Theory and Applications of Graphs

Volume 5 | Issue 2 Article 7

December 2018

Subsets of vertices of the same size and the same maximum distance

Maria Axenovich
Karlsruhe Institute of Technology, maria.aksenovich@kit.edu

Dominik Duerrschnabel
dominik@duerrschnabel-online.de

Recommended Citation
Axenovich, Maria and Duerrschnabel, Dominik (2018) "Subsets of vertices of the same size and the same maximum distance," *Theory and Applications of Graphs*: Vol. 5 : Iss. 2 , Article 7.
DOI: 10.20429/tag.2018.050207
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol5/iss2/7

Abstract

For a connected simple graph $G = (V, E)$ and a subset X of its vertices, let

$$d^*(X) = \max\{\mathrm{dist}_G(x, y) : x, y \in X\}$$

and let $h^*(G)$ be the largest k such that there are disjoint vertex subsets A and B of G, each of size k such that $d^*(A) = d^*(B)$. Let $h^*(n) = \min\{h^*(G) : |V(G)| = n\}$. We prove that $h^*(n) = \lfloor (n+1)/3 \rfloor$, for $n \geq 6$. This solves the homometric set problem restricted to the largest distance exactly. In addition we compare $h^*(G)$ with a respective function $h_{\mathrm{diam}}(G)$, where $d^*(A)$ is replaced with $\mathrm{diam}(G[A])$.

1 Introduction

For a subset X of vertices of a graph G, let $d^*(X) = \max\{\mathrm{dist}_G(x, y) : x, y \in X\}$, where dist_G is the distance in G. We call two subsets of vertices $A, B \subseteq V$ *weakly homometric* if $|A| = |B|$, $A \cap B = \emptyset$, and $d^*(A) = d^*(B)$. Let $h^*(G)$ be the largest k such that G has weakly homometric sets of size k each. Let $h^*(n)$ be the smallest value of $h^*(G)$ over all connected n-vertex graphs. Informally, any connected graph G on n vertices has two disjoint subsets of vertices of the same size at least $h^*(n)$ that have the same largest distance (in G) between their vertices. All graphs considered in this note are simple.

The notion of weakly homometric sets originates from the notion of homometric sets introduced by Albertson et al. [1]. For a subset of vertices X, let $d(X)$ be a multiset of pairwise distances between the vertices of X. Two disjoint sets of vertices A and B are called homometric, if $d(A) = d(B)$. Let $h(G)$ be the largest k such that G has two homometric sets of size k each. Let $h(n)$ be the smallest value of $h(G)$ among all connected n-vertex graphs. The study of homometric sets is partly motivated by the question: "Can one distinguish two vertex sets of equal size only by the respective distances?". The answer depends on the size of these sets. The best known bounds on $h(n)$ are as follows:

$$c\left(\frac{\log n}{\log\log n}\right)^2 \leq h(n) \leq n/4 - c' \log\log n,$$

for positive contants c, c', where the lower bound is due to Alon [2], and the upper one is due to Axenovich and Özkahya [3], both of the bounds are slight improvements of the original bounds by Albertson et al. [1]. There are much better bounds on $h(G)$ known when G is a tree or when G has diameter 2, see Fulek and Mitrović [6] and Bollobás et al. [4], see also an earlier paper by Caro and Yuster [5]. Specifically, a result of Bollobás et al. [4] implies that $h(G)$ is close to $\frac{|V(G)|}{2}$, when G is large and has diameter at most 2. Since a closed neighborhood of a vertex induces a graph of diameter at most 2 and there is such a neighborhood on at least $2|E(G)|/|V(G)|$ vertices, we have that there is a positive constant c such that for any graph G, $h(G) \geq c|E(G)|/|V(G)|$.

Weakly homometric sets are concerned only with one, the largest, distance. In this note we find $h^*(n)$ exactly.

Theorem 1.1. *For any $n \geq 6$, $h^*(n) = \lfloor (n+1)/3 \rfloor$.*

Note that considering connected graphs in the definition of h^* is not an essential restriction. Indeed, if a graph G is not connected and has at least two components of size at least two each, then by taking ∞ as a distance between any two vertices from different components, we see that $h^*(G) \geq \lfloor n/2 \rfloor$. Otherwise, G has two connected components, one of which is a single vertex. Thus by Theorem 1.1 applied to the larger component $h^*(G) \geq \lfloor n/3 \rfloor$.

