4-20-2015

Explorations of the Collatz Conjecture (mod m)

Glenn Micah Jackson Jr
Georgia Southern University

Explorations of the Collatz Conjecture (mod m)

An Honors Thesis submitted in partial fulfillment of the requirements for Honors in Mathematics.

By
Micah Jackson

With the mentorship of Dr. David Stone

ABSTRACT

The Collatz Conjecture is a deceptively difficult problem recently developed in mathematics. In full, the conjecture states: *Begin with any positive integer and generate a sequence as follows: If a number is even, divide it by two. Else, multiply by three and add one. Repetition of this process will eventually reach the value 1.* Proof or disproof of this seemingly simple conjecture have remained elusive. However, it is known that if the generated Collatz Sequence reaches a cycle other than 4, 2, 1, the conjecture is disproven. This fact has motivated our search for occurrences of 4, 2, 1, and other cycles in a Collatz Sequence mod m.

Thesis Mentor: ______________________________
Dr. David Stone

Honors Director: ______________________________
Dr. Steven Engel

November 3, 2014
Department of Mathematics
University Honors Program
Georgia Southern University

Acknowledgements

The following presented work would not be possible if it were not for the efforts of several to guide me along in this process. I am deeply grateful for the mentorship of Dr. David Stone. Without his guidance, this research may have never began. I must also thank the generous donors of several Georgia Southern University groups for aiding me in conference travel and research expenses. Without this opportunity, my work and growth as a student could never have come this far. Finally, to the numerous professors and colleagues that have given suggestions or critiques about this work, I must say thank you. Every idea is valuable in research.

Background

The Collatz Conjectjure (also known as the *3x + 1 Problem, Hailstone Problem, Kakutani's Conjecture, Ulam's Conjecture, Hasse's Algorithm, and the Syracuse Problem*) is usually attributed to Lothar Collatz, an alumnus of the University of Hamburg. While at university, Collatz became interested in iterative graph representations of number-theoretic functions. Though never publishing any of these problems, Collatz is known to have circulated them with his colleagues at the International Congress of Mathematics in 1950. By the early 1950s, the Collatz Conjecture was not just an underground problem known by a select few professors, but it had become known to the math community. Bryan Thwaites, another mathematician that the Collatz Conjecture is occasionally named after, discovered the problem in 1952 and has since offered a £1000 prize for a proof. This offer follows from similar prizes: Harold S. M. Coxeter will provide $50 for a proof while Paul Erdős offers $500.

However, prize money is only a secondary benefit that would come from a proof. A mathematician who can construct a proof of this conjecture will be known for solving a problem for which Erdős has said, "Mathematics is not yet ready for such problems." The method of proof will most likely open up new areas of study as well since the Collatz Problem is tied to much more than just number theory or graphs. As seen in the results of Krager[1], the Collatz Conjecture lends itself to p-adic analysis, analysis where "distance" is determined by the largest power of a given prime p dividing the difference between two rational numbers. The Collatz Function also shares connections to the Mersenne Primes, prime numbers of the form $2^n - 1$, with n a positive integer[3]. Suppose M_i is the i^{th} Mersenne Prime, then Ohira and Watanabe have shown a strong correlation between the *stopping time* (number of iterations for the sequence to reach 1) of M_i and the index i. Another author, Riho Terras, has published results showing that the limit of the stopping time tends to 0 as the fixed initial value is increased[6]. These results were attained using techniques from the field of statistics. Whatever the eventual method of proof will be, it is clear that it will have deep implications and connections to various other areas of mathematics.

Formulation of Problem

The formal statement of the Collatz Conjecture begins with the Collatz Function, $C(k)$, defined as follows:

For a positive integer k,

$$C(k) = \begin{cases} 3k+1, & \text{if } k \text{ is odd} \\ k/2, & \text{if } k \text{ is even} \end{cases}$$

Because the conjecture is focused on iterations of this function, let $C^2(k) = C(C(k)), C^3(k) = C(C^2(k))$, so on and denote k as $C^0(k)$. Furthermore, the sequence generated, $\{C^i(k)\}_{i=1}^{n}$, will be known as the *trajectory.*

Conjecture (Collatz Conjecture)**.** *Beginning with any positive integer k, there exists some n such that $C^n(k)$ = 1.*

Examples:

(a) For $k = 11$. The trajectory of 11, or $\mathrm{Traj}(11) = 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$.

(b) $\mathrm{Traj}(75) = 226, 113, 340, 170, 85, 256, 128, 64, 32, 16, 8, 4, 2, 1$.

