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Reducing the maximum degree of a graph by deleting vertices: the extremal cases

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Reducing the maximum degree of a graph by deleting vertices: the extremal cases

Cover Page Footnote
The authors are greatly indebted to Professor Yair Caro for suggesting the problem in [3] (which motivated this paper) and for several fruitful discussions. The authors also wish to thank the anonymous referees for checking the paper carefully and for constructive remarks which led to an improvement in the presentation.

Abstract

Let $\lambda(G)$ denote the smallest number of vertices that can be removed from a non-empty graph G so that the resulting graph has a smaller maximum degree. In a recent paper, we proved that if n is the number of vertices of G, k is the maximum degree of G, and t is the number of vertices of degree k, then $\lambda(G) \leq \frac{n+(k-1)t}{2k}$. We also showed that $\lambda(G) \leq \frac{n}{k+1}$ if G is a tree. In this paper, we provide a new proof of the first bound and use it to determine the graphs that attain the bound, and we also determine the trees that attain the second bound.

1 Introduction

Unless otherwise stated, we use small letters such as x to denote non-negative integers or elements of a set, and capital letters such as X to denote sets or graphs. The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \{0\} \cup \mathbb{N}$, the set $\{i \in \mathbb{N} \colon i \leq n\}$ is denoted by $[n]$. For a set X, the set $\{\{x, y\} \colon x, y \in X, x \neq y\}$ of all 2-element subsets of X is denoted by $\binom{X}{2}$. All arbitrary sets are assumed to be finite.

We adopt the definitions and notation in [3] for graphs. In particular, we have the following. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G, $N_G[v]$ denotes $N_G(v) \cup \{v\}$, $E_G(v)$ denotes the set of edges of G that are incident to v, and $d_G(v)$ denotes $|N_G(v)|$ ($= |E_G(v)|$) and is called the *degree of v in G*. The *minimum degree of G* is $\min\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\delta(G)$. The *maximum degree of G* is $\max\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\Delta(G)$. The set of vertices of G of degree $\Delta(G)$ is denoted by $M(G)$. For $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v \in X} N_G(v)$, $N_G[X]$ denotes $\bigcup_{v \in X} N_G[v]$, $G[X]$ denotes the graph $(X, E(G) \cap \binom{X}{2})$, and $G - X$ denotes $G[V(G) \backslash X]$. We may abbreviate $G - \{v\}$ to $G - v$. For $v, w \in V(G)$, the *distance of w from v* is denoted by $d_G(v, w)$. Where no confusion arises, the subscript G is omitted from any of the notation above that uses it; for example, $N_G(v)$ is abbreviated to $N(v)$.

If $|V(G)| = k + 1$ and $E(G) = \{xv \colon v \in V(G) \backslash \{x\}\}$ for some $x \in V(G)$, then G is called a *k-star*, or simply a *star*, with *centre x*. The k-star $(\{0\} \cup [k], \{\{0, i\} \colon i \in [k]\})$ is denoted by $K_{1,k}$. The complete graph $([n], \binom{[n]}{2})$, the path $([n], \{\{1, 2\}, \dots, \{n-1, n\}\})$, and the cycle $([n], \{\{1, 2\}, \dots, \{n-1, n\}, \{n, 1\}\})$ are denoted by K_n, P_n, and C_n, respectively.

If $G_1, \dots, G_t$ are graphs such that $V(G_i) \cap V(G_j) = \emptyset$ for every $i, j \in [t]$ with $i \neq j$, then $G_1, \dots, G_t$ are said to be *vertex-disjoint*.

If $k \geq 2$, $S_1, \dots, S_t$ are vertex-disjoint k-stars, and G is a graph such that $V(G) = \bigcup_{i=1}^{t} V(S_i)$, $\bigcup_{i=1}^{t} E(S_i) \subseteq E(G)$, $\Delta(G) = k$, and $|M(G)| = t$ (or, equivalently, $M(G)$ is the set of centres of $S_1, \dots, S_t$), then we call G a *special k-star t-union* and we call $S_1, \dots, S_t$ the *constituents of G*.

