



February 2019

## Singular Ramsey and Turán numbers

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### Recommended Citation

Caro, Yair and Tuza, Zsolt (2019) "Singular Ramsey and Turán numbers," *Theory and Applications of Graphs*: Vol. 6 : Iss. 1 , Article 1.

DOI: 10.20429/tag.2019.060101

Available at: <https://digitalcommons.georgiasouthern.edu/tag/vol6/iss1/1>

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# Singular Ramsey and Turán numbers

## **Cover Page Footnote**

Research of the second author was supported in part by the National Research, Development and Innovation Office -- NKFIH under the grant SNN 129364.

## Abstract

We say that a subgraph  $F$  of a graph  $G$  is singular if the degrees  $d_G(v)$  are all equal or all distinct for the vertices  $v \in V(F)$ . The singular Ramsey number  $\text{Rs}(F)$  is the smallest positive integer  $n$  such that, for every  $m \geq n$ , in every edge 2-coloring of  $K_m$ , at least one of the color classes contains  $F$  as a singular subgraph. In a similar flavor, the singular Turán number  $\text{Ts}(n, F)$  is defined as the maximum number of edges in a graph of order  $n$ , which does not contain  $F$  as a singular subgraph. In this paper we initiate the study of these extremal problems. We develop methods to estimate  $\text{Rs}(F)$  and  $\text{Ts}(n, F)$ , present tight asymptotic bounds and exact results.

## 1 Introduction

In this paper we introduce a new type of Ramsey and Turán numbers, where the classical condition of the occurrence of a specified subgraph in an edge-colored complete graph is combined with restrictions on vertex degrees in the monochromatic host graph.

### 1.1 Brief survey on degree-constrained problems

The smallest particular case of Ramsey's theorem is that on six vertices every graph or its complement contains the triangle  $K_3$ . Starting from here, Albertson [2] proved<sup>1</sup> that for  $n \geq 6$  in every 2-coloring of the edges of  $K_n$  there is a monochromatic  $K_3$  with two equal degrees. Inspired by this result several papers were written, see for example [3, 5, 6, 8, 11].

An obvious step after [2] is to try to generalize this result to other graphs and also to try to bound the difference between the maximum and minimum degree of the specified monochromatic subgraph. The efforts in the direction can be summarized as follows.

In [4] Albertson and Berman showed that  $K_n$  can always be colored red-blue in such a way that no red  $K_4$  occurs and no blue  $K_2$  has equal monochromatic degree at its two vertices. This shows that the phenomenon observed by Albertson is isolated and not extended to other graphs. But the authors of [4] also showed that for  $n \geq 6$  in every 2-coloring of  $K_n$  there is a  $K_3$  with spread of the degrees at most 5, where the spread of a sequence  $D =: \{d_1, \dots, d_m\}$  is defined as  $\max\{|d_i - d_j| \mid 1 \leq i, j \leq m\}$ . An extension of their result is presented in [20].

In the papers [14, 17] Chen, Erdős, Rousseau and Schelp developed the notion of spread explicitly and proved in [17] that every graph on at least  $k + 2$  vertices contains at least  $k + 2$  vertices whose degrees have spread at most  $k$ . This is a non-trivial extension of the popular observation that every graph with more than one vertex has two vertices of the same degree. From the quoted theorem the authors also proved among other things that for every graph  $G$  and every  $n \geq \text{R}(G)$  (the classical Ramsey number) every 2-coloring of  $K_n$  contains a monochromatic copy of  $G$ , whose vertex degrees in the host monochromatic graph have spread at most  $\text{R}(G) - 2$ , and that in a certain sense this upper bound is tight. An easy corollary is that the spread 5 from the Albertson–Berman result mentioned above can be reduced to 4 for  $n \geq 6$ , which is best possible (as already noted in [17]).

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<sup>1</sup>Considerable delay occurred between the birth and the publication of [2], and some of the follow-up papers appeared even several years earlier.

Albertson [1] also introduced the corresponding Turán number, namely the maximum number of edges in a graph on  $n$  vertices having no copy of  $K_m$  with all degrees equal, and presented an exact bound. (In an earlier paper [9] Caccetta, Erdős and Vijayan studied a Turán-type problem concerning the existence of a complete graph  $K_m$  with large degrees.)

A closely related subject is that of constant-degree independent sets, introduced by Albertson and Boutin [5], which was recently further developed by Caro, Hansberg and Pepper [11]. The latter considered various bounds on the constant-degree  $k$ -independent set in trees, forest,  $d$ -degenerate graphs and  $d$ -trees. Yet another direction concerns low-degree independent sets in planar graphs, developed by many authors and best presented in [6].

Further related notions are the so-called fair dominating sets (which actually are regular dominating sets, see Caro, Hansberg and Henning [10]), irregular independence number and irregular domination number (Borg, Caro and Fenech [8]), and the problem of monochromatic degree-monotone paths in 2-colorings of the edges of complete graphs (Caro, Yuster and Zarb [12]).

## 1.2 Singular Ramsey and Turán numbers

Albertson and Berman [4] presented edge 2-colorings of  $K_n$  avoiding a monochromatic copy of  $G$  with all monochromatic degrees in  $K_n$  equal. On the other hand, the opposite possibility of having a monochromatic copy of  $G$  with all its vertices having distinct monochromatic degrees in  $K_n$  is very easy to exclude, by any decomposition of  $K_n$  into two regular spanning graphs  $H$  and  $\overline{H}$ . However, simultaneous exclusion of the two cases is impossible if  $n$  is large. This fact motivates our present study.

**Definition 1.** Let  $k \geq 1$  be an integer. A sequence  $a_1 \leq a_2 \leq \dots \leq a_n$  of integers is called  $k$ -singular if either  $a_1 = \dots = a_n$  or for every  $j = 1, \dots, n-1$ ,  $a_{j+1} - a_j \geq k$ . Also if  $a_1, a_2, \dots, a_n$  are integers (repetitions are allowed), we say that they form a  $k$ -singular set if putting them in increasing order we obtain a  $k$ -singular sequence. (Hence, “set” may mean “multiset” in this particular context.)

**Definition 2.** A subgraph  $H$  of graph  $G$  is called  $k$ -singular if the degree sequence of its vertices in  $G$  — where  $G$  is termed the *host graph* — forms a  $k$ -singular sequence.

For short, in case of  $k = 1$ , a 1-singular sequence is called *singular sequence*, and a 1-singular subgraph is called *singular subgraph*.

Let now  $\mathcal{F}$  be a family of graphs.

**Definition 3.** The  $k$ -singular Ramsey number  $\text{Rs}(\mathcal{F}, k)$  is defined as the smallest integer  $n$  such that in every 2-coloring of the edges of  $K_m$  for any  $m \geq n$ , one of the graphs induced by the color classes contains a  $k$ -singular member of  $\mathcal{F}$ .

**Remark 4.** If in a graph  $G$  the subsequence of degrees belonging to a set  $B$  of vertices is  $k$ -singular, then so does the subsequence belonging to  $B$  in the complement of  $G$  as well. Hence, in case of two colors, the vertex sets of  $k$ -singular subgraphs in color 1 coincide with those in color 2 (for any  $k \geq 1$ ).

In a similar flavor, as a little deviation, we also introduce a Turán-type function.

**Definition 5.** Given  $\mathcal{F}$ , and a natural number  $k$ , the  $k$ -singular Turán number — as a function of the order  $n$  — denoted by  $\text{Ts}(n, \mathcal{F}, k)$  is defined as the maximum number of edges in a graph  $G$  on  $n$  vertices that contains no  $k$ -singular copy of any  $F \in \mathcal{F}$ . In particular, let  $\text{Ts}(n, q)$  be the maximum number of edges in a graph  $G$  of order  $n$  that contains no singular copy of  $K_q$ .

For *singular Ramsey numbers* we shall use the simpler notation  $\text{Rs}(\mathcal{F}) = \text{Rs}(\mathcal{F}, 1)$  for  $k = 1$ , and we write  $\text{Rs}(F)$  for  $\text{Rs}(\{F\})$ .

It is also natural to introduce non-diagonal and multicolored versions of  $\text{Rs}(F)$ .

**Definition 6.** If  $F_1$  and  $F_2$  are two graphs, their singular Ramsey number  $\text{Rs}(F_1, F_2)$  is the smallest  $n$  such that, for every  $m \geq n$ , every 2-coloring of  $K_m$  contains a singular copy of  $F_1$  in the first color or a singular copy of  $F_2$  in the second color. More generally, also for an integer  $s > 2$ , one may consider  $s$  families  $\mathcal{F}_1, \dots, \mathcal{F}_s$  of graphs and define the  $k$ -singular Ramsey number  $\text{Rs}(\mathcal{F}_1, \dots, \mathcal{F}_s, k)$  as the smallest integer  $n$  with the property that, for any  $m \geq n$ , in every coloring of the edges of  $K_m$  with  $s$  colors, there is an  $i$  ( $1 \leq i \leq s$ ) such that the graph induced by the  $i$ th color class contains a  $k$ -singular<sup>2</sup> member of  $\mathcal{F}_i$ .

**Remark 7.** (*Non-monotonicity.*) Let the number of colors be fixed. If every coloring of  $K_n$  contains a monochromatic singular copy of some  $F \in \mathcal{F}$ , still there is no guarantee that so does every coloring of  $K_{n+1}$  as well. This issue concerning (non-) monotonicity was observed already in the first papers by Albertson, and ever since; it is treated by imposing the condition for every  $m \geq n$  in the definition of  $\text{Rs}(\mathcal{F})$ , rather than just taking the smallest  $n$  forcing a singular monochromatic  $F$  in every 2-coloring of  $K_n$ .

**Remark 8.** (*Monotonicity Principle.*) It is obvious — but will be applied at some point below — that the function  $\text{Rs}$  is monotone with respect to inclusion, for any fixed number of colors; for instance, if  $F_1 \subseteq G_1$  and  $F_2 \subseteq G_2$ , then  $\text{Rs}(F_1, F_2) \leq \text{Rs}(G_1, G_2)$  holds.

**Remark 9.** In the classical version of Ramsey and Turán numbers, isolated vertices are practically irrelevant, namely  $\text{R}(G \cup mK_1) = \max(\text{R}(G), m + |V(G)|)$ ; but this is not at all the case in the singular version. For instance, it can easily be shown (partly following also from some later observations) that for the graph  $G = P_3 \cup K_1$  — the path on 3 vertices plus an isolated vertex — we have  $\text{Rs}(P_3 \cup K_1) = 10$  while  $\text{R}(P_3 \cup K_1) = 4$ , moreover  $\text{Rs}(P_3) = 5$  and  $\text{R}(P_3) = 3$ . (Also, one may observe that  $\text{Rs}(3K_1) = 5$  while  $\text{R}(3K_1) = 3$ .) Similarly, the Turán number of  $K_2 \cup K_1$  is zero for every  $n \geq 3$ , but  $K_4 - e$  does not contain it as a singular subgraph, therefore  $\text{Ts}(4, K_2 \cup K_1, 1) = 5$ .

In this paper we will mostly consider Ramsey-type results for two colors, and develop a couple of methods suitable for determining the exact value of singular Ramsey numbers in both the diagonal and non-diagonal cases, provided that the specified graphs satisfy certain properties. We also present asymptotic estimates, and the  $k$ -singular version will be touched, too. In a section after the Ramsey-type results we provide tight asymptotics for the  $k$ -singular Turán number of a graph.

<sup>2</sup>See the concluding section for some possible interpretations of this definition more precisely.

### 1.3 Our results

While the star graphs can be considered as the easiest infinite class of graphs concerning the classical Ramsey numbers (they almost admit a one-line proof), they turn out to be a bit complicated in the singular version. For this reason, although we present a complete solution, we do not discuss them earlier than in Section 5. Before that, we give some general lower and upper bounds (Section 2), describe some methods to derive tight estimates (Section 3), and determine exact results for all, but one, graphs with at most four vertices and edges, with the unique exception of  $C_4$  (Section 4). Tight asymptotics for singular Turán numbers are given in Section 6. Some open problems are mentioned in the concluding section.

### 1.4 Terminology and notation

*Particular graphs.* We use standard notation  $P_n$  and  $C_n$  for the path and the cycle on  $n$  vertices;  $K_{p,q}$  for the complete bipartite graph with  $p$  and  $q$  vertices in its classes; and  $mK_2$  for the matching with  $m$  edges. The *claw* is the graph  $K_{1,3}$ . The *paw*, which we abbreviate in formulas as  $PW$ , is the graph with four vertices and four edges obtained from  $K_3$  by adding a pendant vertex (or from  $K_4$  by deleting the edges of a  $P_3$ ). The *bull* is the graph obtained from  $K_3$  by adding two pendant vertices which are adjacent to two of its distinct vertices (a self-complementary graph with five vertices and five edges).

*Vertex degrees.* The degree of a vertex  $v$  in a graph  $G$  is denoted by  $d_G(v)$ , or simply  $d(v)$  if  $G$  is clear from the context. Minimum and maximum degree are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. A *degree class* consists of all vertices having the same degree; hence the degree classes partition  $V(G)$ , and their number is equal to the number of distinct values which occur in the degree sequence of  $G$ . Given a vertex partition  $V_1 \cup \dots \cup V_k = V(G)$ , and a vertex  $v \in V_i$ , the *internal degree* of  $v$  is the number of its neighbors inside  $V_i$ , and its *external degree* is the number of its neighbors in  $V(G) \setminus V_i$ .

*Ramsey number.* We denote the Ramsey number by  $R(\mathcal{F})$ , that is the smallest  $n$  such that in every 2-coloring of the edges of  $K_n$ , one of the color classes contains a monochromatic member of  $\mathcal{F}$ .

