Reducing the maximum degree of a graph by deleting vertices: the extremal cases

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Reducing the maximum degree of a graph by deleting vertices: the extremal cases

Cover Page Footnote
The authors are greatly indebted to Professor Yair Caro for suggesting the problem in [3] (which motivated this paper) and for several fruitful discussions. The authors also wish to thank the anonymous referees for checking the paper carefully and for constructive remarks which led to an improvement in the presentation.
Abstract

Let \( \lambda(G) \) denote the smallest number of vertices that can be removed from a non-empty graph \( G \) so that the resulting graph has a smaller maximum degree. In a recent paper, we proved that if \( n \) is the number of vertices of \( G \), \( k \) is the maximum degree of \( G \), and \( t \) is the number of vertices of degree \( k \), then \( \lambda(G) \leq \frac{n + (k - 1)t}{2k} \). We also showed that \( \lambda(G) \leq \frac{1}{k}n \) if \( G \) is a tree. In this paper, we provide a new proof of the first bound and use it to determine the graphs that attain the bound, and we also determine the trees that attain the second bound.

1 Introduction

Unless otherwise stated, we use small letters such as \( x \) to denote non-negative integers or elements of a set, and capital letters such as \( X \) to denote sets or graphs. The set \( \{1, 2, \ldots \} \) of positive integers is denoted by \( \mathbb{N} \). For any \( n \in \{0\} \cup \mathbb{N} \), the set \( \{i \in \mathbb{N} : i \leq n\} \) is denoted by \( [n] \). For a set \( X \), the set \( \{(x, y) : x, y \in X, x \neq y\} \) of all 2-element subsets of \( X \) is denoted by \( \binom{X}{2} \). All arbitrary sets are assumed to be finite.

We adopt the definitions and notation in [3] for graphs. In particular, we have the following. For \( v \in V(G) \), \( N_G(v) \) denotes the set of neighbours of \( v \) in \( G \), \( N_G[v] \) denotes \( N_G(v) \cup \{v\} \), \( E_G(v) \) denotes the set of edges of \( G \) that are incident to \( v \), and \( d_G(v) \) denotes \( |N_G(v)| \) (= \( |E_G(v)| \)) and is called the degree of \( v \) in \( G \). The minimum degree of \( G \) is \( \min\{d_G(v) : v \in V(G)\} \) and is denoted by \( \delta(G) \). The maximum degree of \( G \) is \( \max\{d_G(v) : v \in V(G)\} \) and is denoted by \( \Delta(G) \). The set of vertices of \( G \) of degree \( \Delta(G) \) is denoted by \( M(G) \). For \( X \subseteq V(G) \), \( N_G(X) \) denotes \( \bigcup_{v \in X} N_G(v) \), \( N_G[X] \) denotes \( \bigcup_{v \in X} N_G[v] \), \( G[X] \) denotes the graph \( (X, E(G) \cap \binom{X}{2}) \), and \( G - X \) denotes \( G[V(G) \setminus X] \).

We may abbreviate \( G - \{v\} \) to \( G - v \). For \( v, w \in V(G) \), the distance of \( w \) from \( v \) is denoted by \( d_G(v, w) \). Where no confusion arises, the subscript \( G \) is omitted from any of the notation above that uses it; for example, \( N_G(v) \) is abbreviated to \( N(v) \).

If \( |V(G)| = k + 1 \) and \( E(G) = \{xv : v \in V(G) \setminus \{x\}\} \) for some \( x \in V(G) \), then \( G \) is called a \( k \)-star, or simply a star, with centre \( x \). The \( k \)-star \( (\{0\} \cup [k], \{\{0, i\} : i \in [k]\}) \) is denoted by \( K_{1,k} \). The complete graph \( ([n], \binom{[n]}{2}) \), the path \( ([n], \{\{1, 2\}, \ldots, \{n - 1, n\}\}) \), and the cycle \( ([n], \{\{1, 2\}, \ldots, \{n - 1, n\}, \{n, 1\}\}) \) are denoted by \( K_n \), \( P_n \), and \( C_n \), respectively.