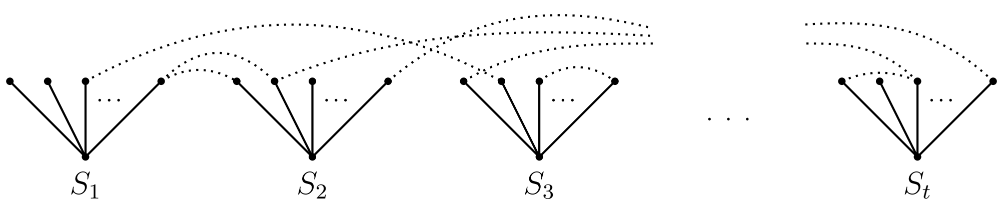

Figure 1: An illustration of a special k-star t-union.

If $S_1, \dots, S_t$ are vertex-disjoint k-stars and T is a tree such that $V(T) = \bigcup_{i=1}^{t} V(S_i)$, $\bigcup_{i=1}^{t} E(S_i) \subseteq E(T)$, and $\Delta(T) = k$, then we call T *k-special* (it is easy to see that T

has $t-1$ edges $e_1,\ldots,e_{t-1}$ such that $E(T)\setminus\bigcup_{i=1}^{t} E(S_i) = \{e_1,\ldots,e_{t-1}\}$ and, for each $i\in[t-1]$, there exist some $j,k\in[t]$ such that $j\neq k$ and $e_i=\{v_j,v_k\}$ for some leaf v_j of S_j and some leaf v_k of S_k).

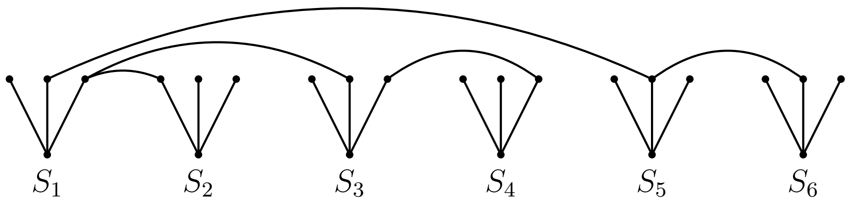

Figure 2: An illustration of a k-special tree with $k=3$ and $t=6$.

We call a subset R of $V(G)$ a Δ-*reducing set of* G if $\Delta(G-R)<\Delta(G)$ or $R=V(G)$ (note that $V(G)$ is the smallest Δ-reducing set of G if and only if $\Delta(G)=0$). Note that R is a Δ-reducing set of G if and only if $M(G)\subseteq N[R]$. Let $\lambda(G)$ denote the size of a smallest Δ-reducing set of G.

A subset D of $V(G)$ is called a *dominating set of* G if $N[D]=V(G)$. The size of a smallest dominating set of G is called the *domination number of* G and is denoted by $\gamma(G)$. A dominating set of G is a Δ-reducing set of G. Thus, the problem of minimizing the size of a Δ-reducing set is a variant of the classical domination problem [4–9]; the aim is to use as few vertices as possible to dominate the vertices of maximum degree rather than all the vertices. Many other variants have been studied; many of the earliest ones are referenced in [9], but nowadays there are several others. If G is k-regular (that is, $d(v)=k$ for each $v\in V(G)$), then our problem is the same as the classical one, that is, $\lambda(G)=\gamma(G)$.

The parameter $\lambda(G)$ was introduced and studied in our recent paper [3]. An application is indicated in [13]. One of our main results in [3] is that if G is a non-empty graph, $n=|V(G)|$, $k=\Delta(G)$, and $t=|M(G)|$, then $\lambda(G)\leq\frac{n+(k-1)t}{2k}$. We remarked that this upper bound can be attained in cases where $\lambda(G)=t$ and also in cases where $\lambda(G)<t$. In this paper, we provide a new proof of the bound, using induction, and use the new argument to determine the graphs that attain the bound.

Theorem 1.1 *If G is a non-empty graph, $n=|V(G)|$, $k=\Delta(G)$, and $t=|M(G)|$, then*

$$\lambda(G)\leq\frac{n+(k-1)t}{2k}.$$

Moreover, equality holds if and only if one of the following holds:
(i) $k=1$ and each component of G is a copy of K_2,
(ii) $k=2$ and each component of G is a copy of P_3 or C_4,
(iii) $k\geq 2$ and G is a special k-star t-union.

In [3], we also proved the following bound for trees.

