vertices of a graph is called *proper* if no two adjacent vertices receive the same color. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to properly color the graph G.

Chapter 2

Gallai Ramsey Numbers

2.1 Introduction

In this section, when it is clear from the context, we will associate a graph G with either its edge set or its vertex set.

Throughout this research, we consider edge colorings of complete graphs on n vertices, denoted K_n, graphs in which an edge is present between every pair of vertices. In particular, K_3 is frequently called a triangle since it consists of three vertices and three edges. A colored graph is called *rainbow* if every edge has a distinct color.

Definition 1. *A coloring of a complete graph G is said to be a* Gallai coloring *if this coloring contains no rainbow triangles.*

The following generalization of Ramsey numbers has gotten much attention in recent years. With a formal definition in [3], the function has been studied in [4, 5, 9, 10, 11, 12] among others, with a survey of known results presented in [6].

Definition 2. *The* Gallai-Ramsey number*, denoted $gr_k(H : G)$, is the minimum number N such that every coloring of a complete graph on at least N vertices using at most k colors will contain either a rainbow copy of the graph G or a monochromatic copy of the graph H.*

A very slight change in the definition, forcing the available colors to appear, yields the following.

Definition 3. *The* exact Gallai-Ramsey number*, denoted $gr'_k(H : G)$, is the minimum number N such that every coloring of a complete graph on at least N vertices using exactly k colors will contain either a rainbow copy of the graph G or a monochromatic copy of the graph H.*

The overall goal of this particular section of research was to find, or at least bound, exact Gallai-Ramsey numbers. This problem stems from what is already known about Gallai-Ramsey numbers. It turns out that the exact Gallai-Ramsey numbers differ from Gallai-Ramsey numbers only if the monochromatic graph G in question is a star, that is, a single vertex with an edge to all other vertices and no other edges in the graph. Call a star with t edges S_t and note that S_t has $t+1$ vertices.

The following theorem from [8] is utilized heavily in this proof. For this statement, a partition is non-trivial if there exist at least two parts.

Theorem 1 (Gallai - [8])**.** *In any Gallai colored complete graph, there exists a non-trivial partition of the vertices such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

Generally, the main issues with applying Theorem 1 stem from not knowing how many pieces there are in the partition other than that there are at least 2. The only thing we can say about the subgraphs within the pieces of the partition is that Theorem 1 can then be reapplied within each piece.

The following result solidifies the Gallai-Ramsey numbers for all stars.

Theorem 2 (Markström, Thomason, Wagner - [17])**.**

$$gr_k(K_3 : S_t) = \begin{cases} \dfrac{5t-3}{2}, & \textit{for } t \textit{ odd} \\ \dfrac{5t-6}{2}, & \textit{for } t \textit{ even.} \end{cases}$$

One of the other goals of this project was to demonstrate the difference between the functions gr and gr'. Certainly sometimes the functions are the same, but several small cases show that is not always true.

2.2 Small Cases

Our first result is the following.

Theorem 3. $gr_3'(K_3 : S_3) = 5$.

Note that this is strictly less than $6 = gr_3(K_3 : S_3)$.

Proof. For the lower bound, Figure 2.1 shows a coloring of a complete graph on four vertices (K_4) using three colors. Note, for this particular coloring, the use of three colors is required. This graph includes a rainbow matching (the dashed and thick edges) and another edge in the third color. In this graph, neither a rainbow colored K_3 or a monochromatic S_3 can be found. Therefore, $gr_3'(K_3 : S_3) \geq 5$.

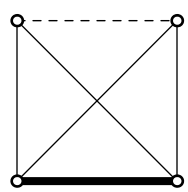

Figure 2.1: A three coloring of K_4.

For the upper bound, consider a complete graph on five vertices (K_5) using three colors. By Theorem 1, there exists a partition of the vertices. Without loss of generality, suppose colors 1 and 2 are the colors used between parts. Note that, to avoid a monochromatic S_3, there cannot be one vertex with three incident edges of the same color. This implies there cannot be a part A containing three vertices since any vertex outside of A would have only one color on edges going to A, creating a monochromatic S_3. Also, there must be a part with at least two vertices since the third color must be used.