When the distance is considered in a subgraph rather than in an original graph, we consider the following function that is of independent interest. For a graph G, $h_{\mathrm{diam}}(G)$ is the largest integer k such that there are disjoint sets $A, B \subseteq V(G)$, each of size k and so that $\mathrm{diam}(G[A]) = \mathrm{diam}(G[B])$.

Theorem 1.2. *Let G be an n-vertex graph, then $h_{\mathrm{diam}}(G) \geq \lfloor (n+1)/3 \rfloor$. Moreover if $\mathrm{diam}(G) \geq 4$ or $\mathrm{diam}(G) = 1$ then $h_{\mathrm{diam}}(G) = \lfloor n/2 \rfloor$.*

In order to prove the main result, we consider an auxiliary coloring of the edges of a complete graph on the vertex set $V = V(G)$ with colors $1, 2, \ldots, \mathrm{diam}(G)$ such that the color of xy is $\mathrm{dist}_G(x, y)$, $x, y \in V$. The result follows from observations about the structure of the color classes. In fact, the proof allows for an algorithm determining large weakly homometric sets.

2 Proofs

Let, for a graph G and $X \subseteq V(G)$, $E_i(X) = \{xy : \ x, y \in X, \mathrm{dist}_G(x, y) = i\}$, i.e., E_i is a set of pairs at distance i in G. We say that $E_i(X)$ is *good* if it contains two disjoint pairs $xy, x'y'$. Note that if a non-empty $E_i(X)$ is not good, i.e., bad, it is a triangle or a star in X. Further observe that if $X = A \cup B$, where A and B are weakly homometric in G, then $E_i(X)$ is good, for $i = d^*(A)$. We say that we *split* a set X of vertices if we form two disjoint subsets of X of size $\lfloor |X|/2 \rfloor$. We denote $d(xy) = \mathrm{dist}_G(x, y)$, $x, y \in V(G)$. We denote the edge set of a star with center x and leaf set X as $S(x, X)$.

Lemma 2.1. *Let G be a graph, $X \subseteq V(G)$, $i = d^*(X)$. If $E_i(X)$ is good or $d^*(X) \leq 2$, then $h^*(G) \geq \lfloor (|X| - 1)/2 \rfloor$.*

Proof. Assume first that $xy, x'y' \in E_i(X)$ are disjoint pairs of vertices and $i = d^*(X)$. Split X such that x, y are in one part and x', y' in another part. The resulting sets are weakly homometric sets. If $d^*(X) = 2$, then either $E_2(X)$ is good implying $h^*(G) \geq \lfloor |X|/2 \rfloor$ or non-edges form a star or a triangle, so deleting one vertex allows us to split the remaining vertices of X in two sets each inducing a clique. Thus $h^*(G) \geq \lfloor (|X| - 1)/2 \rfloor$ in this case. If $d^*(X) = 1$, then X induces a clique and $h^*(G) \geq \lfloor |X|/2 \rfloor$. □

Proof of Theorem 1.1. First we shall show the lower bound on $h^*(n)$. Consider a connected graph G on n vertices. Let $d = \mathrm{diam}(G)$. If $d = 2$, the lower bound follows from the Lemma 2.1. So, we assume that $d \geq 3$. If $E_d(V)$ is good, then by Lemma 2.1 $h^*(G) \geq \lfloor (n-1)/2 \rfloor \geq \lfloor (n+1)/3 \rfloor$. If $E_d(V)$ is bad, it either forms a triangle or a star.

Case 1 $E_d(V)$ forms a triangle xyz.
Let x' and y' be distinct vertices such that $d(xx') = d(yy') = d - 1$. Such x', y' could be chosen on a shortest xy-path. Let A and B be disjoint subsets of $V - z$, each of size

$\lfloor (n+1)/3 \rfloor$, A containing x and x', B containing y and y'. We see that A and B are weakly homometric with maximum distance $d-1$.

Case 2 $E_d(V)$ forms a star.
Let $E_d(V) = S(x_0, Y)$, forming a star with center x_0 and leaf set Y. Let $x_d \in Y$, i.e., $d(x_0x_d) = d$. Consider a shortest x_0-x_d path $x_0, \dots, x_d$ of length d.

