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Colored complete hypergraphs containing no rainbow Berge triangles

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Abstract

The study of graph Ramsey numbers within restricted colorings, in particular forbidding a rainbow triangle, has recently been blossoming under the name Gallai-Ramsey numbers. In this work, we extend the main structural tool from rainbow triangle free colorings of complete graphs to rainbow Berge triangle free colorings of hypergraphs. In doing so, some other concepts and results are also translated from graphs to hypergraphs.

1 Introduction

Edge-colorings of graphs, particularly complete graphs, have been studied from a wide variety of perspectives. Among the most famous of these is certainly Ramsey theory, but even within Ramsey theory, there have been many variants. One particular direction, has been restricting the colorings to those containing no rainbow copy of some specified subgraph. In light of the following structural result, originally proven by Gallai [11] in 1967, it is natural that the rainbow triangle has received the most attention.

To be precise, a subgraph of a colored graph is called *rainbow* if its edges have been assigned all distinct colors. A colored complete graph is called a *Gallai coloring* if it contains no rainbow colored triangle.

Theorem 1 ([4, 11, 14])**.** *Every Gallai coloring of a complete graph of order at least* 2 *contains a non-trivial partition (with at least two parts) of the vertices such that between the parts there is a total of at most two colors on the edges, and between each pair of parts there is only one color on the edges.*

Such a partition is called a *Gallai partition* or *G-partition*.

The goal of this paper is to extend Theorem 1 to hypergraphs and apply it to several related areas of study. When transitioning to hypergraphs, the first question is how to translate the definitions. For example, what does "complete" mean and what shall be called a "triangle"?

Given a positive integer n, let K_n^3 denote the complete 3-uniform hypergraph on n vertices. A hypergraph with at least 4 vertices and three distinct edges e_1, e_2, e_3 is called a *Berge triangle*, denoted by BC_3, if there exist three distinct vertices, say u, v, w, with $u, v \in e_1$, $v, w \in e_2$, and $u, w \in e_3$. As in the case for graphs, a Berge triangle is called *rainbow* if the hyperedges have been assigned distinct colors.

We first consider colorings of K_n^3 containing no rainbow Berge triangles. Our first main result shows that the structure of rainbow-BC_3-free colored 3-uniform hypergraphs is actually much stronger than the structure provided by Theorem 1 for graphs. A color is called *universal* if no two hyperedges of any two other colors share a vertex. In this work, we consider only colorings of the hyperedges (not the vertices) of hypergraphs.

Theorem 2. *For a positive integer n, let H be a copy of K_n^3 and $c(H)$ be a coloring of H containing no rainbow BC_3. Then there exists a universal color.*

There are 4 non-isomorphic 3-uniform Berge triangles. See Figure 1. With this in mind, the hypothesis in Theorem 2 assumes that none of these four subhypergraphs appear using three different colors.

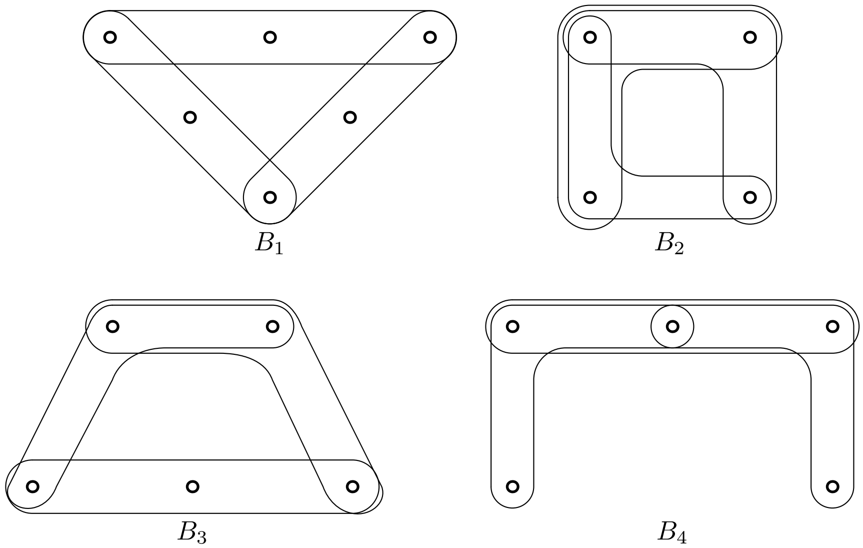

Figure 1: All 3-uniform Berge triangles

Note that B_1 is also known as a "linear" or "loose" C_3 and B_2 can also be regarded as a K_4^3 minus a single hyperedge. With this notation for the different specific Berge triangles, we can be more specific in our statements. In fact, it turns out that not all of the Berge triangles are needed in the statement of Theorem 2.