Let A be a part with order 2 and let u, v, w be the vertices outside A. Without loss of generality, u and v both have the same color, say red, to A. Then w cannot have red edges to A since otherwise there would be a red S_3 centered in A. Similarly, w cannot have red edges to either u or v since otherwise there would be a red S_3 centered at u or v. Thus, w must be the center of a blue S_3, completing the proof. □

2.3 General Results

Theorem 4. *In any colored complete graph G with k vertices using k colors, there exists a rainbow triangle.*

Proof. Let G be a complete colored graph on k vertices. Consider a subgraph H such that every edge of H has a different color and all k colors are used. Since $|V(H)| = |E(H)| = k$, H must contain a cycle. This cycle must be rainbow colored because H itself is rainbow colored, implying there exists a rainbow cycle in G. Let C be the smallest rainbow cycle in G. If C is a rainbow triangle, then the proof is complete. If C is not a rainbow triangle, choose a chord e in C. This chord may share a color with at most one edge of C. If e does share a color with an edge in C, say f, then travel the other way (avoiding f) around the cycle, this creates a shorter rainbow cycle, contradicting the choice of C. If e does not share a color with an edge in C, then using e and traveling either way around the cycle C will create a shorter rainbow cycle, contradicting the choice of C. □

Chapter 3

Vertex and Total Proper Connection Numbers

3.1 Introduction

All graphs considered in this work are simple, finite and undirected. Unless otherwise noted, by a coloring of a graph, we mean a vertex-coloring, not necessarily proper.

Now well studied, the *(edge) rainbow k-connection number* of a graph is the minimum number of colors c such that the edges of the graph can be colored so that between every pair of vertices, there exist k internally disjoint rainbow edge-colored paths. See [13, 14] for surveys of results about the rainbow connection number. Note that the rainbow 1-connection number is related, at least conceptually, to the diameter of the graph.

The *total rainbow k-connection number*, defined in [15], is defined to be the minimum number of colors c such that the edges and vertices of the graph can be colored with c colors so that between every pair of vertices, there exist k internally disjoint rainbow paths where here rainbow means all interior vertices and edges have distinct colors. Note that we cannot require the end-vertices of the paths to also have distinct colors as that would reduce the problem to edge rainbow k-connectivity since every vertex would then be required to have a distinct color.

The *edge proper connection number* $\mathrm{pc}_k(G)$, defined in [2] and further studied in [7], is defined to be the minimum number of colors c such that the edges of the graph G can be colored with c colors such that between each pair of vertices, there exist k internally disjoint, properly edge-colored paths. One feature of edge-proper connection that makes the results extremely complicated is that proper edge-colored paths are not transitive in the sense that if there is a proper path from u to v and a proper path from v to w, there may not be a proper path from u to w. For example, let G be a path on three vertices, uvw and color both edges red.

In this work, we consider a vertex version of the edge proper connection number. For a positive integer k, a colored graph G is called *(vertex) properly k-connected* if, between every pair of vertices, there exist at least k internally disjoint properly colored paths. Note that each path, including end-vertices, must be properly colored. Given a graph G, the *vertex proper k-connection number* of the graph G, denoted $\mathrm{vpc}_k(G)$, is the minimum number of colors needed to produce a properly k-connected coloring of G. For ease of notation, let $\mathrm{vpc}(G) = \mathrm{vpc}_1(G)$.

The function $\mathrm{vpc}_k(G)$ is clearly well defined if and only if $\kappa(G) \geq k$. Also note that $\mathrm{vpc}_k(G) \leq \chi(G)$ for every k-connected graph G. Furthermore, the following fact is immediate.

Fact 1. *For all $k \geq 2$ and every k-connected graph G,* $\mathrm{vpc}_k(G) \geq \mathrm{vpc}_{k-1}(G)$.

A graph G is called *minimally k-connected* if G is k-connected but the removal of any edge from G leaves a graph that is not k-connected. A classical result of Mader [16] (also found in [1]) will immediately give us one of our upper bounds.

Theorem 5 ([16, 1])**.** *A minimally k-connected graph is $k+1$ colorable and this bound is sharp.*

3.2 General Classification

Our first observation demonstrates the transitivity of the vertex proper connection, a fact that is not true in the case of edge proper connection.

Fact 2. *In a colored graph G, if there is a proper path from u to v and a proper path from v to w, then there is a proper path from u to w.*

Proof. The proof is trivial if the $u-v$ path and the $v-w$ path intersect only at v so suppose the paths intersect elsewhere and let x be the first vertex on the path from u to v that is also on the $v-w$ path. Note that we may have $x=u$. Then the subpath of the $u-v$ path that goes from u to x and the subpath of the $v-w$ path that goes from x to w is a properly colored path and completes the proof. □

Clearly the addition of edges cannot increase the vertex proper connection number of a graph so the following fact is trivial.

Fact 3. *Given a positive integer k and a k-connected graph G, if H is a spanning k-connected subgraph of G, then $\mathrm{vpc}_k(G) \leq \mathrm{vpc}_k(H)$.*

Our main result solidifies the link between the vpc_k function and the chromatic number of the graph. It turns out that $\mathrm{vpc}_k(G)$ always equals the chromatic number of a particular subgraph of G. Let $s\chi_k(G)$ denote the smallest, over all spanning k-connected subgraphs H of G, chromatic number of H.