(c) $\mathrm{Traj}(16384) = 8192, 4096, 2048, 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1.$.

(d) $\mathrm{Traj}(2^j) = 2^{j-1}, 2^{j-2}, \ldots, 2^2, 2, 1$.

Since inception of this conjecture, the conclusion has been verified up to 2^{58}, or roughly 5.8×10^{18}[5]. However, progress towards a proof has not fared so well. In attempts to prove the Collatz Conjecture, researchers have examined various properties and measures of the Collatz Function. The most natural measure to look at would be the number of iterations needed for the sequence to reach 1. This is known as the *stopping time*. Note that this is the length of trajectory. Provided below are graphs mapping the integers k to their corresponding stopping times.

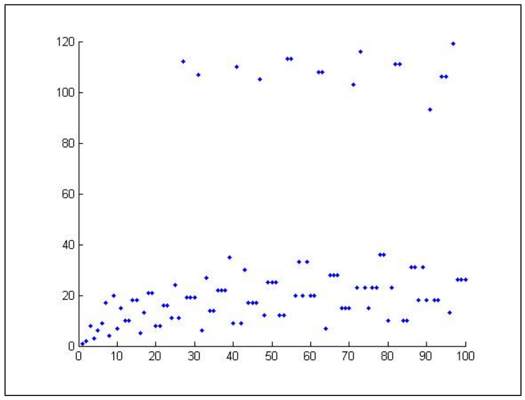

Figure 1: Stopping times up to n = 100

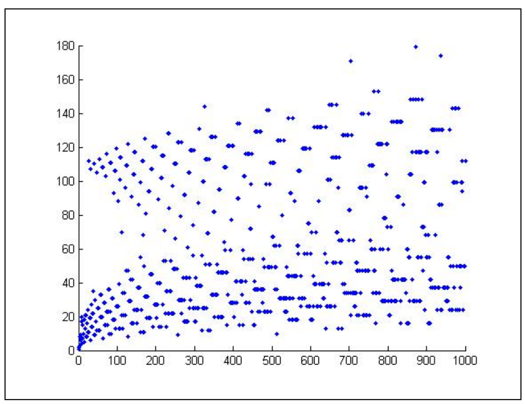

Figure 2: Stopping times up to n = 1000

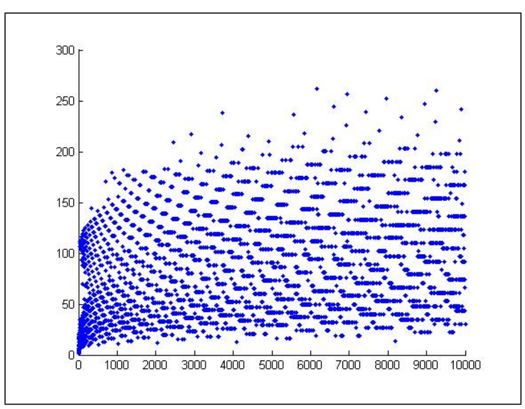

Figure 3: Stopping times up to n = 10000

In Figures 1, 2, and 3, it appears that there are two distinct groupings of trajectories converging into one large periodic distribution. Unfortunately, periodicity and nice pictures do not prove the conjecture and we are left with another mystery of the Collatz Conjecture.

Another characteristic of interest is the maximum value attained during the trajectory of k. Graphed below, in Figures 4, 5, 6, are integers k and their corresponding maximum trajectory value.