*Substitution.* Let  $H$  be a graph with  $k$  vertices  $v_1, \dots, v_k$ , and let  $F_1, \dots, F_k$  be  $k$  non-null graphs ( $F_i = K_1$  is allowed). The *substitution* of  $F_1, \dots, F_k$  into the “host graph”  $H$ , denoted by  $H[F_1, \dots, F_k]$ , is the graph whose vertex set is the disjoint union  $V(F_1) \cup \dots \cup V(F_k)$ , each  $V(F_i)$  induces the graph  $F_i$  itself, and two vertices  $x \in V(F_i)$  and  $y \in V(F_j)$  ( $i \neq j$ ) are adjacent in  $H[F_1, \dots, F_k]$  if and only if  $v_i v_j$  is an edge in  $H$ . In this construction we say that the graph  $F_i$  is substituted for  $v_i$ .

In a graph  $G = (V, E)$ , the subgraph induced by a set  $Y \subset V$  is denoted by  $G[Y]$ .

## 2 Singular Ramsey numbers: General bounds

We start with the following easy lemma.

**Lemma 10.** *Every sequence of  $k(n-1)^2 + 1$  integers contains a  $k$ -singular subsequence of cardinality at least  $n$ .*

**Proof.** Suppose we have no  $n$  equal elements in the sequence. Then we must have at least  $k(n-1) + 1$  elements of distinct values. Reorder them in increasing order, say  $a_1 < \dots < a_{k(n-1)+1}$ . Take the subsequence  $a_{jk+1}$  for  $j = 0, \dots, n-1$ . Clearly this is a  $k$ -singular  $n$ -term sequence.  $\square$

**Theorem 11.** *For any two families  $\mathcal{F}_1, \mathcal{F}_2$  of graphs and every natural number  $k \geq 1$  the following general upper bound holds:*

$$\text{Rs}(\mathcal{F}_1, \mathcal{F}_2, k) \leq k(\text{R}(\mathcal{F}_1, \mathcal{F}_2) - 1)^2 + 1.$$

**Proof.** Consider a 2-coloring of the edges of  $K_m$ , for any  $m \geq k(\text{R}(\mathcal{F}_1, \mathcal{F}_2) - 1)^2 + 1$ . Let  $G_1$  and  $G_2$  be the subgraphs obtained by the edges of color 1 and color 2, respectively. By Lemma 10 the sequence of degrees of the vertices of  $G_1$  contains a  $k$ -singular subsequence of cardinality  $\text{R}(\mathcal{F}_1, \mathcal{F}_2)$ . The degrees of the corresponding vertices form a  $k$ -singular subsequence also in  $G_2$ . Now consider the 2-coloring induced on the complete graph on those  $\text{R}(\mathcal{F}_1, \mathcal{F}_2)$  vertices. By definition there is either a monochromatic copy of a graph  $G \in \mathcal{F}_1$  in color 1 or of a graph  $H \in \mathcal{F}_2$  in color 2. Hence the degrees of  $G$  (in the first case) form a  $k$ -singular subsequence in the host graph  $G_1$  or the degrees of  $H$  (in the second case) form a  $k$ -singular subsequence in the host graph  $G_2$ . Thus a required  $k$ -singular subgraph occurs whenever  $m \geq k(\text{R}(\mathcal{F}_1, \mathcal{F}_2) - 1)^2 + 1$ , which means  $\text{Rs}(\mathcal{F}_1, \mathcal{F}_2, k) \leq k(\text{R}(\mathcal{F}_1, \mathcal{F}_2) - 1)^2 + 1$ .  $\square$

An immediate corollary is:

**Corollary 12.**

- (i) *For every graph  $G$  we have  $\text{Rs}(G) \leq (\text{R}(G) - 1)^2 + 1$ , and also  $\text{Rs}(G, H) \leq (\text{R}(G, H) - 1)^2 + 1$  for any two graphs  $G$  and  $H$ .*
- (ii) *Every 2-coloring of  $K_{k(n-1)^2+1}$  contains a monochromatic  $k$ -singular tree of order at least  $n$ .*

**Proof.** (i) This is just the case  $\mathcal{F} = \{G\}$ , or  $\mathcal{F}_1 = \{G\}$  and  $\mathcal{F}_2 = \{H\}$ , with  $k = 1$  in Theorem 11.

(ii) Consider the degree sequence in the graph induced by the edges colored 1. By Lemma 10 there is a  $k$ -singular subsequence of  $n$  degrees. Consider now the induced coloring on the complete graph  $K_n$  whose vertices are those forming the  $k$ -singular sequence. Since every graph or its complement is connected, it follows that there is a connected monochromatic subgraph of order  $n$  whose degree sequence is  $k$ -singular in the host graph, and hence such a tree occurs.  $\square$

Having proved a general upper bound, we next supply a general quadratic lower bound.

**Theorem 13.** *Let  $G$  be any graph on  $n \geq 3$  vertices. Then  $\text{Rs}(G) \geq \max\{\text{R}(G), (n-1)^2+1\}$ .*

**Proof.** Trivially  $\text{Rs}(G) \geq \text{R}(G)$ , so we only have to show  $\text{Rs}(G) \geq (n-1)^2 + 1$ .

We will construct a graph  $H$  on  $(n-1)^2$  vertices whose vertex set  $V$  is partitioned into  $n-1$  subsets  $V_0, \dots, V_{n-2}$ , each of cardinality  $n-1$ , such that all vertices in  $V_i$  have the same degree ( $i = 0, 1, \dots, n-2$ ) but vertices from distinct subsets have distinct degrees. Then

clearly no copy of  $G$  in  $H$  and  $\overline{H}$  can be singular, as it must take at least two vertices in the same class and at least two vertices in distinct classes.

If  $n - 1$  is even, then we simply insert any  $i$ -regular graph inside  $V_i$ . (Such graphs exist, e.g. by taking  $i$  perfect matchings from any 1-factorization of  $K_{n-1}$ .)

If  $n - 1$  is odd, then depending on residue modulo 4, one of the sequences  $0, 1, \dots, n - 2$  and  $1, 2, \dots, n - 1$  contains an *even* number of *odd* terms. If it is  $0, 1, \dots, n - 2$ , then we insert a regular graph of degree  $2 \cdot \lfloor \frac{i}{2} \rfloor$  inside  $V_i$  (e.g., the union of  $\lfloor i/2 \rfloor$  edge-disjoint Hamiltonian cycles of  $K_{n-1}$ ). Moreover we insert a perfect matching between  $V_1$  and  $V_3$ , between  $V_5$  and  $V_7$ , ..., between  $V_{n-5}$  and  $V_{n-3}$ . Else, if it is  $1, 2, \dots, n - 1$ , then we insert a regular graph of degree  $2 \cdot \lfloor \frac{i+1}{2} \rfloor$  inside  $V_i$  (e.g., the union of  $\lfloor \frac{i+1}{2} \rfloor$  edge-disjoint Hamiltonian cycles of  $K_{n-1}$ ) and take a perfect matching between  $V_0$  and  $V_2$ , between  $V_4$  and  $V_6$ , ..., between  $V_{n-4}$  and  $V_{n-2}$ .

These graphs satisfy the requirements, proving the lower bound for all  $n$ .  $\square$

**Remark 14.** *An alternative proof — which also works in the  $k$ -singular case for  $k \geq 2$  — can be obtained from the Erdős–Gallai characterization of graphical sequences. We note that for some combinations of  $k$  and  $n$  (both even) an analogous construction with  $k(n - 1)^2$  vertices is not possible, because a graph cannot have an odd number of odd-degree vertices.*

In particular, the following bounds are obtained from the above estimates.

**Corollary 15.** *If  $G$  is a graph of order  $n \geq 3$ , then*

$$\max\{\mathbf{R}(G), (n - 1)^2 + 1\} \leq \mathbf{Rs}(G) \leq (\mathbf{R}(G) - 1)^2 + 1.$$

If  $\mathcal{G}$  is a class of graphs in which  $\mathbf{R}(G)$  is a linear function of  $|V(G)|$  over all graphs  $G \in \mathcal{G}$ , then the growth order of both estimates in Corollary 15 is quadratic in  $n$ . In particular, applying the theorem of [15] on the Ramsey numbers of graphs with bounded maximum degree, we obtain:

**Theorem 16.** *Let  $\mathcal{G}$  be the class of graphs with bounded degree  $\Delta$  fixed. Then for all  $G \in \mathcal{G}$  of order  $n$  we have  $\mathbf{Rs}(G) = \Theta(n^2)$ , as  $n \rightarrow \infty$ .*

We conclude this section with a sufficient condition ensuring that the lower bound in Corollary 15 holds with equality. This result also exhibits a significant difference between the classical and the singular versions of Ramsey numbers concerning the role of isolated vertices.

**Proposition 17.** *Let  $G = H \cup mK_1$ , i.e. the graph obtained from a graph  $H$  by adding  $m$  isolated vertices. If  $|V(G)| \geq \mathbf{R}(H)$ , then  $\mathbf{Rs}(G) = (|V(G)| - 1)^2 + 1$ .*

**Proof.** We only have to prove that  $(|V(G)| - 1)^2 + 1$  is an upper bound on  $\mathbf{Rs}(G)$ . If  $n \geq (|V(G)| - 1)^2 + 1$ , then in every 2-coloring of  $K_n$  the subgraph of color 1 contains a singular subgraph, say  $G^*$ , on  $|V(G)| = |V(H)| + m \geq \mathbf{R}(H)$  vertices. Thus, a singular monochromatic copy of  $H$  occurs, either in color 1 or in color 2, which can be supplemented to a singular copy of  $G$  because the  $m$  isolated vertices put no restriction on the color distribution in the rest of  $G^*$ .  $\square$



### 3 Some methods

Assume that a graph  $G$  has been fixed, for which we wish to find estimates on  $\text{Rs}(G)$ . We say that a graph  $F$  is  $G$ -free if  $F$  does not contain any subgraph isomorphic to  $G$ . Moreover, let us call  $F$  an  $R$ -graph for  $G$  if both  $F$  and  $\overline{F}$  are  $G$ -free. Analogously, we say that  $F$  is an  $SR$ -graph ('S' standing for 'singular') for  $G$  if neither  $F$  nor  $\overline{F}$  contains a singular subgraph isomorphic to  $G$ .

Lower bounds on  $\text{Rs}(G)$  will be obtained by constructing  $SR$ -graphs from several (smaller)  $R$ -graphs. We call this the technique of canonical colorings. Possible different approaches will be described in the next two subsections, and a kind of combination of them afterwards.

The fourth subsection presents a method to derive upper bounds when some favorable information concerning the structure of  $R$ -graphs of order  $\text{R}(G)-1$  is available. This approach will lead to exact results in several cases. Finally we mention another approach to upper bounds, based on vertex degrees.

#### 3.1 Non-regular Canonical Coloring, NRCC

This approach is useful when 'large'  $R$ -graphs are not regular. For instance, the claw  $K_{1,3}$  and its complement  $K_3 \cup K_1$  are the two  $R$ -graphs of order 4 for  $G = 2K_2$ , and also for  $P_4$ , but neither of them is regular. We apply this method in Section 4.2.

Let  $G$  be a graph on  $n$  vertices, and let  $t \leq n - 1$ . Consider  $t$  copies of (not necessarily isomorphic)  $R$ -graphs over mutually disjoint vertex sets  $V_1, \dots, V_t$ . Suppose that we can insert edges between the vertex classes (but not inside them) to obtain a graph  $H$  with the following properties:

1. In each vertex class  $V_i$  ( $i = 1, \dots, t$ ) all the degrees  $d_H(v)$  are equal.
2. Degrees of vertices belonging to distinct vertex classes are distinct.

**Lemma 18.** *With the assumptions above, we have*

$$\text{Rs}(G) \geq |V(H)| + 1 = 1 + \sum_{i=1}^t |V_i|.$$

**Proof.** In such a case  $H$  and  $\overline{H}$  have exactly  $t$  classes of distinct degrees and  $t \leq n - 1$ , hence no copy of  $G$  with all degrees distinct is possible (there are too few distinct degree classes). Also, since each set  $V_i$  induces an  $R$ -graph in  $H$ , no copy of  $G$  with all degrees equal is possible as it should be contained in a unique degree class. Hence  $H$  is an  $SR$ -graph, showing  $\text{Rs}(G) \geq |V(H)| + 1$ .  $\square$

#### 3.2 Regular Canonical Coloring, RCC

We can apply this approach when there exist 'large'  $R$ -graphs which are regular. (The first classical example is  $G = K_3$  whose unique largest  $R$ -graph is  $C_5$ .) We apply this method in Section 4.3.

Let  $F$  be an R-graph on  $q$  vertices  $v_1, \dots, v_q$ , and let  $H_1, \dots, H_q$  be  $q$  further R-graphs. Denote by  $H = F[H_1, \dots, H_q]$  the graph obtained by taking the vertex-disjoint copies of  $H_1, \dots, H_q$  and making all the vertices of  $H_i$  adjacent to all the vertices of  $H_j$  if and only if the vertices  $v_i$  and  $v_j$  are adjacent in  $F$ .

Suppose that  $H$  has the following properties:

1. Each  $H_i$  ( $i = 1, \dots, q$ ) is a regular induced subgraph of  $H$ .
2. If  $i \neq j$ , then the degrees  $d_H(v)$  for vertices  $v$  in  $H_i$  and  $H_j$  are not the same.

**Lemma 19.** *With the assumptions above, we have*

$$\text{Rs}(G) \geq |V(H)| + 1 = 1 + \sum_{i=1}^q |V(H_i)|.$$

**Proof.** Observe first that since all vertices of  $H_i$  are connected to the same vertices outside  $H_i$  and also have the same degree inside  $H_i$ , it follows that  $H_i$  is a regular subgraph in  $H$  (hence the name Regular Canonical coloring). Since  $H_i$  and its complement  $\overline{H_i}$  are  $G$ -free, property 2 implies that there is no copy of  $G$  with all degrees equal.