Theorem 1.2 *If T is a tree, $n=|V(T)|$, and $k=\Delta(T)$, then*

$$\lambda(T)\leq\frac{n}{k+1}.$$

We noted that the bound is sharp; for example, it is attained by k-stars. In this paper, we determine the trees which attain the bound.

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Theorem 1.3 *The bound in Theorem 1.2 is attained if and only if T is k-special.*

As pointed out above, a dominating set is a Δ-reducing set, so $\lambda(G) \leq \gamma(G)$. We conclude this section with a brief discussion on how the bounds above compare with well-known domination bounds. First we note that our bound $\frac{n+(k-1)t}{2k}$ on $\lambda(G)$ is at most Ore's upper bound $\frac{n}{2}$ on $\gamma(G)$ (for $\delta(G) \geq 1$) [11], and it is equal to it if and only if G is k-regular (in which case $\lambda(G) = \gamma(G)$). However, taking $\delta = \delta(G)$, we see that our bound for $k \geq 2$ is at most the classical upper bound $\frac{1+\ln(\delta+1)}{\delta+1}n$ on $\gamma(G)$ [1, 2, 10, 12] if and only if $t \leq \frac{n}{\delta+1}\left(1 + 2\ln(\delta+1) + \frac{2}{k-1}\ln(\delta+1) + \frac{k-\delta}{k-1}\right)$. Thus, the improvement offered by our bound is limited. It is interesting that, on the other hand, the upper bound $\frac{n}{k+1}$ in Theorem 1.2 is a basic lower bound for the domination number of any graph G with $\Delta(G) = k$ (see [6]), meaning that no domination number upper bound is better than it.

2 Proofs of the results

We now prove Theorems 1.1 and 1.3. We will make use of the following two structural results from [3] ([3, Propositions 3.4 and 3.5]).

Proposition 2.1 ([3]) *If G is a graph and $v \in V(G)\backslash N[M(G)]$, then $\lambda(G - v) = \lambda(G)$.*

Proposition 2.2 ([3]) *If v is a vertex of a graph G, then $\lambda(G) \leq 1 + \lambda(G - v)$.*

The next result implies that the bound in Theorem 1.1 is attained by special k-star t-unions, and that the bound in Theorem 1.2 is attained by k-special trees.

Lemma 2.3 *If $S_1, \dots, S_t$ are vertex-disjoint k-stars and G is a graph such that $V(G) = \bigcup_{i=1}^t V(S_i)$, $\bigcup_{i=1}^t E(S_i) \subseteq E(G)$, and $\Delta(G) = k$, then $|V(G)| = (k+1)t$ and $\lambda(G) = t$.*

Proof. We have $|V(G)| = \sum_{i=1}^t |V(S_i)| = (k+1)t$. For each $i \in [t]$, there exists a vertex x_i of S_i such that $N_{S_i}[x_i] = V(S_i)$ and $E(S_i) = E_{S_i}(x_i)$. Let $X = \{x_1, \dots, x_t\}$. Since $V(G) = \bigcup_{i=1}^t V(S_i) = N_G[X]$, X is a Δ-reducing set of G, so $\lambda(G) \leq |X| = t$. Now let R be a Δ-reducing set of G of size $\lambda(G)$. For each $i \in [t]$, we have $k = |V(S_i)\backslash\{x_i\}| = |N_{S_i}(x_i)| \leq |N_G(x_i)| \leq \Delta(G) = k$, so $N_G(x_i) = V(S_i)\backslash\{x_i\}$, $x_i \in M(G)$, and hence $R \cap N_G[x_i] \neq \emptyset$. We have $|R| = |R \cap V(G)| = |R \cap \bigcup_{i=1}^t V(S_i)| = \sum_{i=1}^t |R \cap V(S_i)|$ as $V(S_1), \dots, V(S_t)$ are pairwise disjoint. Thus, $|R| = \sum_{i=1}^t |R \cap N_G[x_i]| \geq \sum_{i=1}^t 1 = t$. We have $t \leq \lambda(G) \leq t$, so $\lambda(G) = t$. □

We need the following notation from [3]. For a graph G, let $M_1(G)$ denote the set $\{v \in M(G) \colon d(v, w) \leq 2 \text{ for some } w \in M(G)\backslash\{v\}\}$, and let $M_2(G)$ denote $M(G)\backslash M_1(G)$. Thus, $M_2(G) = \{v \in M(G) \colon d(v, w) \geq 3 \text{ for each } w \in M(G)\backslash\{v\}\}$.