If \( G_1, \ldots, G_t \) are graphs such that \( V(G_i) \cap V(G_j) = \emptyset \) for every \( i, j \in [t] \) with \( i \neq j \), then \( G_1, \ldots, G_t \) are said to be vertex-disjoint.

If \( k \geq 2 \), \( S_1, \ldots, S_t \) are vertex-disjoint \( k \)-stars, and \( G \) is a graph such that \( V(G) = \bigcup_{i=1}^t V(S_i) \), \( \bigcup_{i=1}^t E(S_i) \subseteq E(G) \), \( \Delta(G) = k \), and \( |M(G)| = t \) (or, equivalently, \( M(G) \) is the set of centres of \( S_1, \ldots, S_t \)), then we call \( G \) a special \( k \)-star \( t \)-union and we call \( S_1, \ldots, S_t \) the constituents of \( G \).

![Figure 1: An illustration of a special k-star t-union.](image-url)

If \( S_1, \ldots, S_t \) are vertex-disjoint \( k \)-stars and \( T \) is a tree such that \( V(T) = \bigcup_{i=1}^t V(S_i) \), \( \bigcup_{i=1}^t E(S_i) \subseteq E(T) \), and \( \Delta(T) = k \), then we call \( T \) a \( k \)-special (it is easy to see that \( T \)
has \( t - 1 \) edges \( e_1, \ldots, e_{t-1} \) such that \( E(T) \setminus \bigcup_{i=1}^{t-1} E(S_i) = \{e_1, \ldots, e_{t-1}\} \) and, for each \( i \in [t-1] \), there exist some \( j, k \in [t] \) such that \( j \neq k \) and \( e_i = \{v_j, v_k\} \) for some leaf \( v_j \) of \( S_j \) and some leaf \( v_k \) of \( S_k \).

![Figure 2: An illustration of a \( k \)-special tree with \( k = 3 \) and \( t = 6 \).](image)

We call a subset \( R \) of \( V(G) \) a \( \Delta \)-reducing set of \( G \) if \( \Delta(G - R) < \Delta(G) \) or \( R = V(G) \) (note that \( V(G) \) is the smallest \( \Delta \)-reducing set of \( G \) if and only if \( \Delta(G) = 0 \)). Note that \( R \) is a \( \Delta \)-reducing set of \( G \) if and only if \( M(G) \subseteq N[R] \). Let \( \lambda(G) \) denote the size of a smallest \( \Delta \)-reducing set of \( G \).

A subset \( D \) of \( V(G) \) is called a dominating set of \( G \) if \( N[D] = V(G) \). The size of a smallest dominating set of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A dominating set of \( G \) is a \( \Delta \)-reducing set of \( G \). Thus, the problem of minimizing the size of a \( \Delta \)-reducing set is a variant of the classical domination problem [4–9]; the aim is to use as few vertices as possible to dominate the vertices of maximum degree rather than all the vertices. Many other variants have been studied; many of the earliest ones are referenced in [9], but nowadays there are several others. If \( G \) is \( k \)-regular (that is, \( d(v) = k \) for each \( v \in V(G) \)), then our problem is the same as the classical one, that is, \( \lambda(G) = \gamma(G) \).

The parameter \( \lambda(G) \) was introduced and studied in our recent paper [3]. An application is indicated in [13]. One of our main results in [3] is that if \( G \) is a non-empty graph, \( n = |V(G)| \), \( k = \Delta(G) \), and \( t = |M(G)| \), then \( \lambda(G) \leq \frac{n + (k-1)t}{2k} \). We remarked that this upper bound can be attained in cases where \( \lambda(G) = t \) and also in cases where \( \lambda(G) < t \).

In this paper, we provide a new proof of the bound, using induction, and use the new argument to determine the graphs that attain the bound.