If there was a copy of  $G$  with all degrees distinct in  $H$ , then no  $H_i$  would contain more than one vertex from  $G$ . Hence, by the construction, there would be a copy of  $G$  in  $F$ , but this is impossible because  $F$  and  $\overline{F}$  are  $G$ -free.

Thus  $H$  is an SR-graph for  $G$ , and hence  $\text{Rs}(G) \geq |V(H)| + 1$ . □

### 3.3 A mixed construction

We apply this method in Sections 4.3 and 4.4.

The construction starts with a graph  $H$  such that  $H$  contains no singular  $G_1$  and  $\overline{H}$  contains no singular  $G_2$ . Partition  $V(H)$  into some number of subsets, say  $V(H) = X_1 \cup \dots \cup X_k$ ; many of those  $X_i$  may also be singletons. As a generalization of substitution, we replace those  $X_i$  with mutually vertex-disjoint graphs  $Q_1, \dots, Q_k$  such that each  $Q_i$  is regular,  $G_1$ -free, and  $\overline{Q_i}$  is  $G_2$ -free. The plan is to create a graph  $F$  whose degree classes are the sets  $V(Q_i)$ , using the structure of  $H$ . If  $X_i$  consists of  $q_i$  vertices from  $H$ , then we partition  $V(Q_i)$  into  $q_i$  subsets. The vertices in the  $j^{\text{th}}$  part of  $Q_i$  are completely adjacent to those classes  $Q_\ell$  which correspond to the neighbors of the  $j^{\text{th}}$  vertex of  $X_i$  in  $H$ . (In particular, if  $X_i$  and  $X_\ell$  are singletons adjacent vertices, then we take complete bipartite adjacency between  $Q_i$  and  $Q_\ell$ .)

A delicate detail in this approach is to ensure that two vertices have the same degree if and only if they are in the same  $Q_i$ . This needs a careful choice of the orders  $|V(Q_i)|$ , the internal degree of each  $Q_i$ , and also the sizes of the partition classes inside  $Q_i$ .

### 3.4 Ramsey-stable graphs

The tool described in this subsection will turn out to be substantial, in the proofs of upper bounds in several results below.

Let  $G$  be a given graph for which we wish to determine or estimate the value of  $\text{Rs}(G)$ . Consider an R-graph  $H$  for  $G$ , say with  $k$  vertices  $v_1, \dots, v_k$ . Let  $N_i$  denote the set of vertices adjacent to  $v_i$  (the neighborhood of  $v_i$ ).

**Definition 20.** We call  $H$  a *Ramsey-stable graph* for  $G$  if, for each  $1 \leq i \leq k$ , the unique way to obtain an R-graph of order  $k$ , in which  $H - v_i$  is an induced subgraph, is to join a new vertex to all vertices of  $N_i$ , and not to join it to any other vertex of  $H - v_i$ . Ramsey-stable graphs for a pair  $(G_1, G_2)$  of graphs can be defined analogously.

**Example 21.** The 5-cycle is Ramsey-stable for  $K_3$ , and also for  $K_{1,3}$ , because the only way to extend  $P_4$  to an R-graph for  $K_3$ , or for  $K_{1,3}$ , is to join a new vertex to the two ends of  $P_4$ .

**Remark 22.** *More generally than the previous example, if we know that all  $n$ -vertex R-graphs for a given  $G$  are regular, then every R-graph  $H$  of order  $n$  is Ramsey-stable for  $G$  because exactly the vertices of minimum degree in  $H - v$  have to be joined by an edge to the new vertex.*

Assume that  $F$  is an SR-graph for a given graph  $G$ , and that the degree sequence of  $F$  contains precisely  $k$  distinct values. We partition  $V(F)$  into the degree classes  $V_1, \dots, V_k$ . Pick one (any) vertex  $v_i$  from each class  $V_i$ , and denote by  $H$  the graph induced by  $\{v_1, \dots, v_k\}$  in  $F$ . Since the set  $\{v_1, \dots, v_k\}$  is irregular in  $F$ , we see that  $H$  is an R-graph for  $G$ .

The significance of Ramsey-stable graphs is shown by the following lemma, which will be crucial in several proofs later on. As a side product, it also implies that if a suitable choice of  $\{v_1, \dots, v_k\}$  gives us a Ramsey-stable  $H$ , then all possible choices of the  $v_i \in V_i$  ( $i = 1, \dots, k$ ) yield the same  $H$ .

**Lemma 23.** (*Regular Substitution Lemma.*) *Let  $F, G, H$  be graphs as above. If  $H$  is Ramsey-stable for  $G$ , then  $F$  is obtained from  $H$  by substituting a regular R-graph for each vertex  $v_i$  of  $H$ . The same structure is valid when  $H$  is Ramsey-stable for a pair  $(G_1, G_2)$ .*

**Proof.** Assume that  $H$  is Ramsey-stable for  $G$ ; the case of  $(G_1, G_2)$  can be handled in exactly the same way. Then for any  $i$ , replacing the vertex  $V(H) \cap V_i$  with any  $v \in V_i$ , the neighborhood remains the same, by assumption. Hence every  $v_j \in N_i$  (which has been taken from the degree class  $V_j$ ) is completely adjacent to  $V_i$ . This is true also when we view the edge  $v_i v_j$  from the other side, from  $v_j$ ; therefore  $v_i$  — and each of its replacement vertices,  $v \in V_i$  — is adjacent to the entire  $V_j$ . Consequently, for each edge  $v_i v_j$  of  $H$ , the edges between  $V_i$  and  $V_j$  in  $F$  form a complete bipartite graph spanning  $V_i \cup V_j$ . On the other hand, by the analogous argument for the non-edges of  $H$ , we see that if  $v_i v_j$  is not an edge in  $H$ , then there are no edges between  $V_i$  and  $V_j$  in  $F$ . Thus,  $F$  is generated by the operation of substitution. As a quantitative consequence, the external degrees of vertices in any one  $V_i$  are all equal.

Equal external degrees imply for a degree class that the internal degrees must also be equal. This implies regularity inside each  $V_i$ .  $\square$

### 3.5 Vertex degrees

In some cases the following approach is useful in deriving upper bounds on  $\text{Rs}(G)$ . We apply it in Section 4.3.

**Lemma 24.** *If, for a given graph  $G$ , every SR-graph of order  $n$  has minimum degree  $\delta$ , then there can be at most  $n - 2\delta$  degree classes.*

**Proof.** Consider any SR-graph  $F$  of order  $n$ , and let  $k$  denote the number of its degree classes. Then, concerning the minimum and maximum degree we have

$$\delta + k - 1 \leq \delta(F) + k - 1 \leq \Delta(F) = n - 1 - \delta(\overline{F}) \leq n - 1 - \delta$$

from where we obtain  $k \leq n - 2\delta$ . □

Typically one can use this in the way that if  $n$  is large then an SR-graph should have not only large minimum degree but also a large number of degree classes, from which a contradiction is derived to the above inequality, concluding that  $\text{Rs}(G) \leq n$ .

## 4 Exact results on $\text{Rs}(G)$ for small graphs

The smallest nontrivial cases are the path  $P_3 = K_{1,2}$  and its subgraphs; they allow a simple solution for  $k$ -singular Ramsey numbers for all  $k$ , which we present in the first subsection. In this way  $K_3$  remains the unique graph  $G$  of order three for which we do not know  $\text{Rs}(G, k)$  over the entire range of  $k$ .

All other subsections of this section deal with the case  $k = 1$  for small graphs, determining  $\text{Rs}$  for every graph with at most four vertices and at most four edges, except for  $C_4$  where we have a non-trivial lower bound. This also includes small star graphs (the claw  $K_{1,3}$ , and the  $K_{1,2}$  which is treated under the name  $P_3$ ); a general theorem for stars will be presented in Section 5.

### 4.1 The path $P_3$ for general $k$ of singularity

**Theorem 25.**  $\text{Rs}(3K_1, k) = \text{Rs}(K_2 \cup K_1, k) = \text{Rs}(P_3, k) = 4k + 1$ .

**Proof.** Clearly, by Theorem 11 above we get  $\text{Rs}(P_3, k) \leq 4k + 1$  as  $\text{R}(P_3) = 3$ .

For the lower bound consider the graph  $H(k)$  on  $4k$  vertices defined as follows:  $V(H(k)) = A \cup B$ , where  $A = \{a_1, \dots, a_{2k}\}$ ,  $B = \{b_1, \dots, b_{2k}\}$ , and  $a_i$  is adjacent to  $b_j$  precisely when  $i \leq j$ .

In this graph, which treats the lower bound for the three graphs  $3K_1$ ,  $K_2 \cup K_1$ ,  $P_3$  together, every degree between 1 and  $2k$  is repeated exactly twice, i.e. no triple can have equal degrees. Also there cannot occur any  $k$ -singular subgraph of order three, because this would require that  $\Delta(H(k)) - \delta(H(k)) \geq 2k$ , however in  $H(k)$  and hence also in its complement the difference is just  $2k - 1$ . □

### 4.2 The path $P_4$ and the 2-matching $2K_2$

Here we prove:

**Theorem 26.**  $\text{Rs}(2K_2) = \text{Rs}(2K_2, P_4) = \text{Rs}(P_4) = 13$ .

**Proof.** By the Monotonicity Principle we have  $\text{Rs}(2K_2) \leq \text{Rs}(2K_2, P_4) \leq \text{Rs}(P_4)$ , therefore it suffices to prove that  $\text{Rs}(2K_2) \geq 13$  and  $\text{Rs}(P_4) \leq 13$ .

For the lower bound on  $\text{Rs}(2K_2)$  we construct an SR-graph on 12 vertices. Consider  $V_1, V_2, V_3$ , where  $|V_i| = 4$  for  $i = 1, 2, 3$ . Let each of  $V_1, V_2, V_3$  induce a  $K_3$  with an isolated vertex. The vertices are labeled as  $V_1 = \{x_1, x_2, x_3, x\}$  where  $x$  is the isolated vertex not in the  $K_3$ , similarly  $V_2 = \{y_1, y_2, y_3, y\}$  with  $y$  not in the  $K_3$ , and  $V_3 = \{z_1, z_2, z_3, z\}$  with  $z$  not in the  $K_3$ .

We complete these vertex classes to a graph  $G$  (color 1) such that all degrees in  $V_1$  are 7, all degrees in  $V_2$  are 5, and all degrees in  $V_3$  are 4. Once this shall be done, there will be no copy of  $2K_2$  with all degrees equal in  $G$  and neither in  $\overline{G}$  because each  $V_i$  induces  $K_3 \cup K_1$  in  $G$  and  $K_{1,3}$  in  $\overline{G}$ . Also there will be no  $2K_2$  with all degrees distinct since this would require four different degrees, while in both  $G$  and  $\overline{G}$  there are only three. We shall do the construction step by step.

First, connect  $x_1$  to  $y_1, y_2, y$ ;  $x_2$  to  $y_2, y_3, y$ ;  $x_3$  to  $y_1, y_3, y$ ; and  $x$  to  $y_1, y_2, y_3, y$ . The degrees are now 4 for  $x, y$ ; 5 for  $x_1, x_2, x_3, y_1, y_2, y_3$ ; 0 for  $z$ ; and 2 for  $z_1, z_2, z_3$ . Next, connect  $x_1$  to  $z_1, z$ ;  $x_2$  to  $z_2, z$ ;  $x_3$  to  $z_3, z$ ;  $x$  to  $z_1, z_2, z_3$ . Then the degrees are 7 for  $x, x_1, x_2, x_3$ ; 5 for  $y_1, y_2, y_3$ ; 4 for  $y, z_1, z_2, z_3$ ; and 3 for  $z$ . Finally, connect  $y$  to  $z$ , and we are done.

For the upper bound on  $\text{Rs}(P_4)$  we will apply the Regular Substitution Lemma. On four vertices precisely two graphs are R-graphs for  $P_4$ : the claw  $K_{1,3}$  and its complement, the triangle  $K_3$  with an isolated vertex. On five vertices every edge 2-coloring contains a monochromatic  $P_4$ . Observe that each of  $K_{1,3}$  and  $K_3 \cup K_1$  is a Ramsey-stable graph for  $\text{Rs}(P_4)$ , because a 3-vertex subgraph with zero or two edges is extendable only to the claw, whereas that with one or three edges is extendable only to the triangle; either extension is unique also concerning the set of neighbors of the new vertex.

Suppose now for a contradiction that there exists an SR-graph  $F$  for  $P_4$  on at least 13 vertices. There can be at most four degree classes in  $F$ , each on at most four vertices. It follows that there are precisely four vertex classes. Due to Lemma 23, each degree class should induce a regular R-graph; but this is impossible for a class with four vertices, which must occur if  $|V(F)| > 12$ . This contradiction completes the proof.  $\square$

### 4.3 The triangle $K_3$ and the claw $K_{1,3}$

Although there is no containment relation between  $K_3$  and  $K_{1,3}$ , the unique R-graph of order 5 for both of them is the 5-cycle. Moreover, on four vertices, every R-graph has positive minimum degree. These facts allow us to treat the two graphs together, and prove the following theorem.

**Theorem 27.**  $\text{Rs}(K_3) = \text{Rs}(K_{1,3}) = 22$ .

**Proof of Lower Bound 22.** We construct an SR-graph of order 21. Consider the 5-cycle  $v_1v_2v_3v_4v_5$  as host graph, and substitute  $F_1, \dots, F_5$  for  $v_1, \dots, v_5$  as follows:

$$F_1 \cong F_2 \cong F_3 \cong C_5, \quad F_4 \cong 2K_2, \quad F_5 \cong K_2.$$

Then the degrees are:

- for  $F_1$ , internal: 2, external:  $2 + 5 = 7$ , total: 9;

- for  $F_2$ , internal: 2, external:  $5 + 5 = 10$ , total: 12;
- for  $F_3$ , internal: 2, external:  $5 + 4 = 9$ , total: 11;
- for  $F_4$ , internal: 1, external:  $5 + 2 = 7$ , total: 8;
- for  $F_5$ , internal: 1, external:  $4 + 5 = 9$ , total: 10.