Theorem 3. *For a positive integer n, let H be a copy of K_n^3 and let $c(H)$ be a coloring of H containing no rainbow B_3. Then there exists a universal color.*

We have yet to find any indication whether this also holds true for B_1, B_2, or B_4. It is also unclear what one might find when forbidding any other Berge hypergraphs.

In order to extend these results to hypergraphs that are not 3-uniform, we first define what it means to be "complete". A hypergraph H is called 3-*complete* if, for each set of 3 vertices, there is a distinct hyperedge containing those 3 vertices. Notice that this definition does not require H to be uniform. With this definition, we get the following extension of Theorem 2.

Theorem 4. *Let H be a 3-complete hypergraph on $n \geq 4$ vertices and let $c(H)$ be a coloring of H. If $c(H)$ contains no rainbow Berge triangle, then there is a universal color.*

Note that Theorem 2 is a corollary of Theorem 4 but Theorem 3 is not. After some preliminary results though, see Lemmas 1-4, it turns out that Theorem 2 and Theorem 3 are equivalent, meaning that Theorem 4 also implies Theorem 3. We therefore only include the proof of Theorem 4, provided in Section 3.

Also observe that the conclusions of Theorems 3 and 4 imply that if H is a hyperedge-colored hypergraph satisfying either of the hypotheses, then there is a nice partition of the vertices.

Corollary 5. *Let $n \geq 4$ and let H be a hypergraph and $c(H)$ be a coloring of H containing no rainbow B_3 such that either*

- *H is a copy of K_n^3, or*
- *H is a copy of an arbitrary hypergraph (perhaps not even uniform) in which for every set of 3 distinct vertices, there is a distinct hyperedge containing those 3 vertices,*

then, up to renumbering the colors, there is a partition of the vertices of H into $H_1, H_2, \ldots, H_k$ such that all hyperedges containing vertices from more than one part H_i must have color 1 and all hyperedges contained within H_i have either color 1 or color i.

Conversely, if such a partition exists, then there is no rainbow BC_3.

Theorem 6. *If H is a hypergraph (not necessarily uniform) of order at least 4 and $c(H)$ a coloring of H for which there is a partition of the vertices into $H_1, H_2, \ldots, H_k$ such that all hyperedges containing vertices from more than one part H_i must have color 1 and all hyperedges contained within H_i have either color 1 or color i, then $c(H)$ contains no rainbow BC_3.*

Section 2 contains several helpful lemmas that are used to show that Theorem 2 and Theorem 3 are equivalent. The proof of Theorem 4 is presented in Section 3 and the proof of Theorem 6 is presented in Section 4. Finally, Section 5 contains several applications of Corollary 5 to other areas of hyperedge-colorings, extending corresponding graph notions.

2 Preliminaries

In this section, we prove some helpful implications between the existence of certain rainbow subhypergraphs in a complete 3-uniform hypergraph K_n^3.

A hypergraph with at least $n+1$ vertices and n distinct hyperedges $e_1, e_2, \ldots, e_n$ is called a *Berge path of length n* if there exist a subset of $n+1$ distinct vertices $v_1, v_2, \ldots, v_n, v_{n+1}$ such that $v_i, v_{i+1} \in e_i$ for each i with $1 \leq i \leq n$. We say that a hypergraph is *connected* if, between every pair of vertices, there is a Berge path. In particular, a path is called *tight* if each pair of consecutive hyperedges shares two vertices. Given a hyperedge-colored hypergraph H containing a hyperedge e, we denote by $c(e)$ the color of the hyperedge e. Note that B_4 is a tight path on 3 hyperedges.

Lemma 1. *For $n \geq 5$, if a coloring $c(H)$ of $H = K_n^3$ contains a rainbow B_4, then $c(H)$ contains a rainbow B_3.*

Proof. Suppose $c(H)$ contains a rainbow B_4, say using hyperedges $e_1 = uvw$ in color 1, $e_2 = uvx$ in color 2, and $e_3 = vwy$ in color 3. If $c(wxy) \notin \{2, 3\}$, then wxy along with e_2 and e_3 form a rainbow B_3. If $c(wxy) = 3$, then wxy along with e_1 and e_2 form a rainbow B_3. These observations together imply that $c(wxy) = 2$, and symmetrically, $c(uxy) = 3$. Then wxy in color 2, uxy in color 3 and e_1 in color 1 form a rainbow B_3, completing the proof. □

Since B_4 is a tight path on 3 hyperedges, the following is a corollary of Lemma 1.

