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Matching Preclusion of the Generalized Petersen Graph

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Matching Preclusion of the Generalized Petersen Graph

Cover Page Footnote

This paper originated from a project started at the Oakland University Summer Mathematics Institute.

Abstract

The matching preclusion number of a graph with an even number of vertices is the minimum number of edges whose deletion results in a graph with no perfect matchings. In this paper we determine the matching preclusion number for the generalized Petersen graph $P(n, k)$ and classify the optimal sets.

Keywords: perfect matching; Petersen graph; generalized Petersen graph.

1 Introduction

A *matching* in a graph is a set of edges such that each vertex is incident to at most one edge in the set. Each vertex incident to an edge in a matching is said to be *saturated* by the matching. A *perfect matching* is a matching which saturates each vertex of the graph. Clearly, any graph with a perfect matching has an even number of vertices; we call such graphs *even*. To study the structure of perfect matchings in even graphs, Brigham et al. [3] introduced the notion of a *matching preclusion set*: a set of edges F is a matching preclusion set of an even graph G if the subgraph of G obtained by removing the edges of F (denoted by $G - F$) has no perfect matching. The *matching preclusion number* of an even graph G, denoted by $mp(G)$, is defined as the minimum cardinality of a matching preclusion set of G. A matching preclusion set which achieves the minimum cardinality is called an *optimal matching preclusion set*.

A natural way to eliminate all perfect matchings in a graph is to remove all edges incident to some vertex. A graph with $mp(G) = \delta(G)$ is said to be *maximally matched*, as it attains the upper bound. A matching preclusion set whose removal isolates a vertex is said to be *trivial*, and a graph where every optimal matching preclusion set is trivial is *super matched*. This property can be viewed as optimal, in the sense that it provides the most stringent possible restriction on the structure of the matching preclusion sets of a graph. We note that if a graph is super matched, then it is necessarily maximally matched, but the converse need not be true. For other graph theory terminology not described here, we refer the reader to [14].

Graph models are used to study various problems in a variety of different areas: examples include molecular chemistry, operations research, activity networks, and species migration. In particular, models of computer and processor networks have a natural representation as graphs. For some applications in high-performance computing, each processor node in a network may need to be paired with a single partner. For such applications, the matching preclusion number gives a natural measure of the robustness of the network in the event of link failure. For more information, see [3]. The matching preclusion problem has been studied for several well-known families of graphs, including the hypercubes, alternating group graphs, and pancake graphs [5–8, 10, 11, 16, 18]. A number of generalizations of matching preclusion have also been studied, including *conditional matching preclusion* [4–7,9,12], where no vertex in the graph may be isolated by the removal of edges, and *strong matching preclusion* [17], where vertices as well as edges may be removed.

2 Generalized Petersen graphs

The *generalized Petersen graphs* $P(n,k)$, as the name suggests, are a family of graphs based on the construction of the well-known Petersen graph. The vertex set of $P(n,k)$ consists of n *outer vertices* $a_0, \ldots, a_{n-1}$ and n *inner vertices* $b_0, \ldots, b_{n-1}$, for $n \geq 3$. For notational convenience, we say that if $i \equiv j \pmod{n}$, then $a_i = a_j$ and $b_i = b_j$. The edges of $P(n,k)$ consist of *outer edges* of the form $a_i a_{i+1}$, *inner edges* $b_i b_{i+k}$, and *spokes* $a_i b_i$, for $0 \leq i \leq n-1$. We can see that each outer edge joins two outer vertices, each inner edge joins two inner vertices, and each spoke joins an inner vertex to an outer vertex.

For fixed n and k, let $d = gcd(n,k)$ and $q = \frac{n}{d}$. From the definition, we observe that $P(n,k)$ consists of a cycle of length n (the *outer cycle*) which is joined by the spokes to d vertex-disjoint cycles of length q (the *inner cycles*). We can also deduce that $P(n,k)$ is isomorphic to $P(n, n-k)$, and that $P(n,k)$ is 3-regular when $n \neq 2k$, so we require that $n > 2k$ in order to eliminate redundant cases and ensure that $P(n,k)$ is a simple graph.