**Theorem 1.1** If \( G \) is a non-empty graph, \( n = |V(G)| \), \( k = \Delta(G) \), and \( t = |M(G)| \), then

\[
\lambda(G) \leq \frac{n + (k-1)t}{2k}.
\]

Moreover, equality holds if and only if one of the following holds:
(i) \( k = 1 \) and each component of \( G \) is a copy of \( K_2 \),
(ii) \( k = 2 \) and each component of \( G \) is a copy of \( P_5 \) or \( C_4 \),
(iii) \( k \geq 2 \) and \( G \) is a special \( k \)-star union.

In [3], we also proved the following bound for trees.

**Theorem 1.2** If \( T \) is a tree, \( n = |V(T)| \), and \( k = \Delta(T) \), then

\[
\lambda(T) \leq \frac{n}{k + 1}.
\]

We noted that the bound is sharp; for example, it is attained by \( k \)-stars. In this paper, we determine the trees which attain the bound.
Theorem 1.3 The bound in Theorem 1.2 is attained if and only if \( T \) is \( k \)-special.

As pointed out above, a dominating set is a \( \Delta \)-reducing set, so \( \lambda(G) \leq \gamma(G) \). We conclude this section with a brief discussion on how the bounds above compare with well-known domination bounds. First we note that our bound \( \frac{n+\lfloor(k-1)t\rfloor}{2k} \) on \( \lambda(G) \) is at most Ore’s upper bound \( \frac{n}{2} \) on \( \gamma(G) \) (for \( \delta(G) \geq 1 \)) \cite{11}, and it is equal to it if and only if \( G \) is \( k \)-regular (in which case \( \lambda(G) = \gamma(G) \)). However, taking \( \delta = \delta(G) \), we see that our bound for \( k \geq 2 \) is at most the classical upper bound \( \frac{1+\ln(\delta+1)}{\delta}\{n/2\} \) on \( \gamma(G) \) \cite{1, 2, 10, 12} if and only if \( t \leq \frac{n}{\delta^2} \{1+2\ln(\delta+1)+\frac{2}{\delta-1}\ln(\delta+1)+\frac{k-2}{k-1}\} \). Thus, the improvement offered by our bound is limited. It is interesting that, on the other hand, the upper bound \( \frac{1}{k+1} \) in Theorem 1.2 is a basic lower bound for the domination number of any graph \( G \) with \( \Delta(G) = k \) (see \cite{6}), meaning that no domination number upper bound is better than it.

2 Proofs of the results

We now prove Theorems 1.1 and 1.3. We will make use of the following two structural results from \cite{3} (\cite{3, Propositions 3.4 and 3.5}).

Proposition 2.1 (\cite{3}) If \( G \) is a graph and \( v \in V(G) \setminus N[M(G)] \), then \( \lambda(G-v) = \lambda(G) \).

Proposition 2.2 (\cite{3}) If \( v \) is a vertex of a graph \( G \), then \( \lambda(G) \leq 1+\lambda(G-v) \).

The next result implies that the bound in Theorem 1.1 is attained by special \( k \)-star \( t \)-unions, and that the bound in Theorem 1.2 is attained by \( k \)-special trees.

Lemma 2.3 If \( S_1, \ldots, S_t \) are vertex-disjoint \( k \)-stars and \( G \) is a graph such that \( V(G) = \bigcup_{i=1}^{t} V(S_i), \bigcup_{i=1}^{t} E(S_i) \subseteq E(G), \) and \( \Delta(G) = k \), then \( |V(G)| = (k+1)t \) and \( \lambda(G) = t \).