Case 2.1 $|Y| \le n - \lfloor (n+1)/3 \rfloor - 1$.
Let A and B be disjoint sets such that $|A| = |B| = \lfloor (n+1)/3 \rfloor$, $A \subseteq V - Y - \{x_1\}$, A contains x_0, x_{d-1}, B contains x_1, x_d. Then A and B are weakly homometric with largest distance $d-1$.

Case 2.2 $|Y| \ge n - \lfloor (n+1)/3 \rfloor$.
In particular $d \le \lfloor (n+1)/3 \rfloor$. Let T be a breadth-first search tree with root x_0. Let L_i's be the layers of T, i.e., sets of vertices at distance i from x_0, $i = 1, \dots, d$. We have that $L_d = Y$, $L_0 = \{x_0\}$.

If T is a broom, i.e., all vertices of Y have a common neighbor, x_{d-1} in T, then $d^*(Y \cup \{x_{d-1}, x_{d-2}\}) = 2$ and by Lemma 2.1 $h^*(G) \ge \lfloor (|Y|+2-1)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 1)/2 \rfloor \ge \lfloor (n+1)/3 \rfloor$.

If T is not a broom, then some layer L_i, $i < d$, has more than one vertex and $d \le \lfloor (n+1)/3 \rfloor - 1$. Let i be the smallest such index, i.e., $L_j = \{x_j\}$ for all $j < i$. Then we see that $S(x_j, Y) \subseteq E_{d-j}(V)$, $j < i$. Let $V_j = V - \{x_0, \dots, x_{j-1}\}$, $j = 1, \dots, d$.

We consider $E_{d-1}(V_1), E_{d-2}(V_2), \dots$ in order and show that each of these sets $E_{d-j}(V_j)$ is either good, allowing to use Lemma 2.1, or is a star with center x_j. If for some j, $0 < j < i$, $S(x_j, Y) \ne E_{d-j}(V_j)$, then for smallest such j, $E_{d-j}(V_j)$ is good and $d^*(V_j) = d - j$, so by Lemma 2.1, $h^*(G) \ge \lfloor (n-j-1)/2 \rfloor \ge \lfloor (n-(d-2)-1)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 2)/2 \rfloor \ge \lfloor (n+1)/3 \rfloor$. Thus, we have that $S(x_j, Y) = E_{d-j}(V_j)$ and $d^*(V_j) = d - j$, $j = 1, \dots, i-1$. Consider $x_i, x_i' \in L_i$. We have that $d(x_ix_d) = d - i$, and the largest distance $d^*(V_i) = d - i$. Moreover, we claim that $d(x_i'y) = d - i$ for each $y \in Y$. Assume not and $d(x_i'y) < d - i$. Then $d(x_{i-1}y) < d - i + 1$, a contradiction. Thus $E_{d-i}(V_i)$ is good, and by Lemma 2.1, we have $h^*(G) \ge \lfloor (n-i)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 1)/2 \rfloor \rfloor \ge \lfloor (n+1)/3 \rfloor$. In all these cases we have that $h^*(G) \ge \lfloor (n+1)/3 \rfloor$.

For the upper bound on $h^*(n)$, let $k = \lfloor (n+1)/3 \rfloor$. Consider a graph G that is a union of a clique K on $n-k$ vertices and a path P on $k+1$ vertices such that K and P share exactly one vertex x that is an end-point of P. Consider two weakly homometric sets A and B in $V(G)$ that have the largest possible size $h^*(G)$. If $(A \cup B) \subseteq V(K)$ then $h^*(G) \le \lfloor (n-k)/2 \rfloor$. So, let's assume that $x' \in V(P) \cap (A \cup B)$ such that x' has the largest distance from x among the vertices of $A \cup B$. Assume further that $x' \in A$ and let $i = d(x'x)$. Then $E_{i+1}(G)$ consists of all pairs $x'y$, $y \in V(K) - \{x\}$ and pairs containing vertices from P that are further from x as x' (if any). Since there are no such vertices in $A \cup B$, we see that $E_{i+1}(A \cup B)$ is a star, so $i + 1 \ne d^*(A)$. Thus $A \subseteq V(P)$. If $d^*(A) > 1$ then $d^*(B) > 1$ and $B \setminus V(K) \ne \emptyset$. Thus at least one vertex in P is from B, so $|A| \le |V(P)| - 1 = k$. If $d^*(A) = 1$, then $|A| = 2$. Thus $h^*(G) \le \max\{\lfloor (n-k)/2 \rfloor, k, 2\} \le \lfloor (n+1)/3 \rfloor$, for $n \ge 6$. □