Proof of Theorem 1.1. If each component of G is a copy of K_2, then $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$. If G has $s_1 + s_2$ components, s_1 components of G are copies of P_3, and s_2 components of G are copies of C_4, then $k = 2$, $n = 3s_1 + 4s_2$, $t = s_1 + 4s_2$, and clearly $\lambda(G) = s_1 + 2s_2 = \frac{n+(k-1)t}{2k}$. If G is a special k-star t-union, then $n = (k+1)t$ and $\lambda(G) = t = \frac{n+(k-1)t}{2k}$ by Lemma 2.3.

We now prove the bound in the theorem and show that it is attained only in the cases above. Since G is non-empty, $n \geq 2$. If $n = 2$, then G is a copy of K_2, so

$\lambda(G) = 1 = \frac{n+(k-1)t}{2k}$. We proceed by induction on n. Thus, consider $n \geq 3$. If $k = 1$, then G is the union of vertex-disjoint copies of K_2, so $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$. Consider $k \geq 2$. Let $v^* \in M(G)$. We have $n \geq |N[v^*]| = k + 1$.

Suppose that $M_2(G)$ has a member u. If $\Delta(G-u) < \Delta(G)$, then $\lambda(G) = 1 \leq \frac{n+(k-1)t}{2k}$ (as $n \geq k+1$). If $\lambda(G) = 1 = \frac{n+(k-1)t}{2k}$, then $V(G) = N[u]$, so G is a special k-star 1-union. Now suppose $\Delta(G-u) = \Delta(G)$. Then, since $u \in M_2(G)$, $M(G-u) = M(G)\backslash\{u\}$ and $v \notin N_{G-u}[M(G-u)]$ for each $v \in N(u)$. Thus, $M(G-N[u]) = M(G-u)$, $\Delta(G-N[u]) = \Delta(G-u) = k$, and $\lambda(G-N[u]) = \lambda(G-u)$ by repeated application of Proposition 2.1. Let $G' = G - N[u]$, $n' = |V(G')| = n - k - 1$, and $t' = |M(G')| = |M(G-u)| = t - 1$. By Proposition 2.2 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G-u) = 1 + \lambda(G') \leq 1 + \frac{n' + (k-1)t'}{2k} = \frac{n + (k-1)t}{2k}.$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then $\lambda(G') = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, G' is a special k-star $(t-1)$-union or each component of G' is a copy of P_3 or C_4. Suppose that each component of G' is a copy of P_3 or C_4. Then $k = 2$. Let u_1 and u_2 be the two members of $N(u)$. Since $u \in M_2(G)$, we have $d(u_1) = d(u_2) = 1$, so $N(u_1) = N(u_2) = \{u\}$. Thus, $G[N[u]]$ is a copy of P_3 and a component of G. Therefore, each component of G is a copy of P_3 or C_4. Now suppose that G' is a special k-star $(t-1)$-union with constituents $S_1, \ldots, S_{t-1}$. Let S_t be the k-star $(N[u], E(u))$. Then $S_1, \ldots, S_t$ are vertex-disjoint, $V(G) = V(G') \cup N[u] = \bigcup_{i=1}^{t} V(S_i)$, and $\bigcup_{i=1}^{t} E(S_i) \subseteq E(G)$. Thus, G is a special k-star t-union.

Now suppose $M_2(G) = \emptyset$. Then $M(G) = M_1(G)$.