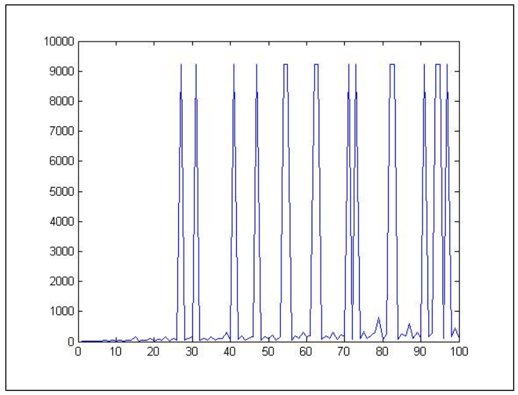

Figure 4: Max values for integers up to n = 100

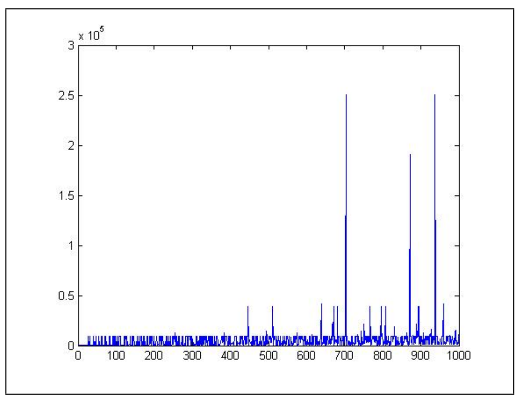

Figure 5: Max values for integers up to n = 1000

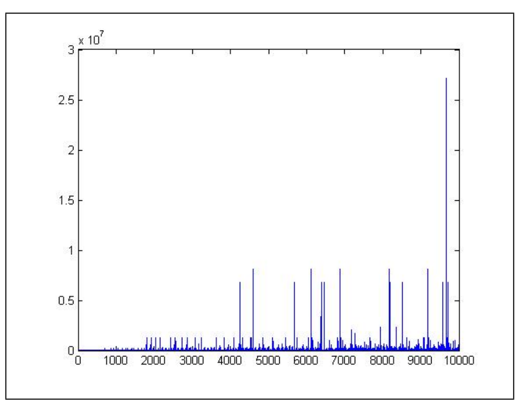

Figure 6: Max values for integers up to n = 10000

From these figures, it is easy to see that most integers reach similar max values with many of the integers actually reaching the same max values. As the magnitude of our initial integer increases, we also see that the number of integers with outlying max values increases. This is due to the intuitive fact that as the starting integer k increases, more divisions by two are necessary and multiplications by three are more likely. It is precisely because of this that some have chosen to examine somewhat normalized max values, or the *expansion factor*. The *expansion factor* is defined as $\dfrac{\max_{j>0}\ C^j(k)}{k}$ for a fixed initial value k. The above results are then transformed into the following.

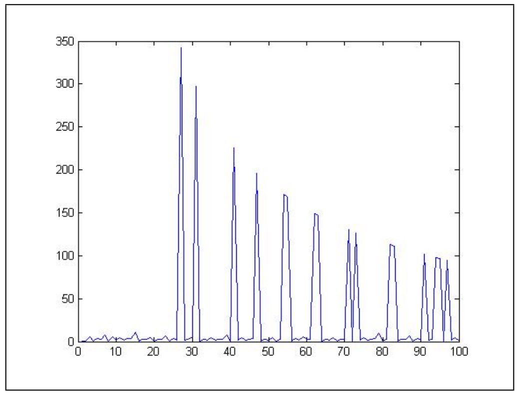

Figure 7: Expansion factors for integers up to n = 100

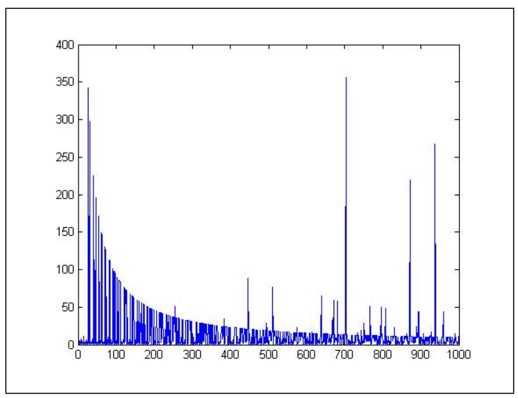

Figure 8: Expansion factors for integers up to n = 1000

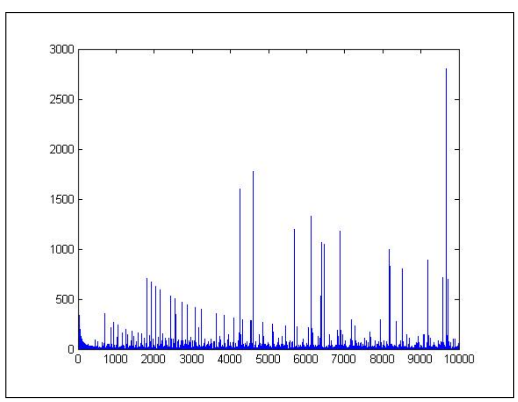

Figure 9: Expansion factors for integers up to n = 10000

From Figures 7, 8, and 9, one can point out that there appears to be a type of exponentially decreasing pattern as k increases. However, as we expand our interval, this characteristic becomes less pronounced and leaves us in the dark once more. Next, one can look at the shape of the actual trajectories for given k. Examples are given in Figures 10, 11, 12 below.

