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On the Planarity of Generalized Line Graphs

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Abstract

One of the most familiar derived graphs is the line graph. The line graph $L(G)$ of a graph G is that graph whose vertices are the edges of G where two vertices of $L(G)$ are adjacent if the corresponding edges are adjacent in G. Two nontrivial paths P and Q in a graph G are said to be adjacent paths in G if P and Q have exactly one vertex in common and this vertex is an end-vertex of both P and Q. For an integer $\ell \geq 2$, the ℓ-line graph $L_\ell(G)$ of a graph G is the graph whose vertex set is the set of all ℓ-paths (paths of order ℓ) of G where two vertices of $L_\ell(G)$ are adjacent if they are adjacent ℓ-paths in G. Since the 2-line graph is the line graph $L(G)$ for every graph G, this is a generalization of line graphs. In this work, we study planar and outerplanar properties of the 3-line graph of connected graphs and present characterizations of those trees having a planar or outerplanar 3-line graph by means of forbidden subtrees.

Key Words: line graph, ℓ-line graph, planar and outerplanar graphs.

AMS Subject Classification: 05C10, 05C75.

1 Introduction

There are many graphs associated with a given graph. We refer to each of these graphs as a "derived graph". For a given graph G, a *derived graph* of G is a graph obtained from G by a graph operation of some type. The study of the structural properties of derived graphs is a popular area of research in graph theory. One of the most familiar graph operations on a graph is that of the line graph. The *line graph* $L(G)$ of a graph G is that graph whose vertices are the edges of G such that two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent.

The concept of line graphs was implicitly introduced in 1932 by Whitney [16]. Prisner described in [14] what makes line graphs a worthwhile concept to study and how the line graph provides another way of analyzing the structure of a graph.

> *Large parts of graph theory could be formulated in terms of line graphs. For instance matching theory deals with independent vertex sets in the line graph, edge coloring is vertex coloring of the corresponding line graph and the edge reconstruction conjecture is the vertex reconstruction conjecture restricted to line graphs.*

As Edelsbrunner [10] expressed it (in the context of computational geometry):

> *There is some magic to ... transformations which has to do with the way human beings understand ... problems. It is fairly obvious that the transformation of a problem to another problem cannot lead to anything new, in particular if the transformation realizes a one-to-one correspondence. Still, there is an impressive large collection of ... problems where the transformations into other problems plays a crucial role in their solutions. One explanation of this empirical fact is*

that the transformation of a problem shifts the emphasis to different aspects of the problem which helps the human investigation to study the problem from a new angle, so to speak.

Since 1932, the study of line graphs has been a classical topic of research in graph theory. For the set $\mathcal{G}$ of all connected graphs and the set $\mathcal{G}'$ of all nonempty connected graphs, we may think of L as a function, namely $L : \mathcal{G}' \to \mathcal{G}$. Whitney [16] showed that this function is nearly injective. A graph G is called *a line graph* if there exists a graph H such that $G = L(H)$. The best known characterization of line graphs is probably a *forbidden subgraph* characterization due to Beineke [5]. A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph*. A characterization of graphs whose line graph is Hamiltonian is due to Harary and Nash-Williams [11] by means of a dominating circuit in the graph. For a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the *kth iterated line graph* $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. Iterated line graphs of almost all connected graphs were shown to be Hamiltonian by Chartrand [6].

A graph G is *planar* if it can be drawn in the plane without any two of its edges crossing. Such a drawing is also called a *planar embedding* of G. Sedlaček [15] characterized those graphs G for which $L(G)$ is planar.

Theorem 1.1 [15] *A nonempty graph G has a planar line graph if and only if (i) G is planar, (ii) $\Delta(G) \leq 4$ and (iii) if $\deg_G v = 4$, then v is a cut-vertex of G.*

A graph G is *outerplanar* if there exists a planar embedding of G so that every vertex of G lies on the boundary of the exterior region. Outerplanar graphs were first studied and named by Chartrand and Harary [8]. Chartrand, Geller and Hedetniemi [7] presented a characterization of graphs whose line graph is outerplanar.