We recover the Petersen graph as $P(5,2)$ (see Figure 1). The properties of the generalized Petersen graphs have been widely studied regarding perfect 2-colorings, the decycling number, and domination number [2, 13, 15]. A number of notable 3-regular graphs arise as generalized Petersen graphs; some examples are listed in Table 1.

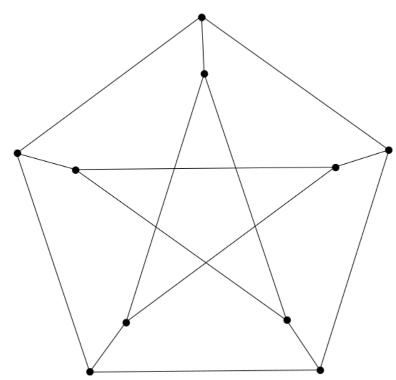

Figure 1: The Petersen graph P(5,2)

The matching preclusion problem has been previously studied for some cases of the generalized Petersen graphs, all of which are even graphs. The Petersen graph $P(5,2)$ is known to be maximally matched but not super matched. The matching preclusion number of $P(n,3)$ has also been determined in [12]. Previous work investigated $P(n,k)$ under the strong matching preclusion problem, in which both edges and vertices can be removed [1]. In this paper, we aim to extend these results and solve the matching preclusion problem for $P(n,k)$ for all $k > 1$ and $n > 2k$. We do not consider the case $k = 1$ here as the matching preclusion problem in the prism graph $P(n,1)$ requires a different, though simpler, analysis than when $k > 1$.

3 Main results

To study the matching preclusion problem for $P(n,k)$, we consider the removal of a set F of three edges and determine the conditions under which $P(n,k) - F$ has a perfect matching.

Graph	generalized Petersen graph
Cubic symmetric graph $F_{048}A$	$P(24,5)$
Cubical graph	$P(4,1)$
Desargues graph	$P(10,3)$
Dodecaheadral graph	$P(10,2)$
Durer graph	$P(6,2)$
Möbius-Kantor graph	$P(8,3)$
Nauru graph	$P(12,5)$
Petersen graph	$P(5,2)$
Prism graph	$P(n,1)$

Table 1: Examples of generalized Petersen graphs [19]

The following result is clear from the construction of $P(n,k)$.

Lemma 3.1. *If F consists of three non-spoke edges, then $P(n,k)-F$ has a perfect matching consisting of all the spokes.*

We consider three additional cases based on the number of spokes in F.

Theorem 3.2. *Let $k>1$. If F consists of one spoke and two non-spokes, and F is not a trivial matching preclusion set, then $P(n,k)-F$ has a perfect matching for any $(n,k)\neq(5,2)$.*

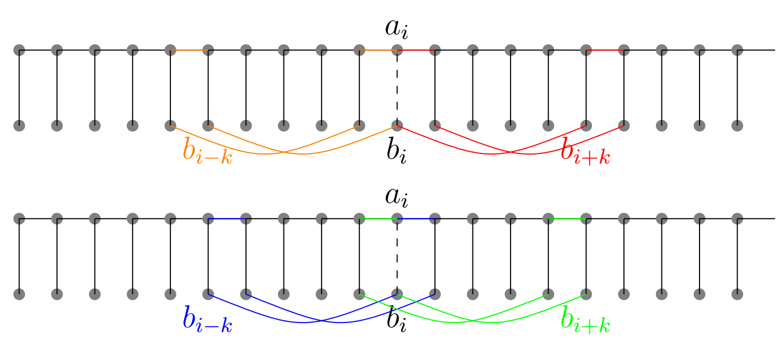

Figure 2: Top: the set S_1 in red, and S_2 in orange. Bottom: the set S_3 in green, and S_4 in blue.

Proof. Suppose that F is a non-trivial matching preclusion set of $P(n,k)$. Let a_ib_i be the spoke in F. Consider the four sets of edges

$$S_1=\{a_ia_{i+1},a_{i+k}a_{i+k+1},b_ib_{i+k},b_{i+1}b_{i+k+1}\},$$

$$S_2=\{a_ia_{i-1},a_{i-k-1}a_{i-k},b_ib_{i-k},b_{i-1}b_{i-k-1}\},$$

$$S_3=\{a_ia_{i-1},a_{i+k-1}a_{i+k},b_ib_{i+k},b_{i-1}b_{i+k-1}\},$$

$$S_4=\{a_ia_{i+1},a_{i-k}a_{i-k+1},b_ib_{i-k},b_{i+1}b_{i-k+1}\},$$

illustrated in Figure 2. Since $n > 2k \geq k+1$, each of these sets consists of four distinct, independent edges, and thus forms a matching in $P(n,k) - F$ which can be extended to a perfect matching by including the remaining spokes not incident to matched vertices. Thus for F to be a matching preclusion set, each of these edge sets must contain at least one of the two non-spoke edges in F.