Lemma 2. *If a coloring $c(H)$ of $H = K_n^3$ with $n \geq 5$ contains a rainbow tight path on 3 hyperedges, then $c(H)$ contains a rainbow B_3.*

Lemma 3. *For $n \geq 6$, if a coloring $c(H)$ of $H = K_n^3$ contains a rainbow B_1, then $c(H)$ contains a rainbow B_3.*

Proof. Suppose $c(H)$ contains a rainbow B_1, say using hyperedges uvw with color 1, wxy with color 2, and yzu with color 3. Then regardless of the color of uwy, there is a rainbow tight path on three hyperedges. By Lemma 2, the proof is complete. □

Lemma 4. *If a coloring $c(H)$ of $H = K_n^3$ with $n \geq 5$ contains a rainbow B_2, then $c(H)$ contains a rainbow B_3.*

Proof. Suppose $c(H)$ is a coloring of $H = K_n^3$ with a rainbow B_2, say with hyperedges $e_1 = abc$ in color 1, $e_2 = bcd$ in color 2, and $e_3 = cda$ in color 3. Let f be another vertex in H and consider the hyperedge bdf. If bdf has color 2 or a new color, then this hyperedge along with e_1 and e_3 form a rainbow B_3 as claimed. This edge therefore has either color 1 or 3 so by symmetry, we may assume $c(bdf) = 1$. Then the edges fbd in color 1, $e_2 = bdc$ in color 2, and $e_3 = dca$ in color 3 forms a rainbow tight path on 3 hyperedges. By Lemma 2, the desired rainbow copy of B_3 is present. □

With these lemmas in place, we observe the equivalence of our first two main results.

Corollary 7. *Theorem 2 and Theorem 3 are equivalent.*

Proof. That Theorem 3 implies Theorem 2 is immediate so suppose we assume the hypothesis of Theorem 3, that the coloring of K_n^3 in question contains no rainbow B_3. By Lemmas 1, 3, and 4, the hypergraph also contains no rainbow B_4, B_1, or B_2 respectively. Thus, there is no rainbow BC_3, which is the hypothesis of Theorem 2, completing the reverse implication. □

3 Proof of Theorem 4

Proof. Let H be a 3-complete hypergraph on $n \geq 4$ and $c(H)$ be a coloring of H containing no rainbow BC_3. If the number of colors used in $c(H)$ is at most 2, then the result is immediate so suppose at least 3 colors appear on the hyperedges of H.

We first prove a claim about the color degree of each vertex.

Claim 1. *Every vertex is contained in hyperedges of at most two different colors.*

Proof. Suppose, for a contradiction, that a vertex v is contained in hyperedges of three different colors, say e_i of color i for $1 \leq i \leq 3$. Since these hyperedges are distinct, there must exist a set of three distinct vertices $v_i \in H \setminus \{v\}$ such that $v_i \in e_i$ for each $1 \leq i \leq 3$. Then a hyperedge e_0 containing vertices $\{v_1, v_2, v_3\}$ must be present and colored so let $c = c(e_0)$. Regardless of the color $c_0 = c(e_0)$, there exist indices i and j such that $c_0 \notin \{c(e_i), c(e_j\}$. Say for example, if $c_0 = 2$, then e_1 and e_3 have different colors than e_0. Then $\{e_0, e_i, e_j\}$ forms a rainbow BC_3, for a contradiction. □

Since every pair of vertices is contained in a hyperedge, certainly every pair of vertices shares at least one color on their incident edges. If every vertex has a color in common, then Claim 1 implies that this is the desired universal color so suppose not. Then, without loss of generality, there must be three vertices v_1, v_2, v_3 such that v_1 has incident hyperedges in

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colors 1 and 2, v_2 has incident hyperedges in colors 2 and 3, and v_3 has incident hyperedges in colors 1 and 3. Then regardless of the color of the edge $v_1v_2v_3$, one of these vertices will have three different colors on incident hyperedges, contradicting Claim 1 and completing the proof of Theorem 4. □

4 Proof of Theorem 6

Proof. For a contradiction, suppose there is a rainbow BC_3 in $c(H)$, and let u, v, w be the representative vertices for this triangle. If all three of u, v, w are in a single part of the assumed partition, then every edge containing any of these three vertices come from only two colors, the universal color and the color corresponding to the part, contradicting the assumption that the BC_3 was rainbow. On the other hand, if at least two of the vertices u, v, w are in different parts, then at least two of the hyperedges in the BC_3 are crossing, and so must have the universal color, again contradicting the assumption that the BC_3 was rainbow. This completes the proof of Theorem 6. □

5 Applications of Corollary 5

In this section, we consider applications of Corollary 5 to other areas related to Ramsey-type questions.

5.1 Gallai-Ramsey Numbers

Using Theorem 1, many authors have considered a variant of Ramsey numbers in which a rainbow triangle is forbidden. Given a positive integer k and graphs G and H, let $gr_k(G : H)$ be the minimum integer N such that every k-coloring of the edges of K_n for $n \geq N$ contains either a rainbow copy of G or a monochromatic copy of H. See [10] for many results on this topic. Naturally, the most common choice for the graph G is the triangle. In particular, the sharp value of $gr_k(K_3 : K_n)$ was conjectured in [7]. The case $n = 3$ was proven in [5] and the case $n = 4$ was recently settled in [20] but the conjecture in general remains open and likely very difficult since it depends on unknown classical 2-color Ramsey numbers for complete graphs.