Since the host graph and also the subgraphs substituted for the degree classes are  $K_3$ -free and  $K_{1,3}$ -free, no singular  $K_3$  or  $K_{1,3}$  occurs.  $\square$

**Proof of Upper Bound 22.** Since  $R(K_3) = R(K_{1,3}) = 6$ , we infer from Theorem 11 that  $Rs(K_3) \leq 26$  as well as  $Rs(K_{1,3}) \leq 26$ . So we have to cover the cases  $n = 22, 23, 24, 25$ , to show that a singular triangle and a singular claw necessarily occurs in each case.

For a contradiction, consider an SR-graph; we know that it can have at most five degree classes, each with at most five vertices. Hence, the following combinations might occur:

- $22 = 5 + 5 + 5 + 5 + 2$
- $22 = 5 + 5 + 5 + 4 + 3$
- $22 = 5 + 5 + 4 + 4 + 4$
- $23 = 5 + 5 + 5 + 5 + 3$
- $23 = 5 + 5 + 5 + 4 + 4$
- $24 = 5 + 5 + 5 + 5 + 4$
- $25 = 5 + 5 + 5 + 5 + 5$

We will show that all of them are impossible.

**First Proof — Degree Counting.** We arrange the degree classes in decreasing order of size  $|V_1| \geq |V_2| \geq |V_3| \geq |V_4| \geq |V_5|$ , and denote the vertices of  $V_i$  as  $v_{i1} v_{i2}, \dots$ . Then consider the seven cases separately.

- $22 = 5 + 5 + 5 + 5 + 2$  — Vertex  $v_{11}$  has precisely two neighbors in each of the sets  $\{v_{12}, v_{13}, v_{14}, v_{15}\}$ ,  $\{v_{21}, v_{31}, v_{41}, v_{51}\}$ ,  $\{v_{22}, v_{32}, v_{42}, v_{52}\}$ ; and has at least one neighbor in each of  $\{v_{23}, v_{33}, v_{43}\}$ ,  $\{v_{24}, v_{34}, v_{44}\}$ ,  $\{v_{25}, v_{35}, v_{45}\}$ . Thus  $d(v_{11}) \geq 9$ . Similarly,  $v_{51}$  has precisely two neighbors in each of the sets  $\{v_{1j}, v_{2j}, v_{3j}, v_{4j}\}$  for  $j = 1, \dots, 5$ , hence  $d(v_{51}) \geq 10$ . This means  $\delta(F) \geq 9$ , as the positions of the other vertices are analogous; and since we have five degree classes,  $\Delta(F) \geq 13$  follows. The same inequalities must hold for  $\overline{F}$ , too. But  $\Delta(F) \geq 13$  implies  $\delta(\overline{F}) = |V(F)| - 1 - \Delta(F) \leq 8$ , a contradiction.
- $22 = 5 + 5 + 5 + 4 + 3$  — Here  $v_{11}$  has two neighbors in  $\{v_{12}, v_{13}, v_{14}, v_{15}\}$  and also in each of the sets  $\{v_{2j}, v_{3j}, v_{4j}, v_{5j}\}$  for  $j = 1, 2, 3$ ; and has at least one neighbor in  $\{v_{24}, v_{34}, v_{44}\}$ , which means  $d(v_{11}) \geq 9$ . Vertex  $v_{41}$  has two neighbors in  $\{v_{1j}, v_{2j}, v_{3j}, v_{5j}\}$  for  $j = 1, 2, 3$ , and at least one neighbor in each of  $\{v_{14}, v_{24}, v_{34}\}$ ,  $\{v_{15}, v_{25}, v_{35}\}$ ,  $\{v_{42}, v_{43}, v_{44}\}$ . Vertex  $v_{51}$  has two neighbors in  $\{v_{1j}, v_{2j}, v_{3j}, v_{4j}\}$  for  $j = 1, 2, 3, 4$ , and at least one neighbor in  $\{v_{15}, v_{25}, v_{35}\}$ . Thus,  $\delta(F) \geq 9$ , a contradiction again.

- $22 = 5 + 5 + 4 + 4 + 4$  — Here the vertices of  $V_1 \cup V_2$  must have degree at least 10, and the vertices of  $V_3 \cup V_4 \cup V_5$  must have degree at least 9.
- $23 = 5 + 5 + 5 + 5 + 3$  — Here the vertices of  $V_5$  have two neighbors in  $\{v_{1j}, v_{2j}, v_{3j}, v_{4j}\}$  for  $j = 1, 2, 3, 4, 5$ , while the other vertices have two neighbors in each of four 4-tuples and one neighbor in each of two triples. Thus  $\delta(F) \geq 10$ ,  $\Delta(F) \geq 14$ ,  $\delta(\overline{F}) < 10$  — a contradiction.
- $23 = 5 + 5 + 5 + 4 + 4$  — Also here, every vertex has degree at least 10, hence the maximum degree should be at least 14.
- $24 = 5 + 5 + 5 + 5 + 4$  — Here  $\delta(F) \geq 11$  and  $\Delta(F) \geq 15$  should hold.
- $25 = 5 + 5 + 5 + 5 + 5$  — This graph should be 12-regular, despite that it has five degree classes.  $\square$

**Second Proof — Ramsey-Stable Graphs.** Since the 5-cycle is the unique R-graph on five vertices, any 5-tuple with one vertex from each degree class must induce  $C_5$ . Thus, by the Regular Substitution Lemma,  $F$  is obtained by substituting *regular* R-graphs into  $C_5$ . In particular, each 5-element  $V_i$  must induce the 2-regular  $C_5$ .

The partition  $22 = 5 + 5 + 5 + 5 + 2$  cannot occur because vertices in both neighbors of the 2-element class along the 5-cycle have external degree  $5 + 2 = 7$  and internal degree 2, contradicting that they are distinct degree classes. The same argument excludes  $23 = 5 + 5 + 5 + 5 + 3$ ,  $24 = 5 + 5 + 5 + 5 + 4$ , and of course  $25 = 5 + 5 + 5 + 5 + 5$  as well.

For  $22 = 5 + 5 + 4 + 4 + 4$  note further that a 4-element  $V_i$  must induce the 2-regular  $C_4$  or the 1-regular  $2K_2$ . Thus, all internal degrees are between 1 and 2, and all external degrees are between 8 and 10, leaving room for no more than four degree classes while we should have five of them. For this reason, the case  $22 = 5 + 5 + 4 + 4 + 4$  cannot occur, and the same argument excludes  $23 = 5 + 5 + 5 + 4 + 4$ .

The only case that remains is  $22 = 5 + 5 + 5 + 4 + 3$ . External degree 7 can only occur for a 5-element class which has internal degree 2. External degree 8 can only occur for a 4-element or a 5-element class, both having internal degree at least 1. All other possibilities yield external degrees at least 9, thus  $\delta(F) \geq 9$ , and we can conclude as in the first proof that  $\Delta(F) \geq 13$  should hold, from which we arrive at the contradiction  $\delta(\overline{F}) \leq 8$ .  $\square$

It turns out that the non-diagonal singular Ramsey number  $\text{Rs}(K_3, K_{1,3})$  is bigger.

**Theorem 28.**  $\text{Rs}(K_3, K_{1,3}) = 29$ .

**Proof of Lower Bound 29.**

We construct a graph  $F$  on 28 vertices, without singular triangles, whose complement  $\overline{F}$  does not contain any singular claws. This  $F$  will have  $k = 5$  degree classes  $V_1, \dots, V_5$ , where  $|V_1| = |V_3| = |V_5| = 6$  and each of those classes induces  $K_{3,3}$ , while  $|V_2| = |V_4| = 5$  and both classes induce  $C_5$ . Hence each degree class is internally regular, with no  $K_3$  in it, and no  $K_{1,3}$  in the complementary graph.

We also partition  $V_5$  into two sets as  $V_5 = V' \cup V''$ , with the only restriction that  $|V'| = 4$  and  $|V''| = 2$ , but no condition on the actual position of vertices. The other edges of  $F$  establish complete adjacencies

- between  $V_1 \cup V_2$  and  $V_3 \cup V_4$ ,
- between  $V_1 \cup V_2$  and  $V'$ ,
- between  $V''$  and  $V_3 \cup V_4$ .

There are no other edges in  $F$ . Then the degrees are:

- in  $V_1$ : internal 3, external  $6 + 5 + 4 = 15$ , total 18;
- in  $V_2$ : internal 2, external  $6 + 5 + 4 = 15$ , total 17;
- in  $V_3$ : internal 3, external  $6 + 5 + 2 = 13$ , total 16;
- in  $V_4$ : internal 2, external  $6 + 5 + 2 = 13$ , total 15;
- in  $V_5$ , for vertices in any of  $V'$  and  $V''$ : internal 3, external  $6 + 5 = 11$ , total 14.

One can observe that every triangle of  $F$  contains two vertices in the same  $V_i$  and one vertex in another class, hence no singular triangles occur. Similarly the complement of  $F$  contains no singular claws. Thus  $F$  satisfies all requirements and yields  $\text{Rs}(K_3, K_{1,3}) \geq 29$ .  $\square$

Concerning the upper bound we first observe some structural properties of the graphs which are R-graphs for  $(K_3, K_{1,3})$ .

**Claim 1.** *We have  $\text{R}(K_3, K_{1,3}) = 7$ , and the unique R-graph of order 6 is  $K_{3,3}$ .*

**Proof.** Observe that  $K_{3,3}$  is the unique triangle-free graph of order 6 whose minimum degree is at least 3. On the other hand, if the minimum degree is smaller than 3, then the complement contains  $K_{1,3}$ .  $\square$

**Claim 2.** *On five vertices there are precisely two graphs  $H$  — namely  $C_5$  and  $K_{2,3}$  — such that  $H$  is triangle-free and  $\overline{H}$  is  $K_{1,3}$ -free. The first one,  $C_5$ , is a Ramsey-stable graph for  $(K_3, K_{1,3})$ .*

**Proof.** All vertex degrees must be at least 2 (otherwise  $\overline{H}$  contains  $K_{1,3}$ ) and at most 3 (otherwise  $H$  contains  $K_3$  or  $\overline{H}$  contains  $K_{1,3}$ ). If  $H$  is 2-regular, then  $H \cong C_5$ . In the remaining case assume that  $d(v) = 3$ . The three neighbors of  $v$  must be mutually non-adjacent, otherwise  $K_3 \subset H$ ; and all of them have to be adjacent to the fifth vertex, since  $\delta(H) \geq 2$ . No further edges can occur, hence  $H \cong K_{2,3}$  in this case. Since no other R-graphs are possible, and  $P_4$  is not an induced subgraph of  $K_{2,3}$ , it is clear that  $C_5$  is Ramsey-stable.  $\square$

**Claim 3.** *Among the regular four-vertex graphs  $H$  there are precisely two — namely  $C_4$  and  $2K_2$  — such that  $H$  is triangle-free and  $\overline{H}$  is  $K_{1,3}$ -free.*

**Proof.** The other two regular graphs of order 4 are  $K_4$  which contains  $K_3$ , and  $4K_1$  whose complement contains  $K_{1,3}$ .  $\square$



**Proof of Upper Bound 29.**

Let  $F$  be an SR-graph for  $(K_3, K_{1,3})$ , say on  $n := \text{Rs}(K_3, K_{1,3}) - 1$  vertices; we need to prove that  $n \leq 28$ . We see from Claim 1 that  $F$  has at most six degree classes  $V_1, \dots, V_k$ , and  $|V_i| \leq 6$  holds for each of them.

**Case 1: Four vertex classes.**

This case is obvious: since  $|V_i| \leq 6$  holds for all  $i$ , we cannot have more than 24 vertices.

**Case 2: Six vertex classes.**

Picking one vertex  $v_i$  from each vertex class  $V_i$  we obtain an R-graph  $H$  of order 6. Due to Claim 1, we have  $H \cong K_{3,3}$ , which is Ramsey-stable. The Regular Substitution Lemma implies that  $F$  is obtained by substituting regular R-graphs for the vertices of  $H$ ; the possible subgraphs with more than three vertices are listed in Claims 1, 2, and 3 (and  $K_{2,3}$  is excluded). Let us denote the subgraphs substituted into the partite sets by  $Q_1, Q_2, Q_3$  and  $R_1, R_2, R_3$ ; and let their respective orders be  $q_1, q_2, q_3, r_1, r_2, r_3$ . Also let us write  $d(q_1), d(q_2), d(q_3), d(r_1), d(r_2), d(r_3)$  for their internal degrees. We fix an indexing such that  $d(q_1) \geq d(q_2) \geq d(q_3)$  and  $d(r_1) \geq d(r_2) \geq d(r_3)$ . Note that all these  $d$  are between 0 and 3 (and 0 can occur only if the substituted graph has at most three vertices).

Denoting  $q = q_1 + q_2 + q_3$  and  $r = r_1 + r_2 + r_3$ , the degree set of  $F$  is

$$q + d(r_1), \quad q + d(r_2), \quad q + d(r_3), \quad r + d(q_1), \quad r + d(q_2), \quad r + d(q_3)$$

with six mutually distinct values. In particular, we must have strict inequalities  $d(q_1) > d(q_2) > d(q_3)$  and  $d(r_1) > d(r_2) > d(r_3)$ . It follows on each side of  $K_{3,3}$  that each of  $K_{3,3}$  and  $C_5$  can be substituted only once, which implies  $\max(q, r) \leq 15$ . Moreover, assuming  $q \geq r$  the degrees cannot be smaller than  $r$  and cannot be larger than  $q + 3$ , hence the presence of six distinct degrees implies  $q + 3 \geq r + 5$ , i.e.  $r \leq q - 2 \leq 13$ . Thus  $n = p + r \leq 28$ .