Proof. We have \( |V(G)| = \sum_{i=1}^{t} |V(S_i)| = (k+1)t \). For each \( i \in [t] \), there exists a vertex \( x_i \) of \( S_i \) such that \( N_{S_i}[x_i] = V(S_i) \) and \( E(S_i) = E_{S_i}(x_i) \). Let \( X = \{x_1, \ldots, x_t\} \). Since \( V(G) = \bigcup_{i=1}^{t} V(S_i) = N_G[X], X \) is a \( \Delta \)-reducing set of \( G \), so \( \lambda(G) \leq |X| = t \). Now let \( R \) be a \( \Delta \)-reducing set of \( G \) of size \( \lambda(G) \). For each \( i \in [t] \), we have \( k = |V(S_i)\setminus \{x_i\}| = |N_{S_i}(x_i)| \leq |N_G(x_i)| \leq \Delta(G) = k \), so \( N_G(x_i) = V(S_i)\setminus \{x_i\}, x_i \in M(G) \), and hence \( R \cap N_G[x_i] \neq \emptyset \). We have \( |R| = |R \cap V(G)| = |R \cap \bigcup_{i=1}^{t} V(S_i)| = \sum_{i=1}^{t} |R \cap V(S_i)| \) as \( V(S_1), \ldots, V(S_t) \) are pairwise disjoint. Thus, \( |R| = \sum_{i=1}^{t} |R \cap N_G[x_i]| \geq \sum_{i=1}^{t} 1 = t \). We have \( t \leq \lambda(G) \leq t \), so \( \lambda(G) = t \). \( \square \)

We need the following notation from \cite{3}. For a graph \( G \), let \( M_1(G) \) denote the set \( \{v \in M(G) : d(v, w) \leq 2 \text{ for some } w \in M(G) \setminus \{v\}\} \), and let \( M_2(G) \) denote \( M(G) \setminus M_1(G) \). Thus, \( M_2(G) = \{v \in M(G) : d(v, w) \geq 3 \text{ for each } w \in M(G) \setminus \{v\}\} \).

Proof of Theorem 1.1. If each component of \( G \) is a copy of \( K_2 \), then \( \lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k} \). If \( G \) has \( s_1 + s_2 \) components, \( s_1 \) components of \( G \) are copies of \( P_3 \), and \( s_2 \) components of \( G \) are copies of \( C_4 \), then \( k = 2, n = 3s_1 + 4s_2, t = s_1 + 4s_2 \), and clearly \( \lambda(G) = s_1 + 2s_2 = \frac{n+(k-1)t}{2k} \). If \( G \) is a special \( k \)-star \( t \)-union, then \( n = (k+1)t \) and \( \lambda(G) = t = \frac{n+(k-1)t}{2k} \) by Lemma 2.3.

We now prove the bound in the theorem and show that it is attained only in the cases above. Since \( G \) is non-empty, \( n \geq 2 \). If \( n = 2 \), then \( G \) is a copy of \( K_2 \), so
\[ \lambda(G) = 1 = \frac{n + (k-1)t}{2k}. \]  
We proceed by induction on \( n \). Thus, consider \( n \geq 3 \). If \( k = 1 \), then \( G \) is the union of vertex-disjoint copies of \( K_2 \), so \( \lambda(G) = \frac{n}{2} = \frac{n + (k-1)t}{2k} \). Consider \( k \geq 2 \). Let \( v^* \in M(G) \). We have \( n \geq |N[v^*]| = k + 1 \).