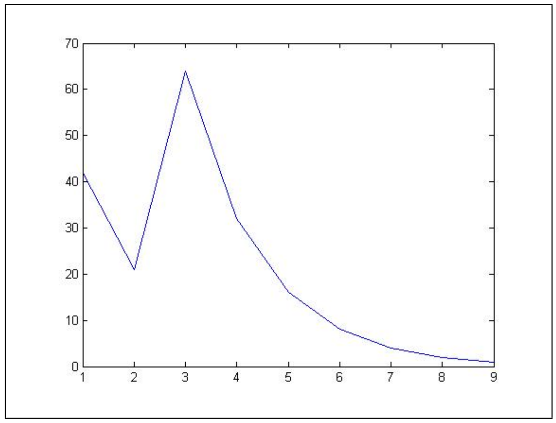

Figure 10: Trajectory of 42

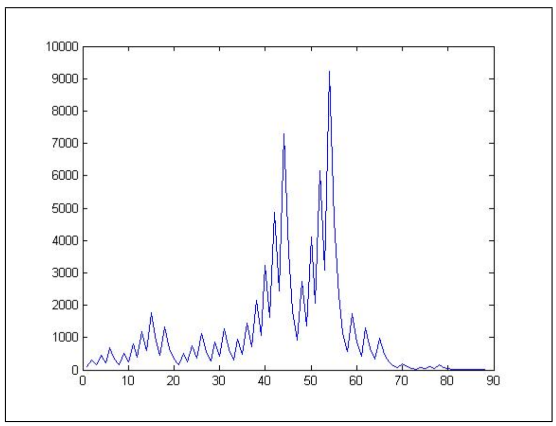

Figure 11: Trajectory of 103

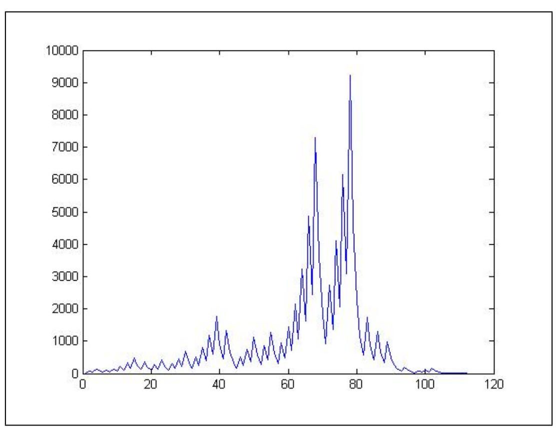

Figure 12: Trajectory of 27

Finally, we can (and will) examine these trajectories when reduced by a modulus m. A priori, one might suspect that the graphs will now have simpler looking results since the range of values that our function can obtain has been reduced to m choices. Unfortunately, Figures 13, 14, and 15 below show this is not the case. The following graphs are even more chaotic-appearing than the original.

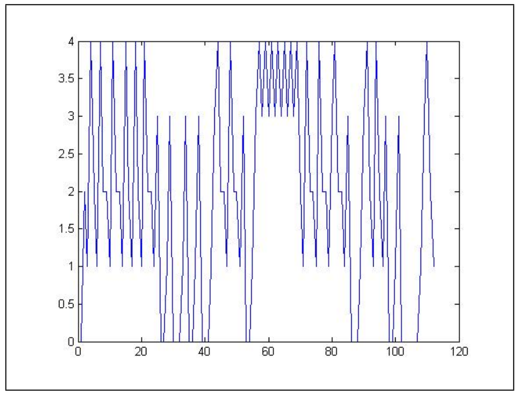

Figure 13: Trajectory of 27 mod 5

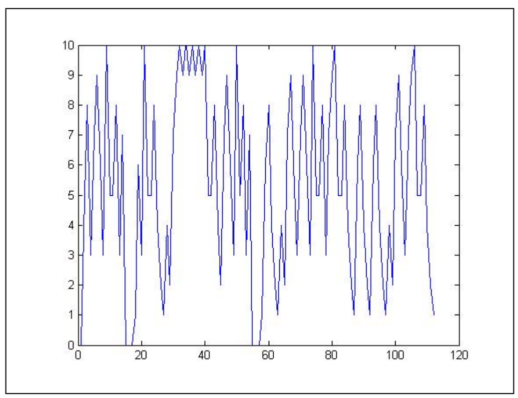

Figure 14: Trajectory of 27 mod 11

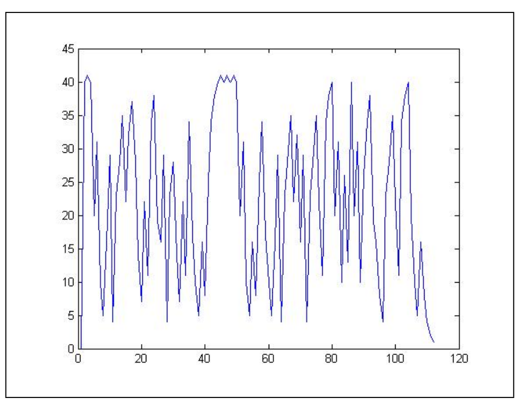

Figure 15: Trajectory of 27 mod 42

Note here that the residue trajectory can not be computed in mod m, but one must determine the original Collatz sequence before reducing it. Also, one can bring some order to the turbulent nature of these sequences. Now, we restate the Collatz Conjecture as the equivalent:

Conjecture (Collatz Conjecture)**.** *Given any positive integer k, the sequence generated by iterations of the Collatz Function will eventually reach and remain in the cycle 4, 2, 1.*

This is the definition that has motivated the present paper's focus. We have examined Collatz sequences reduced by modulus $m > 4$ and corresponding occurrences of the sequence 4, 2, 1. If a Collatz Sequence eventually cycles 4, 2, 1, then the sequence mod m will also reach the cycle. Thus, if it is shown that a reduced Collatz Sequence does not reach 4, 2, 1, then the original sequence will not reach the cycle and the conjecture is disproven. Additionally, if a Collatz Sequence reaches a cycle other than 4, 2, 1, the conjecture is disproven. For this reason, we have examined conditions under which a reduced Collatz Sequence will reach a cycle other than 4, 2, 1.

Congruence mod m

Congruence of integers is a concept developed by Gauss in his classic work, *Disquisitiones Arithmeticae*, published in 1801. For a fixed positive integer m, known as the modulus, we say that two integers a and b are *congruent mod m* if they leave the same remainder upon division by m. We denote this by $a \equiv b \bmod m$. For instance, $32 \equiv 7 \bmod 5$ because each leaves a remainder of 2 when divided by 5.

Any integer n is congruent mod m to a unique integer in the list 0, 1, 2, $\ldots, m-1$. This integer is known as the *residue* of n mod m. For example, if we let $m = 10$, then the residue of 47 mod 10 is 7 since 47 divided by 10 leaves a remainder of 7. Thus, $47 \equiv 7 \bmod 10$.

In particular, $n \equiv 0 \bmod m$ is equivalent to saying that m is a divisor of n. For instance, $n \equiv 0$ mod 2 if and only if n is even. Congruence is very similar to the = relation. They are both *equivalence relations* on the integers and make it possible to establish and use properties of divisibility in an algebraic fashion.

A more rigorous introduction to modular arithmetic and congruences is given in [4].

Results

Result 1. We are first interested in the case of 4, 2, 1 occurring in the reduced trajectories before the cycle appears in the original trajectory. Let $a_i = 4 + s_0 m$, $a_{i+1} = 2 + s_1 m$, and $a_{i+2} = 1 + s_2 m$, so that $a_i, a_{i+1}, a_{i+2} \equiv 4, 2, 1 \bmod m$. Moreover, if a_i, a_{i+1} are even and a_{i+2} is odd, then $a_i, a_{i+1}, a_{i+2} \equiv 0, 0, 1 \bmod 2$. Then we have the following result:

a_i mod 2	m odd	m even
0,0,0	$s_0 \equiv 0 \bmod 8$	Not Possible
0,0,1	$s_0 \equiv 4 \bmod 8$	$s_0 \equiv 1 \bmod 2$
0,1,0	Not Possible	Not Possible
1,0,0	$s_0 \equiv 3 \bmod 4$, $m = 11$	Not Possible
1,0,1	$s_0 \equiv 1 \bmod 4$, $m = 11$	Not Possible

Example: Suppose we wish to generate a trajectory that is congruent to 4, 2, 1 when reduced by a modulus m and this sequence consists of an odd number followed by two evens. This corresponds to the fourth row. From here, we can see that if our modulus is odd, then $s_0 \equiv 3 \bmod 4$ and specifically, $m = 11$. However, if m is even, this case is not possible. Suppose $a_i = 4 + s_0 m$ is odd and m is even. Then $a_i \equiv (0 \bmod 2) + s_0(0 \bmod 2) \equiv 0 \bmod 2$, a contradiction.

Example: Let $a_i = 81 = 4 + (7)(11)$. Then Traj(81) = 244, 122, 61, $\ldots$, so $a_i, a_{i+1}, a_{i+2}, a_{i+3} \equiv 4, 2, 1, 6 \bmod 11$.