**Case 3: Five vertex classes.**

As above, we pick one (any) vertex  $v_i$  from each  $V_i$  ( $1 \leq i \leq 5$ ), and consider the graph  $H$  induced by them in  $F$ . Due to Claim 2, this  $H$  must be  $C_5$  or  $K_{2,3}$ . Since  $C_5$  is Ramsey-stable, the proof for it is easy. Indeed, as above, the Regular Substitution Lemma implies that  $F$  is obtained by substituting regular R-graphs for the vertices. But  $n = 29$  or  $n = 30$  would imply that along the 5-cycle four consecutive substitutions would be  $K_{3,3}$ . The two middle ones of them would have external degree 12, internal degree 3, total degree 15, contradicting the assumption that they form distinct degree classes.

Hence, from now on we assume that  $H \cong K_{2,3}$ . Re-label the indices, if necessary, so that the 2-element class of  $K_{2,3}$  is  $\{v_1, v_2\}$  and the 3-element class is  $\{v_3, v_4, v_5\}$ . Although  $K_{2,3}$  is not Ramsey-stable, vertices  $v_1$  and  $v_2$  have the property that replacing any one of them with a vertex from its class, we must obtain again a  $K_{2,3}$ , which implies that  $\{v_3, v_4, v_5\}$  is completely adjacent to  $V_1 \cup V_2$ . Due to the exclusion of singular  $K_3$ , this also forces that  $V_1$  and  $V_2$  are completely non-adjacent.

For a vertex  $v \in V_i$  from  $V_3 \cup V_4 \cup V_5$  there can occur two situations:  $v$  is adjacent either to  $v_1$  and  $v_2$  — in which case we say that  $v$  is in a *stable position* — or to the two vertices of  $\{v_3, v_4, v_5\} \setminus \{v_i\}$ .

If all  $v \in V_3 \cup V_4 \cup V_5$  are in a stable position, then we have complete adjacency between  $V_1 \cup V_2$  and  $V_3 \cup V_4 \cup V_5$ , moreover no edges can occur between  $V_3$  and  $V_4$ , between  $V_3$  and  $V_5$ , and between  $V_4$  and  $V_5$ , and also between  $V_1$  and  $V_2$  either. This yields regular

external degrees for each  $V_i$ . Hence the internal degrees of  $V_3$ ,  $V_4$ , and  $V_5$  must be regular and mutually distinct, as well as those in  $V_1$  and  $V_2$ , what implies that  $|V_1 \cup V_2| \leq 6 + 5 = 11$  and  $|V_3 \cup V_4 \cup V_5| \leq 6 + 5 + 4 = 15$ , thus  $n \leq 26$ .

The occurrence of vertices in non-stable position requires a little more structural analysis. For this, suppose that a  $v \in V_5$  is adjacent to  $v_3$  and  $v_4$ , instead of  $v_1$  and  $v_2$ . Since  $\overline{F}$  contains no singular claws, and  $v \in V_5$  already has non-neighbors in  $V_1$  and  $V_2$ , all vertices of  $V_3 \cup V_4$  are adjacent to  $v$ . Then no edges can occur between  $V_3$  and  $V_4$ , for otherwise  $F$  would contain a singular  $K_3$ . Similarly,  $v$  has no neighbors in  $V_1 \cup V_2$ , because such a neighbor and  $v$  would form a singular  $K_3$  with  $v_3$  (and also with  $v_4$ ).

The non-adjacency of  $V_3$  and  $V_4$  also implies that all vertices in  $V_3 \cup V_4$  are in a stable position. Thus, we have the following structure:

- there is complete adjacency between  $V_1 \cup V_2$  and  $V_3 \cup V_4$ ;
- $V_5$  admits a partition  $V' \cup V''$  such that  $V_1 \cup V_2$  is completely adjacent to  $V'$  and  $V_3 \cup V_4$  is completely adjacent to  $V''$ ;
- no other edges occur between any  $V_i$  and  $V_j$  for  $1 \leq i < j \leq 5$ .

This structure implies that vertices in  $V_1$  and  $V_2$  have the same external degree, namely  $|V_3| + |V_4| + |V'|$ ; and similarly, both  $V_3$  and  $V_4$  have external degree  $|V_1| + |V_2| + |V''|$ . As a consequence,  $|V_1| + |V_2| \leq 6 + 5 = 11$  and  $|V_3| + |V_4| \leq 6 + 5 = 11$ , and finally  $n \leq 22 + |V_5| \leq 28$ .  $\square$

#### 4.4 The paw graph

As another small graph, we determine the singular Ramsey number of the paw, that is a triangle with a pendant edge. Its Ramsey number is  $R(PW) = 7$ . Let us first summarize some facts about the R-graphs.

**Lemma 29.** *For the paw graph,*

- (i) *every graph on at most three vertices is an R-graph, and among them, the regular ones are  $K_3$  and its complement;*
- (ii) *on four vertices there are two regular R-graphs, the 1-regular  $2K_2$  and the 2-regular  $C_4$ ;*
- (iii) *on five vertices there are three R-graphs, namely  $K_2 \cup K_3$  and its complement  $K_{2,3}$  which are non-regular, and the 2-regular  $C_5$ ;*
- (iv) *on six vertices there are two R-graphs, the 2-regular  $2K_3$  and its 3-regular complement,  $K_{3,3}$ .*

**Proof.** Parts (i) and (ii) are obvious. For (iii) one may note that  $C_5$  is the unique R-graph for  $K_3$  on five vertices, and of course it is an R-graph for the paw, too. If  $G$  contains a triangle and is an R-graph for the paw, then the triangle is a connected component. This implies that on five vertices the complement of  $G$  must contain  $K_{2,3}$ , and on six vertices the complement must contain  $K_{3,3}$ . Then there cannot be any further edges in  $\overline{G}$ , hence  $G \cong K_2 \cup K_3$  or  $G \cong 2K_3$ . Analogously, if a triangle occurs in  $\overline{G}$ , then  $G \cong K_{2,3}$  or  $G \cong K_{3,3}$ .  $\square$

The quadratic formula yields the upper bound 37 on  $\text{Rs}(PW)$ , but in fact the exact value is much smaller.

**Theorem 30.**  $\text{Rs}(PW) = 31$ .

**Proof of Lower Bound 31.**

We construct a graph  $F$  of order 30 which is an SR-graph for the paw. It will have five degree classes  $V_1, \dots, V_5$ , each of cardinality 6. The degree classes induce R-graphs:  $F[V_1] \cong F[V_3] \cong F[V_5] \cong 2K_3$ , and  $F[V_2] \cong F[V_4] \cong K_{3,3}$ . (One may verify in the proof below that it would be equally fine to take  $F[V_5] \cong K_{3,3}$ .) Further, we partition  $V_5$  as  $V_5 = V' \cup V''$ , with  $|V'| = 2$  and  $|V''| = 4$ .

We make complete adjacencies between any two of the three sets  $V_1, V_2, V'$ ; and also between any two of  $V_3, V_4, V''$ . There are no further adjacencies; i.e., the only edges between  $V_1 \cup V_2 \cup V'$  and  $V_3 \cup V_4 \cup V''$  occur inside  $V_5$  (namely between  $V'$  and  $V''$ ).

This  $F$  contains no regular paw, because the degree classes are paw-free; and it has no irregular paw either, because omitting the internal edges of the degree classes (which edges certainly cannot occur in *any* irregular subgraph) we obtain a graph which is generated by substituting independent sets into  $2K_3$ . Now we have the following degrees:

- in  $V_1$ : external  $6 + 2 = 8$ , internal 2, total 10;
- in  $V_2$ : external  $6 + 2 = 8$ , internal 3, total 11;
- in  $V_3$ : external  $6 + 4 = 10$ , internal 2, total 12;
- in  $V_4$ : external  $6 + 4 = 10$ , internal 3, total 13;
- in  $V_5$ : external  $6 + 6 = 12$ , internal 2, total 14.

Hence,  $F$  satisfies all requirements and yields  $\text{Rs}(PW) > 30$ . □

**Proof of Upper Bound 31.** Suppose for a contradiction that there exists an SR-graph  $F$  on at least 31 vertices. We know that each degree class contains at most six vertices, therefore we have exactly six degree classes  $V_1, \dots, V_6$ . Picking one vertex  $v_i$  from each  $V_i$ , we get an R-graph, say  $H$ , of order 6, which must be either  $2K_3$  or  $K_{3,3}$ , due to Lemma 29(*iv*). Turning to the complement of  $F$  if necessary, we may assume without loss of generality that  $H \cong 2K_3$ .

Since  $2K_3$  is Ramsey-stable, we see from the Regular Substitution Lemma that  $F$  is obtained by substituting regular R-graphs for the vertices of  $H$ . We are going to analyze the feasible substitutions which create three distinct degree classes for each of the two components. We shall see that it is not possible to have more than 16 vertices in a component, there is just one way to obtain 16, and there are exactly two ways to obtain 15. Hence only the combinations  $32 = 16 + 16$  and  $31 = 16 + 15$  would yield  $n > 30$ , but the argument below will show that each of them would force equal degrees to at least two of the  $V_i$ , which contradicts the definition of degree class.

18: The unique feasible partition is  $18 = 6+6+6$ . But then two of the degree classes induce the same R-graph ( $2K_3$  or  $K_{3,3}$ ), therefore they have the same degree in  $F$ , a contradiction.

17: The unique feasible partition is  $17 = 6 + 6 + 5$ . Then the vertices in both 6-classes have external degree 11. In order that they have different degrees in  $F$ , one of them must induce  $K_{3,3}$  and the other induce  $2K_3$ . Then their degrees in  $F$  are 14 and 13, respectively. However, the class of five vertices has external degree 12 and internal degree 2, yielding total 14, which is not feasible.

16: The two feasible partitions are  $16 = 6 + 6 + 4$  and  $16 = 6 + 5 + 5$ . The latter is easy to exclude, because both 5-classes have internal degree 2 and external degree 11. Concerning  $16 = 6 + 6 + 4$  we see that the two subgraphs for '6' must have distinct internal degrees, hence one of them is  $2K_3$ , the other is  $K_{3,3}$ . Both have external degree  $6 + 4 = 10$ , hence the vertex degrees are 12 and 13, respectively. This implies that the subgraph for '4', which has external degree 12, must be  $C_4$  because  $2K_2$  with internal degree 1 would repeat the degree 13. Thus the degrees necessarily are

$$12, 13, 14.$$

Of course, this cannot occur on more than one triangle; i.e., the case  $n = 32 = 16 + 16$  is impossible.

15: The possible partitions are  $15 = 6 + 6 + 3$ ,  $15 = 6 + 5 + 4$ ,  $15 = 5 + 5 + 5$ . We can immediately exclude the last one because '5' necessarily means  $C_5$  with internal degree 2, hence in a substitution of the type  $5 + 5 + 5$  the graph would be regular of degree 12. Concerning  $15 = 6 + 6 + 3$  — similarly to the case of  $16 = 6 + 6 + 4$  — we see that  $2K_3$  and  $K_{3,3}$  have to be substituted for  $6 + 6$ , yielding vertex degrees 11 and 12. For '3' we have external degree 12, hence  $3K_1$  is not an alternative, we have to substitute the other regular graph,  $K_3$ , which has internal degree 2. In this way we obtain the degrees

$$11, 12, 14$$

which cannot be coupled with the case (12, 13, 14) of  $16 = 6 + 6 + 4$ .

In  $15 = 6 + 5 + 4$  the '5' class means  $C_5$  with internal degree 2 and external degree 10, i.e. degree 12 in  $F$ . Therefore the '4' class with external degree 11 must be  $C_4$  with internal degree 2 and total degree 13. The external degree for '6' is 9, hence internal degree 3 is infeasible, thus we have to substitute  $2K_3$  which leads to degree 11 and in this way we obtain the degree set

$$11, 12, 13.$$

From this, it is clear that  $16 + 15$  cannot occur, and even  $15 + 15$  would be impossible. (In fact, degree 12 appears in all the three types above, and any two types have two values in common.)  $\square$

**Remark 31.** *The construction on 30 vertices is another example of the mixed principle as described in Section 3.3. Here we start from the graph  $H = 2K_3 + e$ , two vertex-disjoint triangles connected by just one edge  $e$ . Although this  $H$  is not paw-free, still does not contain a singular paw; and its complement  $\overline{H} \cong K_{3,3} - e$  is paw-free. Then the two ends of the edge  $e$  can be viewed together as one partition class, while the other classes are singletons. Each end of  $e$  has two neighbors in  $H$  and this yields two neighbor classes for the corresponding subsets after substitution. In case of the paw, two classes of order 6 with identical neighborhood may occur because their internal degree can (and should) be distinct.*

## 4.5 The 4-cycle $C_4$

In case of  $C_4$ , which seems most problematic among the small graphs, we can derive lower and upper bounds which are quite close to each other, but still the exact value of  $\text{Rs}(C_4)$  is unknown.

Note that the 4-cycle has  $R(C_4) = 6$ , and its two  $R$ -graphs of order 5 are  $C_5$  and the bull. Neither of them is Ramsey-stable. Indeed, removing a vertex from  $C_5$  we obtain  $P_4$ , which is extendable not only to  $C_5$  itself, but also to the bull. Similarly, removing the degree-2 vertex from the bull we obtain  $P_4$  which is extendable to  $C_5$ . Moreover, the removal of a pendant vertex from the bull yields the paw, which can be extended to the bull in two different ways. Also, removing a vertex of degree 3 we obtain  $P_3 \cup K_1$ , whose extension to the bull fixes an edge to the isolated vertex, and another edge to the middle of  $P_3$ , but the last edge can go to either end of  $P_3$ .