Suppose that G has a vertex u such that $N[u]$ contains at least 3 vertices in $M(G)$. If $\Delta(G-u) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq k+1$, $t \geq 3$, and $k \geq 2$. Now suppose $\Delta(G-u) = \Delta(G)$. Let $n' = |V(G-u)| = n - 1$ and $t' = |M(G-u)| \leq t - 3$. By Proposition 2.2 and the induction hypothesis,

$$\begin{aligned}\lambda(G) &\leq 1 + \lambda(G-u) \leq 1 + \frac{n' + (k-1)t'}{2k} \\ &\leq 1 + \frac{(n-1) + (k-1)(t-3)}{2k} = \frac{n + (k-1)t - (k-2)}{2k} \leq \frac{n + (k-1)t}{2k}. \quad (1)\end{aligned}$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then, in (1), equality holds throughout. Thus, $k = 2$ (as $n + (k-1)t - (k-2) = n + (k-1)t$), $t' = t - 3$ (as $n' + (k-1)t' = (n-1) + (k-1)(t-3)$), and $\lambda(G-u) = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, $G - u$ is a special 2-star t'-union or each component of $G - u$ is a copy of P_3 or C_4. If $G - u$ is a special 2-star t'-union, then, by definition, the constituents of $G - u$ are the components of $G - u$ (because, since $k = 2$ and $|M(G-u)| = t'$, $d_{G-u}(z) = 1$ for each leaf z of any constituent), and they are copies of P_3. Therefore, in any case, each component of $G - u$ is a copy of P_3 or C_4. Let s_1 be the number of components of $G - u$ that are copies of P_3, and let s_2 be the number of components of $G - u$ that are copies of C_4. Let u_1 and u_2 be two distinct members of $N(u)$. Since $k = 2$ and $|N[u] \cap M(G)| \geq 3$, $N[u] = \{u, u_1, u_2\} = N[u] \cap M(G)$. Thus, $d(u) = d(u_1) = d(u_2) = \Delta(G) = 2$. For each $i \in [2]$, $d_{G-u}(u_i) = d_G(u_i) - 1 = 1$, so u_i is a leaf of a component H_i of $G - u$ that is a copy $(\{u_i, u_i', u_i''\}, \{u_i u_i', u_i' u_i''\})$ of P_3. Since $N(u) = \{u_1, u_2\}$ and $M_2(G) = \emptyset$, H_1 and H_2 are the only components of $G - u$ that are copies of P_3. Suppose $H_1 \neq H_2$. Then G has $s_2 + 1$ components, s_2 components of G

are copies of C_4, and 1 component of G is a copy of P_7. Thus, $n = 4s_2 + 7$, $t = 4s_2 + 5$, and clearly $\lambda(G) = 2s_2 + 2$. We have $\lambda(G) < 2s_2 + 3 = \frac{n+(k-1)t}{2k}$, a contradiction. Thus, $H_1 = H_2$, and hence each component of G is a copy of C_4.

Now suppose that

$$|N[v] \cap M(G)| \leq 2 \text{ for each } v \in V(G). \tag{2}$$

Suppose that, for each $v \in M(G)$, $N(v)$ contains no member of $M(G)$. Let $x \in M(G)$. Since $M(G) = M_1(G)$, there exists some $w \in N(x)\backslash M(G)$ such that $y \in N(w)$ for some $y \in M(G)\backslash N[x]$. Since $x, y \in N(w)$, $N(w) \cap M(G) = \{x, y\}$ by (2). If $\Delta(G-w) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G-w) = \Delta(G)$. Then $M(G-w) = M(G)\backslash\{x, y\}$. Let $G' = G - \{w, x, y\}$. Since $N(x) \cap M(G) = \emptyset$, $N(y) \cap M(G) = \emptyset$, and $N(w) \cap M(G) = \{x, y\}$, we have $M(G') = M(G)\backslash\{x, y\} = M(G-w)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G - \{w, x\}) = \lambda(G - w)$ by Proposition 2.1 (as $y \notin N_{G-\{w,x\}}[M(G - \{w, x\})]$ and $x \notin N_{G-w}[M(G - w)]$). Let $n' = |V(G')| = n - 3$ and $t' = |M(G')| = t - 2$. By Proposition 2.2 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G - w) = 1 + \lambda(G') \leq 1 + \frac{n' + (k-1)t'}{2k} < \frac{n + (k-1)t}{2k}.$$