Let $h^rR_k(H)$ be the k-colored *hypergraph Ramsey number* for finding a monochromatic copy of a given r-uniform hypergraph H within a k-colored complete r-uniform hypergraph. We refer the interested reader to [2, 12, 13, 15, 16, 17, 18] for several results related to this topic.

We now define the hypergraph version of the Gallai-Ramsey number.

Definition 1. *Given connected r-uniform hypergraphs G and H, define the k-color* hypergraph Gallai-Ramsey number *$h^rgr_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-coloring of the hyperedges of K_n^r contains either a rainbow G or a monochromatic H in some color.*

Note that the hypergraph Gallai-Ramsey number clearly exists since it is bounded from above by the hypergraph Ramsey numbers, which are known to exist. By Corollary 5, we can easily extend 2-color hypergraph Ramsey numbers to any number of colors. This result follows from Corollary 5 by simply merging all non-universal colors into a single color.

Corollary 8. *For an integer $k \geq 2$ and a connected hypergraph H,*

$$hgr_k(B_3 : H) = h^3 R_2(H).$$

5.2 Anti-Ramsey Numbers

Given graphs G and H, the *anti-Ramsey number* $ar(G, H)$ is defined to be the maximum number of colors k such that there exists a coloring of the edges of G using k colors in which every copy of H as a subgraph of G has at least two edges of the same color. In other words, the anti-Ramsey number is the maximum number of colors that can be used within G while avoiding a rainbow copy of H. Since their introduction by Erdős, Simonovits, and Sós [6], anti-Ramsey numbers have been well studied from many perspectives. See [10] for an updated list of results in the area.

Despite a robust literature on anti-Ramsey numbers for graphs, the hypergraph version has seen very little attention, the most visible results being contained in only three publications [1, 19, 24]. For completeness, given hypergraphs G and H, the *hypergraph anti-Ramsey number* $har(G, H)$ is defined to be the maximum number of colors k such that there exists a coloring of the hyperedges of G using k colors in which every copy of H as a subhypergraph of G has at least two edges of the same color.

Using Corollary 5, we easily obtain the precise anti-Ramsey number for avoiding a rainbow B_3 within K_n^3.

Corollary 9. *For all $n \geq 3$,*

$$har(K_n^3, B_3) = \left\lfloor \frac{n}{3} \right\rfloor + 1.$$

5.3 Monochromatic Connectivity

Given fixed positive integer parameters k and m, a positive integer n, and some function $f(k, m)$ (not depending on n), a subgraph R of an m-edge-coloring of K_n is called *almost spanning* if $|R| \geq n - f(k, m)$. The search for almost spanning monochromatic highly connected subgraphs was initiated, in this form, by Bollobás and Gyárfás in 2008 with the following conjecture.

Conjecture 1 (Bollobás and Gyárfás [3])**.** *If $n > 4(k-1)$, then every 2-edge-coloring of K_n contains a monochromatic k-connected subgraph of order at least $n - 2(k-1)$.*

After several partial solutions including some partial results in the original paper [3] along with others in [8, 21, 22], Łuczak [23] recently published a proof of Conjecture 1 but a gap in the proof has been found and not yet fixed.

Introducing forbidden rainbow subgraphs to the problem, in 2013, Fujita and Magnant classified those graphs H with the property that if G is an edge-coloring of K_n without rainbow H, then G contains an almost spanning monochromatic k-connected subgraph.

Theorem 10 (Fujita and Magnant [9]). *Let n, k, m be positive integers with $n \gg m \gg k$. A connected graph H has the property that "any m-edge-coloring of K_n without rainbow H contains an almost spanning monochromatic k-connected subgraph" if and only if $H \in \{K_3, P_4^+, P_6\}$ or a connected subgraph of them.*

This concept can naturally be extended to hypergraphs through the following question. Here we say a hypergraph is k-connected if the removal of any set of at most $k-1$ vertices leaves behind a connected hypergraph.

Question 1. *For which 3-uniform hypergraphs H does the following hold for k and m sufficiently large? If G is a rainbow H-free m-coloring of K_n^3, then G contains a monochromatic k-connected subgraph of order at least $n - f(k, m)$ where f is a function of k and m but not n.*

Let $\mathscr{H}$ denote the set of hypergraphs satisfying the claim in Question 1. Then Corollary 5 implies the following result, meaning that $B_3 \in \mathscr{H}$.

Corollary 11. *Let $k \geq 1$ and $n \geq 4$. If G is an m-coloring of K_n^3 with no rainbow B_3, then G contains a spanning monochromatic k-connected subgraph.*

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