Theorem 1.2 *A nonempty graph G has an outerplanar line graph if and only if $\Delta(G) \leq 3$ and every vertex of degree 3 of G is a cut-vertex.*

Over the years, various generalizations of line graphs have been introduced and studied by many (see [4, 13, 14], for example). Another more general class of derived graphs was inspired by line graphs. Observe that an edge in a graph G can be considered as a subgraph P_2 or a subgraph K_2 of G. This suggests that line graphs can be looked at in a number of ways. For example, we could replace an edge by a path, a cycle, a complete graph or some other prescribed graph. Furthermore, we can think of an edge as the edge set of the path P_2 or of the complete graph K_2 and define adjacency of vertices in the resulting graph in terms of a prescribed property involving sets. These observations give rise to new ways of constructing derived graphs. In particular, a more general concept of derived graphs was introduced by Chartrand in 2016, which is the topic of this work.

Let G be a connected graph of order at least 3. Two nontrivial paths P and Q in G are said to be *adjacent* paths in G if $V(P) \cap V(Q) = \{x\}$ where x is an end-vertex of both P and Q. For an integer $\ell \geq 2$, the *ℓ-line graph* $L_\ell(G)$ of a graph G is that graph whose vertex set is the set of all ℓ-paths (paths of order ℓ) of G where two vertices of $L_\ell(G)$ are adjacent if they are adjacent ℓ-paths in G. In particular, the standard line graph $L(G)$ of a graph G is the

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2-line graph $L_2(G)$ of G. By definition, the vertex set of the 3-line graph $L_3(G)$ of a graph G is the set of 3-paths of G where two vertices of $L_3(G)$ (two 3-paths of G) are adjacent if they have an end-vertex and only an end-vertex in common. The 3-line graphs have been studied extensively in [1, 3], where results have been obtained on the connectedness and Hamiltonicity of graphs. Furthermore, the concept of 3-line graph is related to another concept of generalized line graph called *Schwenk graph*, which was introduced by Schwenk in 2016 and studied in [2].

In this work, we study other important structural properties of graphs, namely planarity and outerplanarity. More precisely, we present characterizations of those trees having a planar or outerplanar 3-line graph by means of forbidden subtrees. We refer to the book [9] for graph theory notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

2 Trees Having a Planar 3-Line Graph

In this section, we determine all of those trees having planar 3-line graphs. First, we state a well-known characterization of planar graphs. A graph H is a *subdivision* of a graph G if H can be obtained from G by inserting vertices of degree 2 into some, all or none of the edges of G. Clearly, a subdivision H of a graph G is planar if and only if G is planar. The graphs K_5 and $K_{3,3}$ and their subdivisions play a pivotal role in the study of planar graphs. Kuratowski [12] obtained the following characterization of planar graphs by means of these two forbidden subgraphs.

Theorem 2.1 [12] *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

The following observation is a consequence of Theorem 2.1 and will be useful to us.

Observation 2.2 *If G is a planar graph, then every subgraph of G is also planar.*

We now determine all of those trees having planar 3-line graphs by means of the following forbidden subtree characterization.

Theorem 2.3 *A tree T of order at least 3 has a planar 3-line graph if and only if T does not contain any of the trees of Figure 1 as a subtree.*

Proof. The 3-line graph $L_3(T_1) = K_5 + 10K_1$ (the union of K_5 and $10K_1$) consists of K_5 and 10 isolated vertices and so $L_3(T_1)$ is nonplanar. For $2 \le i \le 7$, the 3-line graph $L_3(T_i)$ contains a subdivision of $K_{3,3}$ and so $L_3(T_i)$ is nonplanar by Theorem 2.1.

Next, we show that if T is any tree of order at least 3 that does not contain any of the trees T_i, $1 \le i \le 7$, as a subtree, then $L_3(T)$ is planar. We proceed by induction on the order of a tree. The result is clearly true for all trees of order 3, 4 or 5. Assume for an integer $k \ge 5$ that if T' is a tree of order k containing no tree T_i, $1 \le i \le 7$, as a subtree, then $L_3(T')$ is planar. Let T be a tree of order $k+1$ containing no tree T_i, $1 \le i \le 7$, as a subtree. We show that $L_3(T)$ is planar.