Proof of Theorem 1.2. Let G be a graph on n vertices and let $k = \lfloor (n+1)/3 \rfloor$. Assign a color $c(A) = \mathrm{diam}(G[A])$ to each k-element subset A of vertices of G. Then $c(A) \in \{1, 2, \dots, k-1, \infty\}$. So, there are at most k colors used in this coloring. The coloring

c corresponds to a coloring of vertices of the Kneser graph $K(n,k)$. Since the chromatic number $\chi(K(n,k)) = n-2k+2$, see Lovász [7], and the number k of colors used is less than the chromatic number $n-2k+2$, we see that c is not a proper coloring, so there are two disjoint sets A and B of the same color. Thus $h_{\text{diam}}(G) \geq k$. In particular, $h_{\text{diam}}(G) \geq \lfloor (n+1)/3 \rfloor$.

If $\text{diam}(G) = 1$ then G is a complete graph and the conclusion follows trivially. If $\text{diam}(G) \geq 4$, we consider a vertex v that is at distance at least 4 to some other vertex. Consider a breadth first search tree with a root v. Let V_i, $i = 0, 1, 2, \ldots, q$ be the layers of that tree, i.e., V_i is a set of vertices at distance i from v, $V_0 = \{v\}$, $q \geq 4$. We see that there are no edges between any two non-consecutive layers. We shall build two disjoint sets A and B such that $G[A]$ and $G[B]$ are both disconnected, i.e., have diameter ∞.

If each layer has size less than $n/2$, put v and V_2 in A, put V_1 in B and split the remaining vertices (except maybe one) between A and B such that $|A| = |B|$. We see that v is not adjacent to any other vertex of A and we see that any vertex of V_2 is not adjacent to any vertex from $B \setminus V_2$.

If there is a layer, L, of size at least $n/2$ then the total number of vertices in all other layers is less than $n/2$. Consider the layers other than L, in order, and assign all vertices of each layer to the same set, A or B, in an alternating fashion. Split the vertices of L between A and B such that $|A| = |B| = \lfloor n/2 \rfloor$. More precisely, let $\{V_0, V_1, \ldots\} \setminus L = \{V_{i_1}, V_{i_2}, \ldots\}$, where $i_1 < i_2 < \cdots$. Put vertices of V_{i_k} in A if k is even, put vertices of V_{i_k} in B if k is odd. We see that there is always a full layer in A between some two vertices of B and there is a full layer of B between two vertices of A. So, $G[A]$ and $G[B]$ are disconnected. $\square$

Acknowledgements: The authors thank Torsten Ueckerdt for interesting discussions on the topic and the referees for carefully checking the proofs.

References

[1] Albertson, M., Pach, J., Young, M., Disjoint homometric sets in graphs. Ars Mathematica Contemporanea, 4(1), 2011.

[2] Alon, N., Problems and results in Extremal Combinatorics-III. Journal of Combinatorics, 7(2-3):233–256, 2016.

[3] Axenovich, M. and Özkahya, L., On homometric sets in graphs. Electronic Notes in Discrete Mathematics, 38: 83–86, 2011.

[4] Bollobás, B., Kittipassorn, T., Narayanan, B., Scott, A., Disjoint induced subgraphs of the same order and size. European Journal of Combinatorics, 49:153–166, 2015.

[5] Caro, Y. and Yuster, R., Large disjoint subgraphs with the same order and size, European Journal of Combinatorics, 30, no. 4, 813–821, 2009.

[6] Fulek, R. and Mitrović, S., Homometric sets in trees. European Journal of Combinatorics, 35:256–263, 2014.

[7] Lovász, L., Kneser's conjecture, chromatic number, and homotopy, Journal of Combinatorial Theory, Series A, 25, no. 3, 319–324, 1978.