Since F is not a trivial matching preclusion set, F contains at most one of a_ia_{i+1} and a_ia_{i-1}, and at most one of b_ib_{i+k} and b_ib_{i-k}. When $n > 2k+2$, it is clear that $i, i-1, i+1, i+k, i+k-1, i+k+1, i-k, i-k-1$, and $i-k+1$ are all distinct modulo n. This means that the vertices with these labels are all distinct, so it is straightforward to check that no set of two edges can intersect all of the sets S_j other than those that form a trivial matching preclusion set. When $n = 2k+2$ we have $i-k-1 \equiv i+k+1 \pmod{n}$, but all the edges in the sets S_j are distinct, so we can once again check that only a trivial matching preclusion set intersects all of the S_j. When $n = 2k+1$, we have $i-k \equiv i+k+1 \pmod{n}$ and $i-k-1 \equiv i+k \pmod{n}$, so $a_{i+k}a_{i+k+1} = a_{i-k}a_{i-k-1}$ is present in both S_1 and S_2. However, when $k \neq 2$ no other edge is present in both S_3 and S_4, so only a trivial matching preclusion set intersects all of the S_j. When $k = 2$, we have $i+k-1 \equiv i+1 \pmod{n}$ and $i-k+1 \equiv i-1 \pmod{n}$, so the edge $b_{i-1}b_{i+k-1} = b_{i+1}b_{i-k+1} = b_{i-1}b_{i+1}$ is an element of both S_3 and S_4. From this, we can see that $P(5,2)$ has a non-trivial matching preclusion set of the form $\{a_ib_i, b_{i-1}b_{i+1}, a_{i+2}a_{i-2}\}$, but otherwise, when F is not a trivial matching preclusion set and $(n,k) \neq (5,2)$, $P(n,k) - F$ has a perfect matching.

□

Theorem 3.3. *Let $k > 1$ and $n \neq 3k$. If F consists of three spokes, then $P(n,k) - F$ has a perfect matching.*

Proof. Recall that $d = gcd(n,k)$ and $q = \frac{n}{d}$. If $P(n,k) - F$ is spanned by even cycles, a perfect matching exists. This occurs when n and q are both even.

In all other cases, n and d have the same parity, so $n-d$ will be even and each inner cycle has an odd number of vertices. When $n \neq 3k$, since $n > 2k$ we have $q > 3$, thus we can choose d consecutive spokes $a_ib_i, \ldots, a_{i+d-1}b_{i+d-1}$ in $P(n,k) - F$. The subgraph of $P' = P(n,k) - F$ induced by the vertices not incident to these spokes consists of one outer path of $n-d$ vertices joined by the remaining spokes to d inner paths of $q-1$ vertices each. As both $n-d$ and $q-1$ are even, each of these paths is saturated by a matching, so by taking these matchings together we obtain a perfect matching of P'. Taking this matching together with the d consecutive spokes gives a perfect matching of $P(n,k) - F$.

□

Theorem 3.4. *Let $k > 1$ and $n \neq 3k$. If F consists of two spokes and one non-spoke, then $P(n,k) - F$ has a perfect matching.*

Proof. Three different cases can be considered:

Case 1: $d = gcd(n,k) = 1$. This means there is one inner cycle.