**Proposition 32.**  $24 \leq \text{Rs}(C_4) \leq 26$ .

**Proof.** The upper bound is a consequence of Corollary 12. For the lower bound we construct an SR-graph on 23 vertices. Let us take the bull as host graph  $H$ , labeling its vertices as  $v_1, \dots, v_5$  where  $\{v_1, v_2, v_3\}$  induces a triangle, and the two pendant edges are  $v_1v_4$  and  $v_3v_5$ . Let us substitute graphs  $F_i$  for  $v_i$  such that  $F_1 \cong K_3$  and  $F_i \cong C_5$  for all  $2 \leq i \leq 5$ . All internal degrees are equal to 2, and the external degrees are 15 in  $F_1$ , 8 in  $F_2$ , 13 in  $F_3$ , 3 in  $F_4$ , and 5 in  $F_5$ . Neither  $F$  nor its complement contains any singular copy of  $C_4$ , hence  $\text{Rs}(C_4) \geq 24$ .  $\square$

## 4.6 Small graphs with isolates

In this last of the subsections devoted to small graphs we give the values for those graphs of order four which have isolated vertices. There are three such graphs:  $K_2 \cup 2K_1$ ,  $P_3 \cup K_1$ , and  $K_3 \cup K_1$ . Note that some lower bounds can easily be obtained from above:

- Theorem 13 (with reference also to Remark 8) implies

$$\text{Rs}(P_3 \cup K_1) \geq \text{Rs}(K_2 \cup 2K_1) \geq 10.$$

- The construction of Theorem 27 yields

$$\text{Rs}(K_3 \cup K_1) \geq 22.$$

We prove that these bounds are tight.

**Proposition 33.**  $\text{Rs}(P_3 \cup K_1) = \text{Rs}(K_2 \cup 2K_1) = 10$ .

**Proof.** In every graph  $G$  with 10 vertices there exists a singular subgraph of order four. It necessarily contains a  $P_3$  or its complement, which can be extended to a singular  $P_3 \cup K_1$ . Thus  $\text{Rs}(P_3 \cup K_1) \leq 10$ .  $\square$

**Theorem 34.**  $\text{Rs}(K_3 \cup K_1) = 22$ .

**Proof.** Suppose for a contradiction that  $F$  is an SR-graph of order  $n \geq 22$  for  $K_3 \cup K_1$ . We know that a singular  $K_3$  occurs in  $F$  (or in its complement), say it has the vertices  $v_1, v_2, v_3$ . If the degrees of this  $K_3$  are all distinct, say  $d_1 < d_2 < d_3$ , then it would be extendable to a singular  $K_3 \cup K_1$  unless all vertices of  $F$  have their degree from  $\{d_1, d_2, d_3\}$ . But then a degree class would have at least eight vertices, so that  $F$  would contain even a singular  $K_3 \cup 5K_1$ .

Hence suppose that the three vertices of any singular  $K_3$  have the same degree in  $F$ . Since  $R(K_3) = 6$ , there can be at most five degree classes, and we easily find a singular  $K_3 \cup K_1$  unless all degree classes have at most five vertices and the degree class(es) inducing a triangle have exactly three vertices. In particular,  $\{v_1, v_2, v_3\}$  itself is a degree class, moreover its complementary 19 or more vertices form only four degree classes. Now we can only have the following possibilities:

- $n = 22 = 3 + 4 + 5 + 5 + 5$ ,
- $n = 23 = 3 + 5 + 5 + 5 + 5$ .

And then the degree counting method in the proof of Theorem 27 can be repeated for these two cases without any changes, leading to the contradiction  $\delta(F) \geq 9$  for  $n = 22$  and  $\delta(F) \geq 10$  for  $n = 23$ .  $\square$

## 5 Stars of any size

The star with  $s$  edges,  $K_{1,s}$ , is an easy case concerning Ramsey numbers; cf. e.g. Section 5.5 of [19]:

- if  $s$  is odd, then  $R(K_{1,s}) = 2s$ , and the extremal R-graphs are precisely the  $(s - 1)$ -regular graphs of order  $2s - 1$ ;
- if  $s$  is even, then  $R(K_{1,s}) = 2s - 1$ , and the extremal R-graphs are the graphs of order  $2s - 2$  with minimum degree at least  $s - 2$  and maximum degree at most  $s - 1$ . In particular, the largest *regular* R-graphs are those graphs of order  $2s - 2$  which are  $(s - 2)$ -regular or  $(s - 1)$ -regular.

It turns out that the parity of  $s$  is essential also with respect to  $R_s$ . The case of even  $s$  is simpler, the quadratic upper bound always is tight.

**Proposition 35.** *If  $s$  is even, then*

$$R_s(K_{1,s}) = (R(K_{1,s}) - 1)^2 + 1 = (2s - 2)^2 + 1.$$

**Proof.** The upper bound  $(2s - 2)^2 + 1$  follows from Corollary 12. For the same lower bound we construct an RS-graph  $F$  of order  $(2s - 2)^2$  on vertex set

$$V = (A_1 \cup \cdots \cup A_{s-1}) \cup (B_1 \cup \cdots \cup B_{s-1})$$

where all sets  $A_i$  and  $B_i$  are mutually disjoint, each of cardinality  $2s - 2$ . The edges of  $F$  are defined as follows:

- each  $A_i$  induces an  $(s - 2)$ -regular graph;
- each  $B_i$  induces an  $(s - 1)$ -regular graph;
- there is no edge between  $A_i$  and  $A_j$  for  $i \neq j$ ;
- there is no edge between  $B_i$  and  $B_j$  for  $i \neq j$ ;
- every  $A_i$  and  $B_j$  are completely adjacent for  $i \neq j$ ;
- the sets  $A_i$  and  $B_i$  are adjacent by a  $(2i)$ -regular bipartite graph, for all  $1 \leq i \leq s - 1$ .

Then, both in  $F$  and in  $\overline{F}$ , each vertex  $v$  is adjacent to at most  $s - 1$  vertices of distinct degrees different also from the degree of  $v$ ; and to at most  $s - 1$  vertices whose degree is equal to that of  $v$ . Thus,  $F$  is an RS-graph for  $K_{1,s}$ .  $\square$

The case of odd  $s$  is more complicated. The quadratic upper bound is never attained, although the singular Ramsey number is not far from it. For the tightness of the lower bound we give two very different constructions, with the purpose to indicate that — contrary to  $R(K_{1,s})$  — the extremal graphs for  $\text{Rs}(K_{1,s})$  may be quite hard to characterize.

**Theorem 36.** *If  $s$  is odd, then*

$$\text{Rs}(K_{1,s}) = (R(K_{1,s}) - 1)^2 + 1 - (2s - 2) = (2s - 1)(2s - 2) + 2.$$

**Proof of the Upper Bound.** Suppose for a contradiction that there exists an SR-graph  $F$  of order at least  $(2s - 1)(2s - 2) + 2$  for  $K_{1,s}$ . Denote its degree classes by  $V_1, \dots, V_m$ . We know that  $m \leq 2s - 1$ , and also  $|V_i| \leq 2s - 1$  for all  $1 \leq i \leq m$ . Hence  $m = 2s - 1$  must hold. Since all R-graphs of order  $2s - 1$  are regular, all of them are Ramsey-stable, due to Remark 22. Thus, by the Regular Substitution Lemma, each  $V_i$  induces a regular R-graph, and two distinct  $V_i, V_j$  are either completely adjacent or completely nonadjacent. From this we obtain that the structure of adjacencies between the degree classes is an  $(s - 1)$ -regular graph of order  $2s - 1$ , therefore

- every external degree is at most  $(s - 1)(2s - 1)$ , and every internal degree is at most  $s - 1$ , therefore the maximum degree of  $F$  is at most  $2s(s - 1)$  and the minimum degree cannot be larger than  $2s(s - 1) - (2s - 2) = 2(s - 1)^2$ .

Let us define  $r_i := (2s - 1) - |V_i|$  for  $i = 1, \dots, m$ . Then the internal degree inside  $V_i$  is at least  $(s - 1) - r_i$ . Moreover, if a  $V_j$  is adjacent to  $V_i$ , then it contributes to the external degree of every  $v \in V_i$  with exactly  $(2s - 1) - r_j$ . It follows that

- the minimum degree is at least  $2s(s - 1) - \sum_{i=1}^{2s-1} r_i$

from where we obtain that

$$\sum_{i=1}^{2s-1} r_i \geq 2s - 2$$

and

$$|V(F)| = (2s - 1)^2 - \sum_{i=1}^{2s-1} r_i \leq (2s - 1)^2 - (2s - 2) < (2s - 1)(2s - 2) + 2,$$

a contradiction.

**Proof of the Lower Bound.** Let  $s = 2t + 1$ ; then  $R(K_{1,s}) - 1 = 2s - 1 = 4t + 1$ . We start with the graph  $H = (C_{4t+1})^t$  which has vertices  $x_0, x_1, \dots, x_{4t}$  and edges  $x_i x_{i\pm j}$  for  $j = 1, \dots, t$ , where subscript addition is taken modulo  $4t + 1$ . For each  $x_i$  we substitute a  $d_i$ -regular graph  $G_i$  with vertex set  $V_i$  in the following way:

- $2t + 1$  sets  $|V_1| = \dots = |V_{2t+1}| = 4t + 1$  and  $d_1 = \dots = d_{2t+1} = 2t$ ;
- $t$  sets  $|V_{2t+2}| = \dots = |V_{3t+1}| = 4t$  and  $d_{2t+2} = \dots = d_{3t+1} = 2t - 1$ ;
- $t - 1$  sets  $|V_{3t+2}| = \dots = |V_{4t}| = 4t$  and  $d_{3t+2} = \dots = d_{4t} = 2t$ ;
- 1 set  $|V_0| = 2t$  and  $d_0 = t$ .

Recall that  $V_i$  is adjacent to  $V_{i-t}, V_{i-t+1}, \dots, V_{i-1}, V_{i+1}, V_{i+2}, \dots, V_{i+t}$ , where subscript addition is taken modulo  $4t + 1$ . Then the obtained degrees — more precisely their differences from the maximum of the internal / external / total degree — can be summarized as shown in Table 1.

	$V_1$	...	$V_t$	$V_{t+1}$	...	$V_{2t+1}$	$V_{2t+2}$	...	$V_{3t}$	$V_{3t+1}$	$V_{3t+2}$	...	$V_{4t}$	$V_0$
SZ	0	...	0	0	...	0	1	...	1	1	1	...	1	$2t + 1$
ID	0	...	0	0	...	0	1	...	1	1	0	...	0	$t$
ED	$3t$	$\searrow$	$2t + 1$	0	$\nearrow$	$t$	$t$	$\nearrow$	$2t - 2$	$4t - 1$	$4t - 1$	$\searrow$	$3t + 1$	$t$
TD	$3t$	$\searrow$	$2t + 1$	0	$\nearrow$	$t$	$t + 1$	$\nearrow$	$2t - 1$	$4t$	$4t - 1$	$\searrow$	$3t + 1$	$2t$

Table 1: Some parameters of the extremal construction for unrestricted  $s = 2t + 1$ . SZ =  $\{(2s - 1) \text{ minus size}\} = 4t + 1 - |V_i|$ ; ID =  $\{(s - 1) \text{ minus internal degree}\} = 2t - d_i$ ; ED =  $\{(s - 1)(2s - 1) \text{ minus external degree}\}$ ; TD =  $\{2s(s - 1) \text{ minus total degree}\}$ ;  $\searrow, \nearrow$  = decreasing / increasing by 1 in each step.

Then the degrees range between  $2s(s - 1) - 4t = 2s(s - 1) - (2s - 2)$  and  $2s(s - 1)$ , and the number of vertices is  $16t^2 + 4t + 1 = (2s - 1)(2s - 2) + 1$ . Both in the graph and in its complement, each vertex has neighbors only in  $s - 1$  other degree classes, and at most  $s - 1$  neighbors in its degree class. Hence we have an SR-graph of the required order.

**Alternative construction for  $s = 4q + 1$ .** The basic structure  $H = (C_{4t+1})^t = (C_{8q+1})^{2q}$  remains the same, but the size distribution of substituted R-graphs will be substantially different: they will have almost equal sizes, rather than involving a very small degree class. We need a construction on  $(2s - 1)^2 - (2s - 2) = (8q + 1)^2 - 8q$  vertices. This will be achieved by taking  $4q + 1$  degree classes of size  $8q + 1$ , moreover  $2q$  classes of size  $8q$ , and  $2q$  classes of size  $8q - 2$ .



	$V_{4q}$	$V_{4q-1}$	$V_{4q-2}$	$V_{4q-3}$	$V_{4q-4}$	...	$V_{2q+3}$	$V_{2q+2}$	$V_{2q+1}$	$V_{2q}$
SZ	0	0	0	0	0	...	0	0	0	0
ID	0	0	0	0	0	...	0	0	0	0
ED	0	1	2	5	6	...	$4q - 7$	$4q - 6$	$4q - 3$	$4q - 2$
TD	0	1	2	5	6	...	$4q - 7$	$4q - 6$	$4q - 3$	$4q - 2$

	$V_{4q+1}$	$V_{4q+2}$	$V_{4q+3}$	$V_{4q+4}$	...	$V_{6q-3}$	$V_{6q-2}$	$V_{6q-1}$	$V_{6q}$
SZ	0	0	0	0	...	0	0	0	0
ID	0	0	0	0	...	0	0	0	0
ED	3	4	7	8	...	$4q - 5$	$4q - 4$	$4q - 1$	$4q + 2$
TD	3	4	7	8	...	$4q - 5$	$4q - 4$	$4q - 1$	$4q + 2$

	$V_{2q-1}$	$V_{2q-2}$	$V_{2q-3}$	$V_{2q-4}$	$V_{2q-5}$	$V_{2q-6}$	...	$V_3$	$V_2$	$V_1$	$V_0$
SZ	1	1	3	1	3	1	...	3	1	3	1
ID	0	1	1	1	1	1	...	1	1	1	1
ED	$4q$	$4q+3$	$4q+2$	$4q+7$	$4q+6$	$4q+11$	...	$8q - 10$	$8q - 5$	$8q - 6$	$8q - 1$
TD	$4q$	$4q+4$	$4q+3$	$4q+8$	$4q+7$	$4q+12$	...	$8q - 9$	$8q - 4$	$8q - 5$	$8q$

	$V_{6q+1}$	$V_{6q+2}$	$V_{6q+3}$	$V_{6q+4}$	$V_{6q+5}$	$V_{6q+6}$	...	$V_{8q-3}$	$V_{8q-2}$	$V_{8q-1}$	$V_{8q}$
SZ	3	1	3	1	3	1	...	3	1	3	3
ID	1	1	1	1	1	1	...	1	1	1	2
ED	$4q$	$4q+5$	$4q+4$	$4q+9$	$4q+8$	$4q+13$	...	$8q - 8$	$8q - 3$	$8q - 4$	$8q - 3$
TD	$4q+1$	$4q+6$	$4q+5$	$4q+10$	$4q+9$	$4q+14$	...	$8q - 7$	$8q - 2$	$8q - 3$	$8q - 1$

Table 2: Parameters in the following ranges:  $4q-1 \geq i \geq 2q$ ;  $4q+1 \leq i \leq 6q$ ;  $2q-1 \geq i \geq 0$ ;  $6q+1 \leq i \leq 8q$ . Notations SZ, ID, ED, TD are the same as in Table 1.