Result 2. *To generate a repeating sequence of 4, 2, 1, begin with any integer* 2^{3k} *and reduce the sequence by* m = 7.

Example:

$$\text{Traj}(2^{3(5)}) = \text{Traj}(32768) = 16384, 8192, 4096, 2048, 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1.$$

$$\text{Traj}(32768) \bmod 7 = 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1$$

That is, the residue trajectory reduces mod 7 immediately to 4, 2, 1, even though the original sequence does not.

Theorem 1. *If* a_i *is even and* $a_i \equiv a_{i+1} \bmod m$*, then* $a_i \equiv 0 \equiv a_{i+1} \bmod m$.

Theorem 2. *If* a_i *is odd, then* $a_i \equiv a_{i+1}$ *mod* m *iff* m *is odd and* $a_i \equiv -2^{-1}$ *mod* m. *Moreover, if* m = 2t + 1, *then,*

(i) *If* m$\equiv$ 1 *mod 4, then* $a_i \equiv$ t + m *mod* 2m

(ii) *If* m$\equiv$ 3 *mod 4, then* $a_i \equiv$ t *mod* 2m

Examples:

(a) Let $m = 9$. Then $t = 4$ and $m \equiv 1$ mod 4. Suppose $a_i = 49$ which implies $a_{i+1} = 148$ which are both $\equiv 4$ mod 9. Also, $a_i \equiv t + m \bmod 2m$ because $49 \equiv 13$ mod 18.

(b) Let $m = 7$. Then $t = 3$ and $m \equiv 3$ mod 4. Suppose $a_i = 17$ which implies $a_{i+1} = 52$ which are both $\equiv 3$ mod 7. Additionally, $a_i \equiv t \bmod 2m$ because $17 \equiv 3$ mod 14.

Theorem 3. *Suppose* $a_i, a_{i+1}, a_{i+2} \equiv a, b, a$ *mod* m *with* a*,* $b \in \mathbb{Z}$ *and* $a \neq b$*. Then,*

(i) *If* a_i *is odd, then* $a \equiv -1$ *mod* m *and* $b \equiv -2$ *mod* m.

(ii) *If* a_i *is even and* a_{i+1} *is odd, then* $a \equiv -2$ *mod* m *and* $b \equiv -1$ *mod* m *provided gcd*$(m, 2)$ = *1.*

(iii) *If* a_i *and* a_{i+1} *are even, it must be that* $a \equiv b \equiv 0$ *which contradicts our assumption that* $a \not\equiv b$*, so this can not occur.*

Examples:

(a) Let $a_i = 63$ which is $\equiv -1$ mod 32. Traj(63) = 190, 95, 286, 143, $\ldots \equiv -2, -1, -2, -17, \ldots$

(b) Let $a_i = 138$ which is $\equiv -2$ mod 7. Traj(138) = 69, 208, 104, 52, $\ldots \equiv -1, -2, -1, -5, \ldots$

Corollary. Suppose $a_i \equiv a_{i+1} \equiv a_{i+2} \bmod m$. Then,

(i) If a_i is even, then must have $a_i \equiv a_{i+1} \equiv a_{i+2} \equiv 0 \bmod m$. Moreover, one can have an arbitrarily long sequence of $a_i \equiv a_{i+1} \equiv a_{i+2} \equiv \ldots \equiv a_{i+j}$ given $a_i \equiv a_{i+1} \equiv a_{i+2} \equiv \ldots \equiv a_{i+j-1} \equiv 0$ mod m.

(ii) If a_i is odd, $a_i \equiv a_{i+1} \equiv a_{i+2}$ does not occur.

(iii) If $a_i, a_{i+1}, a_{i+2} \equiv a, b, a$ and $a \neq b$, then $a, b, a \equiv -2, -1, -2 \bmod m$ or $a, b, a \equiv -1, -2, -1 \bmod m$. Thus, any repeating pair a, b, a, b with $a \neq b$ must be $-2, -1, -2, -1 \bmod m$ or $-1, -2, -1, -2 \bmod m$.