If n is even, $P(n,k) - F$ consists of an even cycle joined to one path with an even number of vertices, so $P(n,k) - F$ has a perfect matching. Thus, suppose n is odd. Since $gcd(n,k) = 1$, we can construct an automorphism ϕ of $P(n,k)$ by taking $\phi(a_i) = b_{k*i}$ and $\phi(b_i) = a_{c*i}$, where $ck \equiv 1 \pmod{n}$. As we can see that ϕ interchanges the outer and inner cycles, we may assume without loss of generality that the non-spoke edge in F is a_ia_{i+1}. If one of a_ib_i and $a_{i+1}b_{i+1}$ is not an element of F, say a_ib_i, then the outer cycle without a_i and the inner cycle without b_i are each paths with an even number of vertices, so we can construct a perfect matching of $P(n,k) - F$ by taking a matching saturating each of those paths together with the spoke a_ib_i. If both a_ib_i and $a_{i+1}b_{i+1}$ are elements of F, and $n > 3$, then a_{i+n-2} and a_{i+1} are distinct vertices, and a_{i+n-2} is incident to a spoke in $P(n,k) - F$. Then $P_1 = a_{i+1}, a_{i+2}, \ldots, a_{i+n-2}, b_{i+n-2}, b_{i+n-2+k}, \ldots, b_{i+n-2-k}$ and $P_2 = a_{i+n-1}a_i$ are two paths, each with an even number of vertices, that span the vertices of $P(n,k) - F$. Thus taking a matching saturating each of these paths gives a perfect matching of $P(n,k) - F$.

Case 2: $d = gcd(n,k) > 1$, and n and $q = \frac{n}{d}$ are even.

In this case, the vertices of $P(n,k)$ are spanned by the outer and inner cycles, all of which are even. Thus the vertices of $P(n,k) - F$ are spanned by a collection of even cycles and one path with an even number of vertices, so we can find a perfect matching of $P(n,k) - F$.

Case 3: n is odd and $d > 1$, or n, k are even and q is odd.

In this case, we know that n and d have the same parity, so $n - d$ is even, and each inner cycle has an odd number of vertices.

First, we suppose that the non-spoke edge in F is an outer edge, say a_ia_{i-1}. Since $n \geq 3d$, we can find d consecutive spokes $a_jb_j, \ldots, a_{j+d-1}b_{j+d-1}$ in $P(n,k) - F$ such that either $a_{i-1}b_{i-1}$ and a_ib_i are both among them or $j - i$ is even modulo n. To see this, suppose that no collection of d consecutive spokes in $P(n,k) - F$ contains both $a_{i-1}b_{i-1}$ and a_ib_i. Then there are three possibilities: if $a_{i-1}b_{i-1}$ and a_ib_i are both in F, then $a_{i+2}b_{i+2}, \ldots, a_{i+d+1}b_{i+d+1}$ is clearly the desired collection of spokes, with $i + 2 - i \equiv 2 \pmod{n}$ even. If only one of $a_{i-1}b_{i-1}$ or a_ib_i is in F, then one of $a_{i+2}b_{i+2}, \ldots, a_{i+d+1}b_{i+d+1}$ or $a_{i+n-d}b_{i+n-d}, \ldots, a_{i-1}b_{i-1}$ is the desired collection of spokes, as the other spoke in F can only be in one of the two disjoint collections. If neither of $a_{i-1}b_{i-1}$ or a_ib_i is in F, then let $a_{i+s}b_{i+s}$ and $a_{i+t}b_{i+t}$ be the spokes in F, with $0 \leq s < t \leq n - 1$. Then the number of spokes in $a_{i+t+1}b_{i+t+1}, \ldots a_ib_i, \ldots, a_{i+s-1}b_{i+s-1}$ is $n - t + s - 1$, which must be strictly less than d. But then we have $(t-1) - (s+1) + 1 \geq n - d - 1 \geq 2d - 1 \geq d + 1$, so either $a_{i+s+1}b_{i+s+1}, \ldots, a_{i+s+d}b_{i+s+d}$ or $a_{i+s+2}b_{i+s+2}, \ldots, a_{i+s+d+1}b_{i+s+d+1}$ is the desired collection of spokes, as one of $s + 1$ or $s + 2$ is even modulo n.

If the vertices incident to the consecutive spokes are removed, the remaining outer vertices form either one path, if the consecutive spokes include at least one of $a_{i-1}b_{i-1}$ or a_ib_i, or two paths otherwise, each with an even number of vertices, since n and d have the same parity and as there are an even number of vertices in $a_i, \ldots, a_{j-1}$ in the latter case. Additionally, the remaining vertices of each inner cycle form a path with $q - 1$ vertices, which is even since q is odd. Therefore we can find matchings saturating each of these paths, and taking these matchings together with the d consecutive spokes gives a perfect matching of $P(n,k) - F$.