We use the symbol  $G_p^d$  to denote any  $d$ -regular graph on  $p$  vertices. Such graphs exist whenever  $p > d \geq 0$  and  $pd$  is even. In the construction below, the actual structure of a  $G_p^d$  will be irrelevant, one may take different graphs for different appearances of the same pair  $(p, d)$ . Using the notation  $V_i$  and  $G_i$  in the sense as above, we now define:

- for every  $i$  in the range  $2q \leq i \leq 6q$  we take  $|V_i| = 8q + 1$ , and let each  $G_i$  be a  $G_{2s-1}^{s-1}$ ;
- with the only one exception of  $V_{8q}$ , for all  $1 \leq i \leq q$  we take  $|V_{4q \pm (2q+2i)}| = 8q$ , and let each  $G_i$  be a  $G_{2s-2}^{s-2}$ ;
- with the only one exception of  $V_{2q-1}$ , for all  $1 \leq i \leq q$  we take  $|V_{4q \pm (2q+2i-1)}| = 8q - 2$ , and let each  $G_i$  be a  $G_{2s-4}^{s-2}$ ;

- for the two exceptional cases we take  $|V_{8q}| = 8q - 2$  with  $G_{8q} = G_{2s-4}^{s-3}$  and  $|V_{2q-1}| = 8q$  with  $G_{2q-1} = G_{2s-2}^{s-1}$ .

The maximum degree occurs at the vertices of  $V_{4q}$ : they have internal degree  $s - 1$ , external degree  $(s - 1)(2s - 1)$ , and total degree  $2s(s - 1)$ . Relevant parameters of vertices in the other degree classes are summarized in Table 2. One can check that each  $V_i$  has a distinct degree, and consequently we obtained an SR-graph of maximum order.  $\square$

## 6 Asymptotics for singular Turán numbers

In this section we present estimates on the singular Turán numbers  $\text{Ts}(n, F)$ , and compare them to the classical Turán number  $\text{ex}(n, K_s)$ , which is the maximum number of edges in a graph of order  $n$  not containing a complete subgraph of order  $s$ . Assume<sup>3</sup> that  $F$  has order  $p := |V(F)| \geq 3$  and chromatic number  $q := \chi(F) \geq 2$ .

We begin with two general constructions, providing lower bounds on  $\text{Ts}(n, F)$ .

Let us assume that  $n$  is a multiple of  $q - 1$ ; regarding lower bounds for other orders we refer to the simple fact that

$$\text{Ts}(n, F) \geq \text{Ts}\left((q - 1) \left\lfloor \frac{n}{q - 1} \right\rfloor, F\right).$$

This will give a fairly good approximation because the difference between the numbers of edges for two consecutive multiples of  $q - 1$  will be  $O(n)$  only, while  $\text{Ts}$  will be shown to grow with a quadratic function of  $n$ . Even in a more general setting where  $F$  is not a fixed graph and  $q$  varies, say  $F = K_{\lfloor \sqrt{n} \rfloor}$ , the difference between the numbers of edges will grow with  $O(qn) = o(n^2)$ , which is negligible compared to  $\text{Ts}(n, F)$ .

The higher structure of both constructions is a partition of the  $n$ -element vertex set into  $p - 1$  classes  $V_1, V_2, \dots, V_{p-1}$ , where

- each  $|V_i|$  is a multiple of  $q - 1$ , and
- each  $V_i$  induces a Turán graph for  $K_q$ , i.e., the subgraph induced by  $V_i$  in the graph of order  $n$  under construction is a complete multipartite graph with  $q - 1$  vertex classes  $U_{i,1}, U_{i,2}, \dots, U_{i,q-1}$  of equal size,

$$|U_{i,1}| = |U_{i,2}| = \dots = |U_{i,q-1}|,$$

each  $U_{i,j}$  is an independent set, and any two of them are completely adjacent.

In both constructions the degree classes will be  $V_1, V_2, \dots, V_{p-1}$ .

**Construction 37.** Choose the sizes of the degree classes  $V_i$  in such a way that

$$|V_1| < |V_2| < \dots < |V_{p-1}|$$

holds, and under this condition  $|V_1|$  is as large as possible, whereas  $|V_{p-1}|$  is as small as possible.

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<sup>3</sup>If  $p = 2$ , then either  $F = 2K_1$  which is a singular subgraph of every graph with at least two vertices — hence  $\text{Ts}(n, F)$  is meaningless — or  $F = K_2$  and  $\text{Ts}(n, F) = 0$  for all  $n$ . The situation is similar if  $\chi(F) = 1$ , i.e.  $F = pK_1$ , in which case  $\text{Ts}(n, F)$  cannot be defined for  $n > (p - 1)^2$ .

Since the sequence  $|V_1|, |V_2|, \dots, |V_{q-1}|$  is strictly increasing, we must have  $|U_{i,j}| \geq |U_{1,j}| + i - 1$  for all  $1 \leq i < p - 1$  (and all  $1 \leq j \leq q - 1$ ). Then the requirement on  $|V_1|$  and  $|V_{p-1}|$  means that we need to maximize  $|U_{1,1}|$  subject to

$$(p - 1)(q - 1)|U_{1,1}| + (q - 1) \sum_{i=2}^{p-1} (i - 1) \leq n,$$

from where we obtain that  $|U_{1,j}| \approx \frac{n}{(p-1)(q-1)} - \frac{p}{2}$  and  $|U_{p-1,j}| \approx \frac{n}{(p-1)(q-1)} + \frac{p}{2}$ . In fact either  $|U_{p-1,1}| = |U_{1,1}| + p - 2$  or  $|U_{p-1,1}| = |U_{1,1}| + p - 1$ . By construction we also have:

- the vertices of  $V_i$  have degree  $n - |U_{i,1}|$ .

This implies that the degree sets are indeed the classes  $V_i$ , and two types of singular subgraphs can occur:

- subgraphs of a  $V_i$ , thus having chromatic number less than  $q$ ;
- subgraphs with at most one vertex in each  $V_i$ , thus having order less than  $p$ .

It follows that the constructed graph does not contain any singular subgraph isomorphic to  $F$ .

Let us compare the number of edges with that in the Turán graph for  $K_{(p-1)(q-1)+1}$ .

**Proposition 38.** *Let  $F$  be a graph with  $p \geq 3$  vertices and chromatic number  $q \geq 2$ . If  $n$  is a multiple of  $q - 1$ , then*

$$\text{ex}(n, K_{(p-1)(q-1)+1}) - \text{Ts}(n, F) \leq cqp^3.$$

for a constant  $c$ . If  $n \equiv r \pmod{(q - 1)}$  with  $r \neq 0$ , then

$$\text{ex}(n, K_{(p-1)(q-1)+1}) - \text{Ts}(n, F) \leq O(rn).$$

**Proof.** Suppose first that  $n$  is divisible by  $q - 1$ . From the graph obtained in Construction 37 we can obtain the Turán graph if, for every  $1 \leq i \leq \frac{p-1}{2}$  and every  $1 \leq j \leq q - 1$ , we replace the vertex classes  $U_{i,j}$  and  $U_{p-i,j}$  with two classes (independent sets) of sizes  $\lfloor \frac{|U_{i,j}| + |U_{p-i,j}|}{2} \rfloor$  and  $\lceil \frac{|U_{i,j}| + |U_{p-i,j}|}{2} \rceil$ . Due to the identity  $(x - a)(x + a) = x^2 - a^2$ , this operation increases the number of edges proportionally to  $(p/2 - i)^2$ , because the subgraph induced by  $U_{i,j} \cup U_{p-i,j}$  remains a complete bipartite graph on exactly the same set of vertices and with an unchanged number of edges to its exterior. There are  $q - 1$  choices for  $j$ , and  $i$  runs from 1 to  $\lfloor (p - 1)/2 \rfloor$ , hence the total difference grows with the order of  $qp^3$ .

If  $n = t(q - 1) + r$  with  $r \neq 0$ , then we supplement the construction with  $r$  isolated vertices, hence no singular  $F$  will arise while the number of edges does not decrease (actually remains unchanged). On the other hand, the Turán function clearly satisfies the inequality  $\text{ex}(n, H) - \text{ex}(n - r, H) < rn$  for every graph  $H$  and all natural numbers  $n$  and  $r$ .  $\square$

**Construction 39.** For the sake of simpler description assume that  $n$  is a multiple of  $(p-1)(q-1)$ , with  $q \geq 2$  and  $p \geq 4$ . We define all  $U_{i,j}$  to have the same size ( $1 \leq i \leq p-1$ ,  $1 \leq j \leq q-1$ ), i.e.  $|U_{i,j}| = \frac{n}{(p-1)(q-1)}$ , each of them being an independent set; hence in particular  $|V_i| = \frac{n}{p-1}$ , where  $V_i = \bigcup_{j=1}^{q-1} U_{i,j}$ . Start with complete bipartite graphs between any two  $U_{i_1,j_1}, U_{i_2,j_2}$ . Represent the sets  $V_i$  with single vertices  $v_i$ , and view them as the vertices of  $K_{p-1}$ . It was proved by Chartrand et al. [13] that the edges of  $K_{p-1}$  can be assigned with integer weights from  $\{1, 2, 3\}$  in such a way that the weighted degrees of the vertices become mutually distinct. Now, for each vertex pair,

- if the weight of  $v_{i_1}v_{i_2}$  is 1, keep complete adjacency between  $V_{i_1}$  and  $V_{i_2}$ ;
- if the weight of  $v_{i_1}v_{i_2}$  is 2, remove a perfect matching between  $V_{i_1}$  and  $V_{i_2}$ ;
- if the weight of  $v_{i_1}v_{i_2}$  is 3, remove a 2-factor between  $V_{i_1}$  and  $V_{i_2}$ .

By construction, the degree classes are the sets  $V_i$ , hence the graph does not contain any singular subgraph with  $p$  vertices and chromatic number  $q$ ; and the number of removed edges is at most  $(p-2)n$ . In fact, applying the results of [21], this upper bound can be reduced to  $(p/2 + c)n$ , where  $c$  is a universal constant for all  $n$  and  $p$ , which is tight apart from the actual value of  $c$ .

Next, we prove an upper bound which shows that the constructions above give tight asymptotics on  $\text{Ts}(n, F)$  for every fixed graph  $F$  as  $n$  gets large.

**Theorem 40.** If  $F$  is a graph with  $p \geq 3$  vertices and chromatic number  $q \geq 2$ , then

$$\text{Ts}(n, F) \leq \text{ex}(n, K_{(p-1)(q-1)+1}) + o(n^2).$$

Moreover, for the complete graph  $K_p$  (i.e.,  $q = p$ ) we have

$$\text{Ts}(n, K_p) \leq \text{ex}(n, K_{(p-1)^2+1}).$$

Both upper bounds are asymptotically sharp as  $n \rightarrow \infty$ .

**Proof.** We begin with the inequality for  $K_p$ , as it is much simpler to prove. If a graph  $G$  of order  $n$  has more than  $\text{ex}(n, K_{(p-1)^2+1})$  edges, then by definition it contains a complete subgraph on  $(p-1)^2 + 1$  vertices. Among them,  $p$  have the same degree in  $G$  or  $p$  have mutually distinct degrees. Thus,  $G$  contains  $K_p$  as a singular subgraph, which implies that  $\text{Ts}(n, K_p)$  cannot be that large.

In the general case let us assume that  $G$  is a graph of order  $n$ , having as many as  $\text{ex}(n, K_{(p-1)(q-1)+1}) + \epsilon n^2$  edges. We are going to apply the Erdős–Stone theorem [18], which states that for any fixed  $\epsilon > 0$  a graph with  $n$  vertices and  $\text{ex}(n, K_s) + \epsilon n^2$  edges contains not only a  $K_s$  but also a complete multipartite graph with  $s$  vertex classes with  $t$  vertices in each class; here  $t$  can be taken any large as  $n$  increases.<sup>4</sup> We take  $s = (p-1)(q-1) + 1$  and assume that  $n$  is large enough to ensure that also  $t$  is sufficiently large, say  $t \geq p^2$ .

<sup>4</sup>For our purpose with a fixed  $F$ , the classical theorem by Erdős and Stone from 1946 is sufficiently strong. An improved numerical estimate on  $t$  was derived three decades later by Bollobás et al. in [7], and finally Chvátal and Szemerédi proved in [16] that  $t$  grows as fast as  $c \frac{\log n}{\log 1/\epsilon}$ . This version is useful when one takes a sequence of graphs  $F$  whose orders tend to infinity as  $n$  gets large but does not exceed  $c' \sqrt{\log n}$  for a small constant  $c'$ , e.g. in case of  $F = K_{\lceil \log \log n \rceil}$ .