Suppose that \( M_2(G) \) has a member \( u \). If \( \Delta(G - u) < \Delta(G) \), then \( \lambda(G) = 1 \leq \frac{n + (k-1)t}{2k} \) (as \( n \geq k+1 \)). If \( \lambda(G) = 1 = \frac{n + (k-1)t}{2k} \), then \( V(G) = N[u] \), so \( G \) is a special \( k \)-star 1-union. Now suppose \( \Delta(G - u) = \Delta(G) \). Then, since \( u \in M_2(G) \), \( M(G - u) = M(G) \setminus \{u\} \) and it is a copy of \( P_3 \) or \( C_4 \). Suppose that each component of \( G' \) is a copy of \( P_3 \) or \( C_4 \). Then \( k = 2 \). Let \( u_1 \) and \( u_2 \) be the two members of \( N(v^*) \). Since \( u \in M_2(G) \), we have \( d(u_1) = d(u_2) = 1 \), so \( N(u_1) = N(u_2) = \{u\} \). Thus, \( G[N[u]] \) is a copy of \( P_3 \) and a component of \( G \). Therefore, each component of \( G \) is a copy of \( P_3 \) or \( C_4 \). Now suppose that \( G' \) is a special \( k \)-star \((t-1)\)-union with constituents \( S_1, \ldots, S_{t-1} \). Let \( S_0 \) be the \( k \)-star \((N[u], E(u)) \). Then \( S_1, \ldots, S_t \) are vertex-disjoint, \( V(G) = V(G') \cup N[u] = \bigcup_{i=1}^t V(S_i) \), and \( \bigcup_{i=1}^t E(S_i) \subseteq E(G) \). Thus, \( G \) is a special \( k \)-star \((t-1)\)-union.

Now suppose \( M_2(G) = \emptyset \). Then \( M(G) = M_1(G) \).

Suppose that \( G \) has a vertex \( u \) such that \( N[u] \) contains at least 3 vertices in \( M(G) \). If \( \Delta(G - u) < \Delta(G) \), then \( \lambda(G) = 1 < \frac{n + (k-1)t}{2k} \) as \( n \geq k + 1 \), \( t \geq 3 \), and \( k \geq 2 \). Now suppose \( \Delta(G - u) = \Delta(G) \). Let \( n' = |V(G - u)| = n - 1 \) and \( t' = |M(G - u)| = t - 3 \). By Proposition 2.2 and the induction hypothesis,  
\[ \lambda(G) \leq 1 + \lambda(G - u) \leq 1 + \frac{n' + (k-1)t'}{2k} \leq 1 + \frac{n + (k-1)t}{2k} \leq \frac{n + (k-1)t}{2k}. \]
Suppose \( \lambda(G) = \frac{n + (k-1)t}{2k} \). Then, in (1), equality holds throughout. Thus, \( k = 2 \) (as \( n + (k-1)t - (k-2) = n + (k-1)t \)), \( t = t - 3 \) (as \( n' + (k-1)t' = (n-1) + (k-1)(t-3) \)), and \( \lambda(G - u) = \frac{n + (k-1)t}{2k} \). By the induction hypothesis, \( G - u \) is a special 2-star \((t-1)\)-union or each component of \( G - u \) is a copy of \( P_3 \) or \( C_4 \). If \( G - u \) is a special 2-star \((t-1)\)-union, then, by definition, the constituents of \( G - u \) are the components of \( G - u \) (because, since \( k = 2 \) and \( |M(G - u)| = t' \), \( d_{G - u}(z) = 1 \) for each leaf \( z \) of any constituent), and they are copies of \( P_3 \). Therefore, in any case, each component of \( G - u \) is a copy of \( P_3 \) or \( C_4 \). Let \( s_1 \) be the number of components of \( G - u \) that are copies of \( P_3 \), and let \( s_2 \) be the number of components of \( G - u \) that are copies of \( C_4 \). Let \( u_1 \) and \( u_2 \) be two distinct members of \( N(u) \). Since \( k = 2 \) and \( |N[u] \cap M(G)| \geq 3 \), \( N[u] = \{u_1, u_2\} \). Thus, \( d(u) = d(u_1) = d(u_2) = \Delta(G) = 2 \). For each \( i \in [2] \), \( d_{G - u}(u_i) = d_G(u_i) - 1 = 1 \), so \( u_i \) is a leaf of a component \( H_i \) of \( G - u \) that is a copy of \( \{u_i, u_i', u_i''\} \) of \( P_3 \). Since \( N(u) = \{u_1, u_2\} \) and \( M_2(G) = \emptyset \), \( H_1 \) and \( H_2 \) are the only components of \( G - u \) that are copies of \( P_3 \). Suppose \( H_1 \neq H_2 \). Then \( G \) has \( s_2 + 1 \) components, \( s_2 \) components of \( G \).
are copies of $C_4$, and 1 component of $G$ is a copy of $P_7$. Thus, $n = 4s_2 + 7$, $t = 4s_2 + 5$, and clearly $\lambda(G) = 2s_2 + 2$. We have $\lambda(G) < 2s_2 + 3 = \frac{n + (k-1)t}{2k}$, a contradiction. Thus, $H_1 = H_2$, and hence each component of $G$ is a copy of $C_4$.