We now suppose the non-spoke edge in F is an inner edge, say b_ib_{i+k}. Let a_sb_s and a_tb_t be the spokes in F, for some $0 \leq s < t \leq n - 1$. Then there are $t - s - 1$ indices r with $s < r < t$, and $n + s - t - 1$ with $0 \leq r < s$ or $t < r \leq n - 1$. Since $n > 2k \geq 2d$, $\max\{t - s - 1, n + s - t - 1\} \geq d$, so either we can choose d consecutive spokes in $P(n,k) - F$ such that one of a_ib_i or $a_{i+k}b_{i+k}$ is among them, or alternatively that both are elements of F. In the former case, the outer vertices not incident to these spokes form a path with $n - d$ vertices, and the remaining inner vertices form a collection of paths, each with $q - 1$ vertices. Since $n - d$ and $q - 1$ are even, we can find matchings saturating each of these paths, and taking these matchings together with the d consecutive spokes gives a perfect matching of

$P(n,k) - F$.

In the latter case, since $n \neq 3k$, we have $q > 3$, so $q \geq 5$ for odd q. This implies that b_{i-k}, b_i, b_{i+k}, b_{i+2k}, and b_{i+3k} are all distinct vertices, and that we can find d consecutive spokes in $P(n,k) - F$ such that $a_{i+3k}b_{i+3k}$ is among them. The outer vertices not incident to these spokes form a path with $n - d$ vertices, the inner cycle containing b_i forms a path with $q - 5$ vertices after excluding $b_{i-k}, b_i, b_{i+k}, b_{i+2k}$, and b_{i+3k}, and the vertices not incident to the spokes in the remaining inner cycles form paths with $q - 1$ vertices each. Taking these matchings, the d consecutive spokes, and the edges $\{b_{i-k}b_i, b_{i+k}b_{i+2k}\}$ together gives a perfect matching of $P(n,k) - F$. This completes the proof.

□

We note that when $n = 3k$, the edge set $F = \{a_ib_i, a_{i+k}b_{i+k}, a_{i+2k}b_{i+2k}\}$ is a non-trivial matching preclusion set of $P(3k,k)$, as its removal leaves the graph disconnected with two odd components. Furthermore, the edge set $F = \{b_ib_{i+k}, a_ib_i, a_{i+k}b_{i+k}\}$ is also a non-trivial matching preclusion set as the vertices b_i and b_{i+k} have degree 1 in $P(n,k) - F$, and their only neighbor is in common. These constructions justify the condition that $n \neq 3k$ in Theorem 3.3 and Theorem 3.4.

Synthesizing the results above, we obtain the following classification.

Theorem 3.5. *Let $k > 1$ and $n > 2k$. Then $P(n,k)$ is maximally matched. Furthermore, $P(n,k)$ is super matched when $n \neq 3k$ and $(n,k) \neq (5,2)$.*

Proof. Let F be a set of edges of $P(n,k)$. When $|F| = 3$, $n \neq 3k$ and $(n,k) \neq (5,2)$ Theorems 3.2, 3.3, and 3.4 indicate that $P(n,k) - F$ has a perfect matching or F is a trivial matching preclusion set. Furthermore, minor modifications to the proofs of these theorems show that when $|F| \leq 2$, $P(n,k) - F$ has a perfect matching for any $k > 1$ and $n > 2k$. Thus $mp(P(n,k)) = 3 = \delta(P(n,k))$, so $P(n,k)$ is maximally matched for any $k > 1$ and $n > 2k$, and $P(n,k)$ is super matched when $n \neq 3k$ and $(n,k) \neq (5,2)$. □

4 Conclusion

In this paper, we determined the matching preclusion number for the family of generalized Petersen graphs, and classified the optimal matching preclusion sets. A natural extension of this work involves the derivation of similar results for some of the generalizations of the matching preclusion problem, including the strong matching preclusion and conditional matching preclusion problems. We conjecture that these problems have a similar "optimal" solution for the generalized Petersen graph, in that the optimal sets for each problem can be constrained in the best possible way outside of a small number of exceptional cases.

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