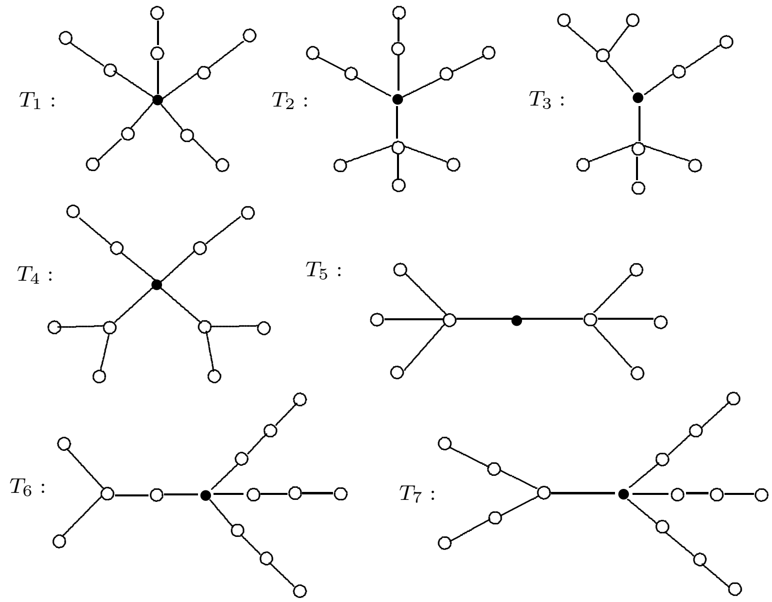

Figure 1: Seven trees T_i for $1 \leq i \leq 7$ for which $L_3(T_i)$ is not planar

Let v be an end-vertex of T and express T as a tree rooted at v and let $T' = T - v$. Let u be the child of v and suppose that $u_1, u_2, \ldots, u_d$ are the children of u. Thus, $\deg_T u = d+1$. Since T does not contain T_1 as a subgraph, at most four children of u are not end-vertices of T. We consider two cases, according to whether exactly four children of u are not end-vertices of T or at most three children of u are not end-vertices of T.

Case 1. *Exactly four children of u are not end-vertices of T.* We may assume that $\deg_T u_1 \geq \deg_T u_2 \geq \deg_T u_3 \geq \deg_T u_4 \geq 2$ and so $\deg_T u_i = 1$ for $5 \leq i \leq d$. The set $S = \{(v, u, u_i) : 1 \leq i \leq d\}$ consists of all 3-paths of T that do not belong to T'. Hence, $S = V(L_3(T)) - V(L_3(T'))$ consists of all 3-paths of T having the interior vertex u. Furthermore, S is a set of independent vertices of $L_3(T)$. In fact, for $i = 5, 6, \ldots, d$, each 3-path (v, u, u_i) is an isolated vertex in $L_3(T)$.

- ⋆ If u_1 has three or more children, then T contains T_2 as a subtree, a contradiction. Thus, u_1 has at most two children. Again, we may assume that u_1 has exactly two children w_1 and w_2. (If u_1 has only one child, then the 3-line graph is the subgraph of the 3-line graph when u_1 has exactly two children.)

- ⋆ If u_2 has two children, then T contains T_4 as a subtree, a contradiction. Thus, u_2 has only one child and so u_3 and u_4 as well have only one child. Let x, y, z be the child of u_2, u_3, u_4, respectively.

- ⋆ If w_1 has three or more children, then T contains T_5 as a subtree, a contradiction. Similarly, if w_2 has three or more children, then T contains T_5 as a subtree, a contradiction. Thus, each of w_1 and w_2 has at most two children. We may assume, without loss of generality, that $\deg_T w_1 \geq \deg_T w_2$. First, suppose that w_1 has exactly two children. If w_2 has one or more children, then T contains T_3 as a subtree, a contradiction. Hence, w_2 must be an end-vertex. Next, suppose that w_1 has exactly one child. Then w_2 has

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at most one child. In this case, we may assume that each of w_1 and w_2 has exactly one child. Thus, either

(i) w_1 has exactly two children and w_2 has no children or

(ii) each of w_1 and w_2 has exactly one child.