Now suppose that
\[
|N[v] \cap M(G)| \leq 2 \text{ for each } v \in V(G).
\] (2)

Suppose that, for each $v \in M(G)$, $N(v)$ contains no member of $M(G)$. Let $x \in M(G)$. Since $M(G) = M_1(G)$, there exists some $w \in N(x) \setminus M(G)$ such that $y \in N(w)$ for some $y \in M(G) \setminus \{x, y\}$. Since $x, y \in N(w)$, $N(w) \cap M(G) = \{x, y\}$ by (2). If $\Delta(G - w) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n + (k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G - w) = \Delta(G)$. Then $M(G - w) = M(G) \setminus \{x, y\}$. Let $G' = G - \{w, x, y\}$. Since $N(x) \cap M(G) = \emptyset$, $N(y) \cap M(G) = \emptyset$, and $N(w) \cap M(G) = \{x, y\}$, we have $M(G') = M(G) \setminus \{x, y\} = M(G - w)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G - w)$ by Proposition 2.1 (as $y \notin N_{G - w}(M(G - w))$) and $x \notin N_{G - w}(M(G - w))$. Let $n' = |V(G')| = n - 3$ and $t' = |M(G')| = t - 2$. By Proposition 2.2 and the induction hypothesis,
\[
\lambda(G) \leq 1 + \lambda(G - w) = 1 + \lambda(G') \leq 1 + \frac{n' + (k - 1)t'}{2k} < \frac{n + (k - 1)t}{2k}.
\]

Finally, suppose that $G$ has a vertex $u$ in $M(G)$ such that $N(u)$ contains a member $w$ of $M(G)$. By (2), $N[u] \cap M(G) = \{u, w\} = N[w] \cap M(G)$. If $\Delta(G - u) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n + (k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G - u) = \Delta(G)$. Then $M(G - u) = M(G) \setminus \{u, w\}$. Let $G' = G - \{u, w\}$. Since $N[u] \cap M(G) = \{u, w\} = N[w] \cap M(G)$, we have $M(G') = M(G) \setminus \{u, w\} = M(G - w)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G - u)$ by Proposition 2.1 (as $w \notin N_{G - u}(M(G - u))$). Let $n' = |V(G')| = n - 2$ and $t' = |M(G')| = t - 2$. By Proposition 2.2 and the induction hypothesis,
\[
\lambda(G) \leq 1 + \lambda(G - u) = 1 + \lambda(G') \leq 1 + \frac{n' + (k - 1)t'}{2k} = \frac{n + (k - 1)t}{2k}.
\]

Suppose $\lambda(G) = \frac{n + (k-1)t}{2k}$. Then $\lambda(G') = \frac{n' + (k-1)t'}{2k}$. By the induction hypothesis, $G'$ is a special $k$-star $(t - 2)$-union or each component of $G'$ is a copy of $P_3$ or $C_4$. Thus, $\delta(G') \geq 1$.

Suppose first that each component of $G'$ is a copy of $P_3$ or $C_4$. Then $\Delta(G') = 2$. Since $\Delta(G) = \Delta(G')$, $d(u) = d(w) = 2$. Thus, $N(u) = \{u', w\}$ for some $u' \in V(G') \setminus \{u, w\} = V(G')$. Since $N[u] \cap M(G) = \{u, w\}$ and $k = 2$, we have $d(u') < 2$, so $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$.