Finally, suppose that G has a vertex u in $M(G)$ such that $N(u)$ contains a member w of $M(G)$. By (2), $N[u] \cap M(G) = \{u, w\} = N[w] \cap M(G)$. If $\Delta(G - u) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G - u) = \Delta(G)$. Then $M(G - u) = M(G)\backslash\{u, w\}$. Let $G' = G-\{u, w\}$. Since $N[u]\cap M(G) = \{u, w\} = N[w]\cap M(G)$, we have $M(G') = M(G)\backslash\{u, w\} = M(G-u)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G-u)$ by Proposition 2.1 (as $w \notin N_{G-u}[M(G - u)]$). Let $n' = |V(G')| = n - 2$ and $t' = |M(G')| = t - 2$. By Proposition 2.2 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G - u) = 1 + \lambda(G') \leq 1 + \frac{n' + (k-1)t'}{2k} = \frac{n + (k-1)t}{2k}.$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then $\lambda(G') = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, G' is a special k-star $(t - 2)$-union or each component of G' is a copy of P_3 or C_4. Thus, $\delta(G') \geq 1$.

Suppose first that each component of G' is a copy of P_3 or C_4. Then $\Delta(G') = 2$. Since $\Delta(G) = \Delta(G')$, $d(u) = d(w) = 2$. Thus, $N(u) = \{u', w\}$ for some $u' \in V(G)\backslash\{u, w\} = V(G')$. Since $N[u] \cap M(G) = \{u, w\}$ and $k = 2$, we have $d(u') < 2$, so $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$.

Now suppose that G' is a special k-star $(t - 2)$-union. Let $S_1, \ldots, S_{t-2}$ be the constituents of G'. Let $X = N(u)\backslash\{w\}$ and $Y = N(w)\backslash\{u\}$. Then $|X| = |Y| = k - 1$ and $d_{G'}(v) < k$ for each $v \in X \cup Y$. For each $i \in [t - 2]$, S_i has a vertex v_i such that $d_{S_i}(v_i) = k$. Since $\Delta(G) = k$, $d(v_i) = d_{S_i}(v_i) = k$ for each $i \in [t - 2]$. Note that

$$X \cup Y \subseteq V(G')\backslash\{v_1, \ldots, v_{t'}\} = V(G')\backslash M(G') = \bigcup_{i=1}^{t-2} N(v_i). \tag{3}$$

Suppose $X \cap Y \neq \emptyset$. Let $x \in X \cap Y$. We have $x \in N(v_p)$ for some $p \in [t - 2]$. Thus, we have $u, w, v_p \in N[x]\cap M(G)$, contradicting (2). Therefore, $X \cap Y = \emptyset$. Recall that we are considering $k \geq 2$. Since $|X| = |Y| = k-1$, $X \neq \emptyset \neq Y$. Let $x^* \in X$. By (3), $x^* \in N(v_p)$

for some $p \in [t-2]$. Consider any $y \in Y$. By (3), $y \in N(v_q)$ for some $q \in [t-2]$. Suppose $q \neq p$. Then $(\{v_1, \dots, v_{t-2}\} \backslash \{v_p, v_q\}) \cup \{x^*, y\}$ is a Δ-reducing set of G of size $t-2$. We have

$$\begin{aligned} t-2 \geq \lambda(G) &= \frac{n+(k-1)t}{2k} = \frac{|\{u,w\} \cup \bigcup_{i=1}^{t-2} V(S_i)| + (k-1)t}{2k} \\ &= \frac{(2+(k+1)(t-2)) + (k-1)t}{2k} = t-1, \end{aligned}$$

a contradiction. Thus, $Y \subseteq N(v_p)$. Let $y^* \in Y$. Then $y^* \in N(v_p)$. By an argument similar to that for x^*, $X \subseteq N(v_p)$. Since $X \cap Y = \emptyset$, we have $2(k-1) = |X \cup Y| \leq |N(v_p)| = k$, so $k \leq 2$. Since $k \geq 2$, $k = 2$. Thus, since $N[u] \cap M(G) = \{u, w\}$, $N(u) = \{w, u'\}$ for some $u' \in V(G) \backslash M(G)$. Since $d(u') < k = 2$, $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$. □

We now prove Theorem 1.3. We make use of the following two well-known facts, which were reproduced in [3] for the proof of Theorem 1.2.