Let  $A_1, \dots, A_{(p-1)(q-1)+1}$  be the vertex classes of a  $K_{t, \dots, t} \subset G$ . Each  $A_i$  contains a singular  $B_i \subset A_i$  with  $|B_i| \geq \sqrt{t} \geq p$ , with vertices whose degrees are all equal or all distinct in  $G$ .

If the degrees are all distinct in at least  $p$  of the sets  $B_i$ , then we can sequentially select one vertex from each  $B_i$  such that in each step the degree of the selected vertex is distinct from all previously selected ones. This yields a singular  $K_p \subset G$ , thus also  $F$  occurs as a singular subgraph of  $G$ .

Suppose that there are only  $h$  sets  $B_i$  (where  $0 \leq h \leq p - 1$ ) inside which the degrees are distinct. We assume that these classes are the ones with largest subscripts, namely  $B_{(p-1)(q-1)-h+2}, \dots, B_{(p-1)(q-1)+1}$ . Then we obtain a sequence  $d_1, d_2, \dots, d_{(p-1)(q-1)-h+1}$  where  $d_i$  is the degree of all vertices in  $B_i$ . If this sequence contains  $q$  equal terms, then from the corresponding sets we can select the vertices for the color classes of  $F$  in a proper  $q$ -coloring, thus  $F$  is a singular subgraph of  $G$  with all degrees equal. Else every value occurs at most  $q - 1$  times, hence the sequence contains at least  $z := \left\lceil \frac{(p-1)(q-1)-h+1}{q-1} \right\rceil$  mutually distinct terms, which can be supplemented at least with further  $\min(h, p - z)$  distinct degrees from the last  $h$  sets  $B_i$ . Now we have

$$\left\lceil \frac{(p-1)(q-1)-h+1}{q-1} \right\rceil + h = p - 2 + h + \left\lceil \frac{q-h}{q-1} \right\rceil \geq p,$$

with equality only if  $h = 0$  or  $h = 1$  or  $q = h = 2$ . Consequently,  $K_p$  occurs as a singular subgraph of  $G$  with all degrees distinct.

Asymptotic tightness follows from the constructions described above, for both cases.  $\square$

For the case  $p = q = 3$  and  $n \equiv 2 \pmod{4}$  we obtained an exact result.

**Corollary 41.** *If  $F = K_3$  (i.e.,  $p = q = 3$ ) and  $n \equiv 2 \pmod{4}$ , then*

$$\text{Ts}(n, K_3) = \text{ex}(n, K_5) = \frac{3}{8}n^2 - \frac{1}{2}.$$

**Proof.** Assume that  $n = 4h + 2$ . Then the Turán graph for  $K_5$  is the complete 4-partite graph in which the vertex classes have respective cardinalities  $h, h, h + 1, h + 1$ . The first  $2h$  vertices have degree  $3h + 2$ , while the last  $2h + 2$  vertices have degree  $3h + 1$ . Hence there are only two degree classes, each of them inducing a complete bipartite graph, therefore the graph certainly is  $K_3$ -free. Thus no singular  $K_3$  occurs, implying  $\text{Ts}(n, K_3) \geq \text{ex}(n, K_5)$ . Also the reverse inequality is valid, by Theorem 40.  $\square$

We close this section with some fairly tight estimates for  $K_3$ .

**Proposition 42.** *For  $F = K_3$  and  $n \geq 3$  we have the following inequalities.*

- (i) *If  $n \equiv 0 \pmod{4}$ , then  $\frac{3}{8}n^2 - 2 \leq \text{Ts}(n, K_3) \leq \frac{3}{8}n^2 - 1$ .*
- (ii) *If  $n \equiv 1 \pmod{4}$ , then  $\frac{3}{8}n^2 - \frac{1}{4}n - \frac{1}{8} \leq \text{Ts}(n, K_3) \leq \frac{3}{8}n^2 - \frac{11}{8}$ .*
- (iii) *If  $n \equiv 3 \pmod{4}$ , then  $\frac{3}{8}n^2 - \frac{1}{4}n - \frac{13}{8} \leq \text{Ts}(n, K_3) \leq \frac{3}{8}n^2 - \frac{11}{8}$ .*

**Proof.** In all cases, the claimed upper bound is a Turán number minus 1, namely  $\text{ex}(n, K_5) - 1$ . Its validity follows from Theorem 40 by the further observation that the corresponding Turán graphs are unique and each of them contains a singular  $K_3$ . For the lower bounds we give constructions as follows.

(i) This is a particular case of Construction 37, with  $|U_{1,1}| = |U_{1,2}| = \frac{1}{4}n - 1$  and  $|U_{2,1}| = |U_{2,2}| = \frac{1}{4}n + 1$ .

(ii) Start with the complete 4-partite graph with equal vertex classes of size  $\frac{1}{4}(n - 1)$ , and join a new vertex, say  $z$ , to all vertices of two classes. Denoting  $n = 4h + 1$ , the two classes adjacent to  $z$  have degree  $3h + 1$ , the other two classes have degree  $3h$ , and  $z$  has degree  $2h$ . There is no singular  $K_3$  because there are only three distinct degrees (degree  $2h$  occurring only on  $z$ ), each degree class is triangle-free, and in every triangle containing  $z$  the other two vertices have degree  $3h + 1$ .

(iii) Assume that  $n = 4h + 3$ . Start with the optimal construction for  $n - 1$ , that is the complete 4-partite graph with vertex classes of respective sizes  $h, h, h + 1, h + 1$ . Similarly to the case of (ii) join a new vertex  $z$  to the  $2h$  vertices of the two smaller classes. Then  $2h$  vertices have degree  $3h + 3$ ,  $2h + 2$  vertices have degree  $3h + 1$ , and  $z$  has degree  $2h$  alone, with all its neighbors having degree  $3h + 3$ .  $\square$

The principle of these constructions can also be applied to obtain improvements of the general lower bounds on  $\text{Ts}(n, F)$  given in Proposition 38, for those  $n$  which are not divisible by  $q - 1$ .

## 7 Concluding remarks and open problems

There are several interesting directions deserving further study, which we only indicate briefly here. In fact some of the preceding results can be directly extended in one way or another, but a more systematic study would be necessary beyond pure generalizations.

**The quadratic bound.** We have seen that  $(\text{R}(G) - 1)^2 + 1$  is an easy upper bound on  $\text{Rs}(G)$ . On the other hand, from the graphs studied here it seems that this naive bound is not very bad. In this direction we propose the following conjecture.

### Conjecture 43.

(i) (weak form) *There exists a constant  $c > 0$  such that*

$$\text{Rs}(G) \geq c(\text{R}(G))^2$$

*holds for all graphs  $G$ .*

(ii) (strong form) *If  $G_1, G_2, \dots$  is an infinite sequence of graphs without isolated vertices, then  $\text{Rs}(G_n) = (1 - o(1))(\text{R}(G_n))^2$  as  $n \rightarrow \infty$ .*

**Remark 44.** *Proposition 17 implies the validity of part (i) for graphs containing very many isolated vertices. On the other hand the same proposition indicates that part (ii) needs the exclusion of isolates — or at least some related condition — because otherwise  $\text{Rs}(G_n)$  is quadratic in  $|V(G_n)|$  rather than in  $\text{R}(G_n)$ .*

**More than two colors.** Instead of 2-coloring the edges of  $K_m$  one may consider  $t \geq 3$  colors. In this case the notion of singular subgraph may be introduced in several ways; here we mention those two of them which can be considered weakest and strongest. In both of them we assume that  $t$  graphs  $G_1, \dots, G_t$  have been specified; moreover in any edge  $t$ -coloring of  $K_m$  we consider the graphs  $F_1, \dots, F_t$  where the edge set of  $F_j$  consists of the edges colored  $j$ . Let us introduce the following notions.

- A monochromatic subgraph  $H$  of color  $i$  is *weakly singular* if  $V(H)$  is singular in  $F_i$ .
- A monochromatic subgraph  $H$  is *strongly singular* if  $V(H)$  is singular in  $F_j$  for all  $1 \leq j \leq t$ .

Then the weak singular Ramsey number  $\text{Rs}_w(G_1, \dots, G_t)$  is the smallest integer  $n$  such that, for every  $m \geq n$ , every edge  $t$ -coloring of  $K_m$  contains a weakly singular subgraph  $G_i$  in the color class  $i$  for some  $1 \leq i \leq t$ ; and the strong singular Ramsey number  $\text{Rs}_s(G_1, \dots, G_t)$  is defined analogously.

It can be proved in various ways that  $\text{Rs}_w$  and  $\text{Rs}_s$  are finite whenever the graphs  $G_i$  are finite. As  $t$  grows, there is an increasing number of possibilities to introduce notions between weak and strong singularity; and in general we have  $\text{Rs}_w(G_1, \dots, G_t) \leq \text{Rs}_s(G_1, \dots, G_t)$  for all  $t$  and all choices of the  $G_i$ .

We expect that  $\text{Rs}_w$  can be estimated more tightly than  $\text{Rs}_s$ . With the notation  $n_j = (|V(G_j)| - 1)^2 + 1$ , a simple argument similar to the proof of Theorem 11 yields

$$\text{Rs}_w(G_1, \dots, G_t) \leq R(K_{n_1}, \dots, K_{n_t})$$

but this is probably quite far from being sharp in general. For small graphs  $G_i$ , however, perhaps the upper bound is not terribly large. In particular, the inequality implies  $\text{Rs}_w(K_3, K_3, K_3) \leq R(K_5, K_5, K_5)$ .

**Problem 45.** Determine  $\text{Rs}_w(K_3, K_3, K_3)$ .

The  $k$ -singular generalization may also be worth studying. For instance, in a way as in Theorem 11, one can easily see that

$$\text{Rs}_w(G_1, \dots, G_t, k) \leq R(K_{n_1(k)}, \dots, K_{n_t(k)})$$

where  $n_j(k) = k(|V(G_j)| - 1)^2 + 1$

**Some simple graphs.** There are some classes of graphs for which the Ramsey number is known. For example, one may consider

**Problem 46.** Determine  $\text{Rs}(G)$  for

- (i)  $G = tK_2$ ,
- (ii)  $G = P_t$ ,
- (iii)  $G = C_t$ ,

for all values of  $t$ .

**The  $k$ -singular version.** So far we have a tight result concerning  $k$ -singular Ramsey numbers only for  $P_3$  and its subgraphs (and for edgeless graphs). On the other hand, some estimates can easily be extended in this direction (cf. Corollary 12(i) and Theorem 13). It would be interesting to see the general effect of  $k$  on the behavior of  $\text{Rs}(G, k)$ , or at least for some particular examples of  $G$ .

**Isolated vertices.** We have shown in Proposition 17 that the quadratic lower bound is tight whenever the number of non-isolated vertices is rather small compared to the order of the graph. Motivated by this, the following problem is of interest.

**Problem 47.** *Given a graph  $G$ , determine the minimum number of isolated vertices which should be added to  $G$  so that the obtained graph  $G^+$  satisfies the equality  $\text{Rs}(G^+) = (|V(G^+)| - 1)^2 + 1$ .*

**Other structures.** Ramsey theory has been studied for various structures, and the notion of singularity can be extended in a meaningful way in some of them. For example, for any family  $\mathcal{F}$  of hypergraphs and for every natural number  $k$ , the inequality  $\text{Rs}(\mathcal{F}, k) \leq k(\text{R}(\mathcal{F}) - 1)^2 + 1$  of Theorem 11 remains valid.

**Singular Turán numbers.** We have determined tight asymptotics for  $\text{Ts}(n, F)$  for all graphs  $F$  having at least one edge, but the exact value is known in a small number of cases only. This leaves several interesting problems open.

**Problem 48.** *Determine  $\text{Ts}(n, K_3)$  for  $n \not\equiv 2 \pmod{4}$ .*

Let us note that the upper bounds in Proposition 42 are tight for  $n = 4$  and  $n = 5$ , while it seems plausible to guess that for every other  $n$  divisible by 4 the lower bound of (i) gives the correct value. In the other cases the lower bounds may turn out to be tight, at least asymptotically.

**Problem 49.** *Determine  $\text{Ts}(n, C_4)$ .*

**Problem 50.** *Prove or disprove: If  $p \geq 3$  is fixed and  $m$  is sufficiently large, then the complete  $(p-1)^2$ -partite graph, in which each of  $m, m+1, \dots, m+p-2$  is the size of exactly  $p-1$  vertex classes, is extremal for singular  $K_p$ , i.e. has  $\text{Ts}(n, K_p)$  edges where  $n$  is the corresponding number of vertices, namely for  $n = (p-1) \cdot \sum_{i=0}^{p-2} (m+i) = (p-1)^2 \cdot (m+p/2-1)$ .*

**Conjecture 51.** *For every graph  $F$  with  $p$  vertices and chromatic number  $q$ , and every residue class  $r \not\equiv 0$  modulo  $q-1$ , there exists a constant  $c(F, r)$  such that*

$$\lim_{n \rightarrow \infty; n \equiv r \pmod{q-1}} \frac{\text{ex}(n, K_{(p-1)(q-1)+1}) - \text{Ts}(n, F)}{n} = c(F, r).$$

**Problem 52.** *Determine the value of the constants  $c(F, r)$  for particular classes of graphs  $F$ , including complete graphs, complete bipartite graphs, paths and cycles.*

**Problem 53.** *Given a constant  $c$  in the range  $0 < c < 1$ , find tight asymptotics on  $\text{Ts}(n, K_{cn})$ .*

**Problem 54.** *Given  $F$ , find tight asymptotics on the  $k$ -singular Turán numbers  $\text{Ts}(n, F, k)$ .*



**Acknowledgements.** We are grateful to the referees for their careful reading and helpful advices that improved the presentation of the paper.

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