Now suppose that $G'$ is a special $k$-star $(t - 2)$-union. Let $S_1, \ldots, S_{t - 2}$ be the constituents of $G'$. Let $X = N(u) \setminus \{w\}$ and $Y = N(w) \setminus \{u\}$. Then $|X| = |Y| = k - 1$ and $d_{G'}(v) < k$ for each $v \in X \cup Y$. For each $i \in [t - 2]$, $S_i$ has a vertex $v_i$ such that $d_{S_i}(v_i) = k$. Since $\Delta(G) = k$, $d(v_i) = d_{S_i}(v_i) = k$ for each $i \in [t - 2]$. Note that
\[
|X \cup Y \subseteq V(G') \setminus \{v_1, \ldots, v_r\} = V(G') \setminus M(G') = \bigcup_{i=1}^{t - 2} N(v_i).
\] (3)

Suppose $X \cap Y \neq \emptyset$. Let $x \in X \cap Y$. We have $x \in N(v_p)$ for some $p \in [t - 2]$. Thus, we have $u, w, v_p \in N[x] \cap M(G)$, contradicting (2). Therefore, $X \cap Y = \emptyset$. Recall that we are considering $k \geq 2$. Since $|X| = |Y| = k - 1$, $X \notin \emptyset$. Let $x^* \in X$. By (3), $x^* \in N(v_p)$.
for some $p \in [t - 2]$. Consider any $y \in Y$. By (3), $y \in N(v_q)$ for some $q \in [t - 2]$. Suppose $q \neq p$. Then $\{v_1, \ldots, v_{t-2}\} \setminus \{v_p, v_q\} \cup \{x^*, y\}$ is a $\Delta$-reducing set of $G$ of size $t - 2$. We have
\[
t - 2 \geq \lambda(G) = \frac{n + (k - 1)t}{2k} = \frac{|\{u, w\} \cup \bigcup_{i=1}^{k-2} V(S_i)| + (k - 1)t}{2k} = \frac{2 + (k + 1)(t - 2) + (k - 1)t}{2k} = t - 1,
\]
a contradiction. Thus, $Y \subseteq N(v_p)$. Let $y^* \in Y$. Then $y^* \in N(v_p)$. By an argument similar to that for $x^*$, $X \subseteq N(v_p)$. Since $X \cap Y = \emptyset$, we have $2(k - 1) = |X \cup Y| \leq |N(v_p)| = k$, so $k \leq 2$. Since $k \geq 2$, $k = 2$. Thus, since $N[u] \cap M(G) = \{u, w\}$, $N(u) = \{w, u'\}$ for some $u' \in V(G) \setminus M(G)$. Since $d(u') < k = 2$, $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$.

We now prove Theorem 1.3. We make use of the following two well-known facts, which were reproduced in [3] for the proof of Theorem 1.2.

**Lemma 2.4** Let $x$ be a vertex of a tree $T$. Let $m = \max\{d(x, y) : y \in V(T)\}$, and let $D_i = \{y \in V(T) : d(x, y) = i\}$ for each $i \in \{0\} \cup [m]$. For each $i \in [m]$ and each $v \in D_i$, $N(v) \cap \bigcup_{j=0}^m D_j = \{u\}$ for some $u \in D_{i-1}$.

**Lemma 2.5** If $T$ is a tree, $x, z \in V(T)$, and $d(x, z) = \max\{d(x, y) : y \in V(T)\}$, then $z$ is a leaf of $T$.

**Proof of Theorem 1.3.** By Lemma 2.3, $\lambda(T) = \frac{n}{k+1}$ if $T$ is $k$-special. We now prove the converse. This is trivial if $n \leq 2$. We proceed by induction on $n$. Suppose $n \geq 3$ and $\lambda(T) = \frac{n}{k+1}$. Since $T$ is a connected graph, we clearly have $k \geq 2$.