Lemma 2.4 *Let x be a vertex of a tree T. Let $m = \max\{d(x,y) \colon y \in V(T)\}$, and let $D_i = \{y \in V(T) \colon d(x,y) = i\}$ for each $i \in \{0\} \cup [m]$. For each $i \in [m]$ and each $v \in D_i$, $N(v) \cap \bigcup_{j=0}^{i} D_j = \{u\}$ for some $u \in D_{i-1}$.*

Lemma 2.5 *If T is a tree, $x, z \in V(T)$, and $d(x,z) = \max\{d(x,y) \colon y \in V(T)\}$, then z is a leaf of T.*

Proof of Theorem 1.3. By Lemma 2.3, $\lambda(T) = \frac{n}{k+1}$ if T is k-special. We now prove the converse. This is trivial if $n \leq 2$. We proceed by induction on n. Suppose $n \geq 3$ and $\lambda(T) = \frac{n}{k+1}$. Since T is a connected graph, we clearly have $k \geq 2$.

Suppose that T has a leaf z whose neighbour is not in $M(T)$. Then $M(T-z) = M(T)$ and, by Proposition 2.1, $\lambda(T-z) = \lambda(T)$. By Theorem 1.2, $\lambda(T-z) \leq \frac{n-1}{k+1} < \frac{n}{k+1}$. Thus, we have $\lambda(T) < \frac{n}{k+1}$, a contradiction.

Therefore, each leaf of T is adjacent to a vertex in $M(T)$. Let x, m, and $D_0, D_1, \dots, D_m$ be as in Lemma 2.4. Let $z \in V(T)$ such that $d(x,z) = m$. By Lemma 2.5, z is a leaf of T. Let w be the neighbour of z. Then $w \in M(T)$. By Lemma 2.4, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $E(T) = \{xz_1, \dots, xz_k\}$ for some distinct vertices $z_1, \dots, z_k$ of T. Thus, T is a k-star and hence k-special.

Now suppose $w \neq x$. Together with Lemma 2.4, this implies that $N(w) = \{v, z_1, \dots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \dots, z_{k-1}$ in D_m. By Lemma 2.5, $z_1, \dots, z_{k-1}$ are leaves of T. Let $T' = T - v$. Then each component of T' is a tree. Let $\mathcal{K}$ be the set of components of T' whose maximum degree is k, and let $\mathcal{H}$ be the set of components of T' whose maximum degree is less than k. Let $W = \{w, z_1, \dots, z_{k-1}\}$. Note that $(W, \{wz_1, \dots, wz_{k-1}\}) \in \mathcal{H}$, and hence $W \cap \bigcup_{C \in \mathcal{K}} V(C) = \emptyset$. Let S_0 be the k-star $(W \cup \{v\}, \{wv, wz_1, \dots, wz_{k-1}\})$.

Suppose $\mathcal{K} = \emptyset$. Then $\{v\}$ is a Δ-reducing set of T, and hence $\lambda(T) = 1$. Since $\lambda(T) = \frac{n}{k+1}$, we have $n = k+1$, so $T = S_0$. Thus, T is k-special.

Now suppose $\mathcal{K} \neq \emptyset$. Let $T_1, \dots, T_r$ be the distinct members of $\mathcal{K}$. For each $i \in [r]$, let R_i be a Δ-reducing set of T_i of size $\lambda(T_i)$. By Theorem 1.2, $|R_i| \leq \frac{|V(T_i)|}{k+1}$ for each $i \in [r]$. Now $\{v\} \cup \bigcup_{i=1}^{r} R_i$ is a Δ-reducing set of T. Thus, we have

$$\lambda(T) \leq 1 + \sum_{i=1}^{r} |R_i| \leq \frac{|V(S_0)|}{k+1} + \sum_{i=1}^{r} \frac{|V(T_i)|}{k+1} \leq \frac{n}{k+1}.$$

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Since $\lambda(T) = \frac{n}{k+1}$, it follows that $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} V(T_i)$ and $\lambda(T_i) = \frac{|V(T_i)|}{k+1}$ for each $i \in [r]$. By the induction hypothesis, for each $i \in [r]$, T_i is k-special, so there exist vertex-disjoint k-stars $S_{i,1}, \dots, S_{i,t_i}$ such that $V(T_i) = \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $\bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T_i)$. Therefore, we have $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $E(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T)$. Since $S_0, T_1, \dots, T_r$ are vertex-disjoint, $S_0, S_{1,1}, \dots, S_{1,t_1}, \dots, S_{r,1}, \dots, S_{r,t_r}$ are vertex-disjoint. Since $\Delta(T) = k$, T is k-special. $\square$

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