Proof of Theorems

Proof of Theorem 1. Let a_i be even so $a_{i+1} = b$ and $a_i = 2b$. Then,

$$\begin{aligned} 2b &\equiv b \bmod m \\ b &\equiv 0 \bmod m \\ \text{i.e. } b &= cm \\ \text{so, } a_i &= (2c)m \\ a_{i+1} &= cm \\ \text{Thus, } a_i &\equiv 0 \equiv a_{i+1} \ \blacksquare \end{aligned}$$

Proof of Theorem 2. Let a_i be odd. Then,

$$\begin{aligned} a_{i+1} &\equiv 3a_i + 1 \bmod m \\ 2a_i + 1 &\equiv 0 \bmod m \\ \text{That is } 2a_i + 1 &= cm \text{ for some odd integer c.} \\ \text{Thus, } m \text{ is odd and } 2a_i &\equiv -1 \bmod m \\ a_i &\equiv -2^{-1} \bmod m. \ \square \end{aligned}$$

For the converse, take m to be odd and $v = 2^{-1} \bmod m$. That is, $2v \equiv 1 \bmod m$. Furthermore, suppose $a_i \equiv -v \bmod m$ and a_i is odd. Then,

$$\begin{aligned} a_{i+1} &= 3a_i + 1 \\ &\equiv 3(-v) + 1 \bmod m \\ &\equiv -v - 2v + 1 \bmod m \\ &\equiv -v - (2v - 1) \bmod m \\ &\equiv -v \bmod m. \ \blacksquare \end{aligned}$$

Now, assuming m to be odd, let $m = 2t+1$, so $t \equiv -2^{-1} \bmod m$. Then $a_i \equiv t \bmod m$, so $a_i = t + km$ for some integer k. However, t and k must have opposite parities for a_i to be odd. Thus, if t is even, $m \equiv 1 \bmod 4$ and for an integer j,

$$\begin{aligned} a_i &= t + (2j+1)m \\ &= (t+m) \bmod 2m. \ \square \end{aligned}$$

Now, if t is odd, $m \equiv 3 \bmod 4$ and

$$\begin{aligned} a_i &= t + 2jm \\ &= t \bmod 2m. \ \blacksquare \end{aligned}$$

Proof of Theorem 3. Define $a_i = a + s_0 m$, $a_{i+1} = b + s_1 m$, $a_{i+2} = a + s_2 m$, and $a_{i+3} = a + s_3 m$.

(i) Let a_i be odd. Then, $a_{i+1} = 3a_i + 1$ and $a_{i+2} = \frac{1}{2}(a_{i+1}) = \frac{1}{2}(3a_i + 1)$. Now,

$$\begin{aligned} b + s_1 m &= 3(a + s_0 m) + 1 \\ &= (3a + 1) + 3s_0 m \\ \text{so, } b &\equiv 3a + 1 \bmod m. \end{aligned}$$

$$\begin{aligned} \text{Furthermore, } a + s_2 m &= \frac{1}{2}(3a_i + 1) \\ 2a + 2s_2 m &= 3a + 1 + 3s_0 m \\ 2s_2 m &= a + 1 + 3s_0 m \\ \text{thus, } a + 1 &= (2s_2 - 3s_0)m. \end{aligned}$$

So $a \equiv -1 \bmod m$ and $b \equiv -2 \bmod m$. ■

(ii) Let a_i and a_{i+1} be even. So,

$$\begin{aligned} a_{i+1} &= \frac{1}{2}a_i \\ b + s_1 m &= \frac{1}{2}(a + s_0 m) \\ 2b + 2s_1 m &= a + s_0 m \\ a &= 2b + (2s_1 - s_0)m \\ \text{so, } a &\equiv 2b \bmod m \end{aligned}$$

$$\begin{aligned} \text{Then, } a_{i+2} &= \frac{1}{2}a_{i+1} \\ a + s_2 m &= \frac{1}{4}(a + s_0 m) \\ 4a + 4s_2 m &= a + s_0 m \\ 3a &= (s_0 - 4s_2)m \\ \text{Thus, } 3a &\equiv 0 \bmod m. \end{aligned}$$

Then, if $gcd(m, 3) = 1$, $a \equiv 0 \bmod m$. ■

(iii) Let a_i be even and a_{i+1} be odd. From (ii), we have that $a \equiv 2b \bmod m$. Now,

$$\begin{aligned} a_{i+2} &= 3a_{i+1} + 1 \\ a + s_2 m &= \frac{3}{2}(a + s_0 m) + 1 \\ 2a + 2s_2 m &= 3a + 3s_0 m + 2 \\ (2s_2 - 3s_0) &= a + 2 \\ \text{So, } a &\equiv -2 \bmod m \\ \text{Thus, } 2b &\equiv -2 \bmod m. \end{aligned}$$

Then, if $gcd(m, 2) = 1$, $b \equiv -1 \bmod m$. ■

Proof of Corollary. Proof follows from Theorem 1. Furthermore, sequences of the form below will be $\equiv 0 \bmod m$ for $j+1$ iterations.