⋆ If one of x, y or z has three or more children, then T contains T_5 as a subtree, which is a contradiction. If each of x, y, z has a child and (i) occurs, then T contains T_6 as a subtree, a contradiction. If each of x, y, z has a child and (ii) occurs, then T contains T_7 as a subtree, a contradiction. Thus, we may assume, without loss of generality, that each of x and y has at most two children and z is an end-vertex. Furthermore, we may assume that each of x and y has exactly two children. Let x_1 and x_2 be the children of x and y_1 and y_2 the children of y.

Thus, either T contains a subtree as shown in Figure 2(i) or T contains a subtree as shown in Figure 2(ii).

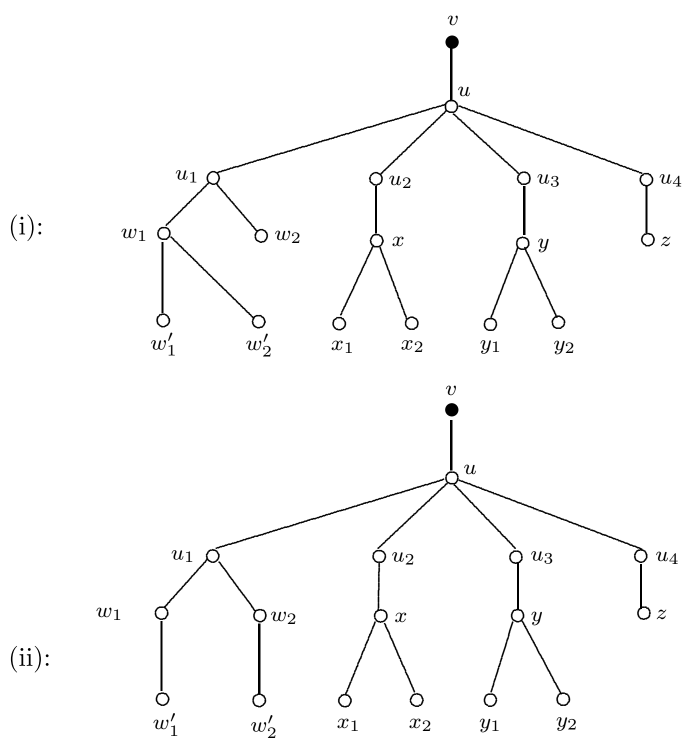

Figure 2: Two possible subtrees in the tree T

By the induction hypothesis, the tree $T' = T - v$ has a planar 3-line graph. Let there be given a planar embedding of $L_3(T')$. Next, we extend this planar embedding to a planar embedding of $L_3(T)$. There are four vertices in $L_3(T)$ that are not in $L_3(T')$, namely $P_i = (v, u, u_i)$, where $i = 1, 2, 3, 4$. Since $P_4 = (v, u, u_4)$ is an isolated vertex of $L_3(T)$, we need only determine how to insert the three vertices

$$P_i = (v, u, u_i), \text{ where } i = 1, 2, 3$$

into the planar embedding of $L_3(T')$ without edges crossing. Consider the following two observations.

(B1) Each 3-path P_i $(1 \leq i \leq 3)$ has degree 2 in $L_3(T)$.

(B2) Each of the 3-paths $(u_1, u, u_4), (u_2, u, u_4)$ and (u_3, u, u_4) has degree 2 in $L_3(T)$.

We consider two cases, according to whether T contains the subtree in Figure 2(i) or T contains the subtree in Figure 2(ii).

Case (*i*). *T contains the subtree in Figure* 2(*i*).