Suppose that $T$ has a leaf $z$ whose neighbour is not in $M(T)$. Then $M(T-z) = M(T)$ and, by Proposition 2.1, $\lambda(T-z) = \lambda(T)$. By Theorem 1.2, $\lambda(T-z) \leq \frac{n-1}{k+1} < \frac{n}{k+1}$. Thus, we have $\lambda(T) < \frac{n}{k+1}$, a contradiction.

Therefore, each leaf of $T$ is adjacent to a vertex in $M(T)$. Let $x$, $m$, and $D_0, D_1, \ldots, D_m$ be as in Lemma 2.4. Let $z \in V(T)$ such that $d(x, z) = m$. By Lemma 2.5, $z$ is a leaf of $T$. Let $w$ be the neighbour of $z$. Then $w \in M(T)$. By Lemma 2.4, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $E(T) = \{xz_1, \ldots, xz_k\}$ for some distinct vertices $z_1, \ldots, z_k$ of $T$. Thus, $T$ is a $k$-star and hence $k$-special.

Now suppose $w \neq x$. Together with Lemma 2.4, this implies that $N(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \ldots, z_{k-1}$ in $D_m$. By Lemma 2.5, $z_1, \ldots, z_{k-1}$ are leaves of $T$. Let $T' = T - v$. Then each component of $T'$ is a tree. Let $\mathcal{K}$ be the set of components of $T'$ whose maximum degree is $k$, and let $\mathcal{H}$ be the set of components of $T'$ whose maximum degree is less than $k$. Let $W = \{w, z_1, \ldots, z_{k-1}\}$. Note that $(W, \{wz_1, \ldots, wz_{k-1}\}) \in \mathcal{H}$, and hence $W \cap \bigcup_{C \in \mathcal{K}} V(C) = \emptyset$. Let $S_0$ be the $k$-star $(W \cup \{v\}, \{vw, wz_1, \ldots, wz_{k-1}\})$.

Suppose $\mathcal{K} = \emptyset$. Then $\{v\}$ is a $\Delta$-reducing set of $T$, and hence $\lambda(T) = 1$. Since $\lambda(T) = \frac{n}{k+1}$, we have $n = k + 1$, so $T = S_0$. Thus, $T$ is $k$-special.

Now suppose $\mathcal{K} \neq \emptyset$. Let $T_1, \ldots, T_r$ be the distinct members of $\mathcal{K}$. For each $i \in [r]$, let $R_i$ be a $\Delta$-reducing set of $T_i$ of size $\lambda(T_i)$. By Theorem 1.2, $|R_i| \leq \frac{|V(T_i)|}{k+1}$ for each $i \in [r]$. Now $\{v\} \cup \bigcup_{i=1}^r R_i$ is a $\Delta$-reducing set of $T$. Thus, we have
\[
\lambda(T) \leq 1 + \sum_{i=1}^r |R_i| \leq \frac{|V(S_0)|}{k+1} + \sum_{i=1}^r \frac{|V(T_i)|}{k+1} \leq \frac{n}{k+1}.
\]
Since $\lambda(T) = \frac{n}{k+1}$, it follows that $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} V(T_i)$ and $\lambda(T_i) = \frac{\lfloor V(T_i) \rfloor}{k+1}$ for each $i \in [r]$. By the induction hypothesis, for each $i \in [r]$, $T_i$ is $k$-special, so there exist vertex-disjoint $k$-stars $S_{i,1}, \ldots, S_{i,t_i}$ such that $V(T_i) = \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $\bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T_i)$. Therefore, we have $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $E(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T)$. Since $S_0, T_1, \ldots, T_r$ are vertex-disjoint, $S_{i,1}, \ldots, S_{i,t_i}, \ldots, S_{r,1}, \ldots, S_{r,t_r}$ are vertex-disjoint. Since $\Delta(T) = k$, $T$ is $k$-special. \hfill \Box

References