Theorem 6 (Classification)**.** *Given a k-connected graph G, $\mathrm{vpc}_k(G) = s\chi_k(G)$.*

Proof. Given a k-connected spanning subgraph H of G with chromatic number ℓ, color this subgraph properly with ℓ colors. Then between every pair of vertices in H, there are at least k internally disjoint properly colored paths. Thus, using Fact 3, $\mathrm{vpc}_k(G) \leq \mathrm{vpc}_k(H) = \ell$ so $\mathrm{vpc}_k(G) \leq s\chi_k(G)$.

Now let $\ell = \mathrm{vpc}_k(G)$ and consider an ℓ-coloring of G which is properly k-connected. Let $\mathscr{P}$ be the set of all proper paths between pairs of vertices (k paths for each pair of vertices). Then the subgraph H of G induced on all the edges of $\mathscr{P}$ spans G, is k-connected and has chromatic number at most ℓ. This means $\mathrm{vpc}_k(G) \geq s\chi_k(G)$, completing the proof. $\square$

3.3 Consequences of Theorem 6

Theorem 6 shows that every statement about vpc_k is a statement about the chromatic number of a minimally k-connected subgraph. Particularly, if G is minimally k-connected, then $\mathrm{vpc}_k(G) = \chi(G)$. When the graph is bipartite, we get the following easy observation.

Corollary 7. *If G is k-connected and bipartite, then for all $t \leq k$, we have* $\mathrm{vpc}_t(G) = 2$.

In light of the classification theorem, we immediately get equivalent colored "fan lemma" and "disjoint paths between k-sets" versions of the definition of vertex proper connectivity.

Corollary 8. *A colored graph G is properly k-connected if and only if for every vertex v and k-set of vertices $\{u_1, u_2, \ldots, u_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where P_i goes from v to u_i and $P_i \cap P_j = \{v\}$ for all i, j.*

Corollary 9. *A colored graph G is properly k-connected if and only if for every $2k$-set of vertices $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots v_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where P_i goes from u_i to v_j for some j and $P_i \cap P_\ell = \emptyset$ for all i, ℓ.*

Theorem 6, along with Theorem 5, also gives us the following general upper bound. The sharpness of Theorem 5 and Corollary 7 yield the sharpness of both bounds here.

Corollary 10. *If G is k-connected, then for $t \leq k$, we have $2 \leq \mathrm{vpc}_t(G) \leq t + 1$ and both bounds are sharp.*

When $k = 1$, Corollary 10 reduces to the following.

Corollary 11. *For every connected graph G on at least 2 vertices,* $\text{vpc}(G) = 2$.

3.4 Total Proper Connection

A natural definition of a total proper connection number is the following. Let $\text{tpc}(G)$ be the minimum number of colors needed to color the vertices and edges of G so that between every pair of vertices u, v, there is a path $P = P_{u,v}$ such that the vertices of P induce a properly (vertex-)colored path and the edges of P also induce a properly (edge-)colored path. Furthermore, we define $\text{tpc}_k(G)$ to be the minimum number of colors needed to produce k internally disjoint such paths between every pair of vertices.

One might think that $\text{tpc}_k(G)$ might simply be the maximum of $\text{pc}_k(G)$ and $\text{vpc}_k(G)$ but this is not obvious even when $k = 1$ since the edge path (for pc) and the vertex path (for vpc) must be the same path. Indeed, in Question 1, we ask whether this equality holds in general. Our results concerning the function tpc support a positive answer to this question.

Question 1. *Is it true that* $\text{tpc}_k(G) = \max\{\text{pc}_k(G), \text{vpc}_k(G)\}$*?*

First we recall a result of Borozan et al. [2] which was originally stated in a stronger form.

Theorem 12 ([2])**.** *If G is bipartite and 2-connected, then* $\text{pc}(G) = 2$.

Proposition 1. *If $\kappa(G) \geq 3$, then* $\text{tpc}(G) = 2$.

Proof. With $\kappa(G) \geq 3$, there is a spanning 2-connected bipartite subgraph B. Color the vertices with two colors according to this subgraph. By Theorem 12, the proper connection number of B is 2. Color the edges of B with 2 colors to be properly connected. For any pair of vertices in B, there is a properly edge-colored path between them which induces a properly vertex-colored path as well since the vertices are properly colored. This means $\mathrm{tpc}(B) = 2$. Since $B \subseteq G$, we must also have $\mathrm{tpc}(G) = 2$ as well. $\square$

Using a similar argument and Corollary 7, we easily get the following result.

Corollary 13. *If G is k-connected and bipartite, then for $t \leq k$,*

$$\mathrm{tpc}_t(G) = \max\{\mathrm{pc}_t(G), \mathrm{vpc}_t(G)\} = \mathrm{pc}_t(G).$$

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