(i)

$$\begin{aligned}
a_i &= 2^j cm \\
a_{i+1} &= 2^{j-1} cm \\
a_{i+2} &= 2^{j-2} cm \\
&\vdots \\
a_{i+j-1} &= 2cm \\
a_{i+j} &= cm \\
\text{So, } a_i &\equiv a_{i+1} \equiv \ldots \equiv a_{i+j} \equiv 0 \bmod m. \ \blacksquare
\end{aligned}$$

(ii) Let a_i be odd, then $m = 2t+1$ is odd and $a_i \equiv a_{i+1} \equiv t \bmod m$. Moreover, a_{i+1} is even, so $a_{i+1} \equiv a_{i+2}$ which implies $a_{i+1} \equiv a_{i+2} \equiv 0 \bmod m$ by Theorem 1. So $t \equiv 0 \bmod m$. But,

$$\begin{aligned}
m &= 2t+1 \\
0 &\equiv 2 \cdot 0 + 1 \bmod m \\
0 &\equiv 1 \bmod m
\end{aligned}$$

Thus, we have reached a contradiction, so this case is not possible. $\blacksquare$

Special Case

Suppose we allow the case that $m = 3$ so that the terminal sequence 4, 2, 1, 4, 2, 1, ... becomes 1, 2, 1, 1, 2, 1, ... If some term a_i in a Collatz sequence is odd, then the next term $a_{i+1} = 3a_i + 1 \equiv 1 \bmod 3$ regardless of the residue of a_i. Otherwise, if a_i is even, then $a_{i+1} = \frac{1}{2}a_i \equiv 2a_i \bmod 3$. Thus, an algorithm for generating the sequence of residues mod 3 is

$$a_{i+1} \bmod 3 = \begin{cases} 1, & \text{if } a_i \text{ is odd,} \\ 2(a_i \bmod 3), & \text{if } a_i \text{ is even.} \end{cases}$$

Because mod 3 is a special case, we examined the behavior of the residue sequences for repeating pairs a, b, a. It is simple to find numbers that generate at least a single occurrence of 1, 2, 1. It also turns out that numbers of the form $35 \cdot 2^{2k+1}$ will produce $k+6$ pairs of 1, 2 followed by infinite repetitions of 1, 2, 1.

Examples:

- Let $a_i = 12s + 4$ for s in the integers. Then, $a_{i+1} = 6s + 2$ and $a_{i+2} = 3s + 1$. Thus, $a_i, a_{i+1}, a_{i+2} \equiv 1, 2, 1 \bmod 3$.

- $$\begin{aligned}
\text{Traj}(1120) = \text{Traj}(35 \cdot 2^{2(2)+1}) &= 560, 280, 140, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1. \\
&\equiv 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1 \bmod 3. \\
\text{Or more clearly,} \quad & 2, (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2, 1).
\end{aligned}$$

Summary

If the Collatz Conjecture is true, then every Collatz sequence eventually produces $4, 2, 1, 4, 2, 1$. A direction for future study is the investigation of sequence whose sequence of residues eventually becomes $a, b, c, a, b, c, \ldots$ for some modulus m. If we could find a, b, c different from $4, 2, 1$, that would disprove the Collatz Conjecture. More generally, if there exists an n-tuple, $a_1, a_2, \ldots, a_n$ different from 4, 2, 1, which is eventually infinitely repeated in the sequence of residues, then this would constitute a disproof. Moreover, if we could prove that the only possible n-tuple is $4, 2, 1$, that would only add to the body of evidence supporting the conjecture.

Bibliography

References

[1] E. Krager. *A 2-adic Extension of the Collatz Function.* http://www.math.uchicago.edu/ may/VIGRE/VIGRE2011/REUPapers/Karger.pdf. (2011)

[2] J. Lagarias. *The 3x + 1 Problem and Its Generalizations.* American Mathematical Monthly, 92, (1985), 1-23.

[3] T. Ohira and H. Watanabe. *A Conjecture on the Collatz-Kakulani Path Length for the Mersenne Primes.* arXiv:1104.2804v3 [math.NT]. (2012).

[4] A. Pettofrezzo and D. Byrkit. *Elements of Number Theory: Second Edition.* Orange Publishers, Florida, (1994), Chapter 3.

[5] T. Oliveira e Silva. *Computational Verification of the 3x + 1 Conjecture.* http://www.ieeta.pt/ tos/3x+1.html. (2008).

[6] R. Terras. *A Stopping Time Problem on the Positive Integers.* Acta Arithmetica, 30, (1976).

[7] R. Terras. *On the Existence of a Density.* Acta Arithmetica, 35, (1979).