(1) For $P_1 = (v, u, u_1)$, let

$$Q_{1,1} = (u_1, w_1, w_1'),\ Q_{1,2} = (u_1, w_1, w_2'),$$
$$Z_{1,1} = (u_1, u, u_2),\ Z_{1,2} = (u_1, u, u_3) \text{ and } Z_{1,3} = (u_1, u, u_4).$$

Let $U_1 = \{Q_{1,1}, Q_{1,2}\}$ and $W_1 = \{Z_{1,1}, Z_{1,2}, Z_{1,3}\}$. The subgraph H_1 of $L_3(T)$ induced by $U_1 \cup W_1$ is isomorphic to $K_{2,3}$ such that $Z_{1,3}$ has degree 2 in $L_3(T')$.

(2) For $P_2 = (v, u, u_2)$, let

$$Q_{2,1} = (u_2, x, x_1),\ Q_{2,2} = (u_2, x, x_2),$$
$$Z_{2,1} = (u_2, u, u_1),\ Z_{2,2} = (u_2, u, u_3) \text{ and } Z_{2,3} = (u_2, u, u_4).$$

Let $U_2 = \{Q_{2,1}, Q_{2,2}\}$ and $W_2 = \{Z_{2,1}, Z_{2,2}, Z_{2,3}\}$. The subgraph H_2 of $L_3(T)$ induced by $U_2 \cup W_2$ is isomorphic to $K_{2,3}$ such that $Z_{2,3}$ has degree 2 in $L_3(T')$.

(3) For $P_3 = (v, u, u_3)$, let

$$Q_{3,1} = (u_3, y, y_1),\ Q_{3,2} = (u_3, y, y_2),$$
$$Z_{3,1} = (u_3, u, u_1),\ Z_{3,2} = (u_3, u, u_2) \text{ and } Z_{3,3} = (u_3, u, u_4).$$

Let $U_3 = \{Q_{3,1}, Q_{3,2}\}$ and $W_3 = \{Z_{3,1}, Z_{3,2}, Z_{3,3}\}$. The subgraph H_3 of $L_3(T)$ induced by $U_3 \cup W_3$ is isomorphic to $K_{2,3}$ such that $Z_{3,3}$ has degree 2 in $L_3(T')$.

For each integer $i = 1, 2, 3$, each of the vertices $Z_{i,3} = (u_i, u, u_4)$ and $P_i = (v, u, u_i)$ has degree 2 (see B1 and B2). Furthermore each of the vertices $Z_{i,3} = (u_i, u, u_4)$ and $P_i = (v, u, u_i)$ are adjacent only to $Q_{i,1}$ and $Q_{i,2}$ in $L_3(T)$. Hence, we can appropriately place P_i near $Z_{i,3}$ and join P_i to $Q_{i,1}$ and $Q_{i,2}$, creating the two 4-cycles

$$C_4^1 = (Q_{i,1}, P_i, Q_{i,2}, Z_{i,3}, Q_{i,1}) \text{ and } C_4^2 = (Q_{i,1}, P_i, Q_{i,2}, Z_{i,2}, Q_{i,1})$$

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without edges crossing. This produces a planar embedding of $L_3(T)$. Therefore, $L_3(T)$ is planar.

Case (ii). T contains the subtree in Figure 2(ii). Since $P_1 = (v, u, u_1)$ is adjacent only to the two 3-paths $Q_{1,1} = (u_1, w_1, w_1')$ and $Q_{1,2} = (u_1, w_2, w_2')$ in $L_3(T)$ and $Q_{1,1}$ and $Q_{1,2}$ are adjacent in $L_3(T)$, we can place P_1 near the edge $Q_{1,1}Q_{1,2}$ and join P_1 to both $Q_{1,1}$ and $Q_{1,2}$ without edges crossing. For the 3-path P_i where $i = 2, 3$, we can proceed as in Case (i) to join P_i to both $Q_{i,1}$ and $Q_{i,2}$ without edges crossing. This produces a planar embedding of $L_3(T)$. Therefore, $L_3(T)$ is planar.

Case 2. At most three children of u are not end-vertices of T. Then T is a subgraph of a tree T^* (as described in Case 1) having exactly four children of u are not end-vertices of T^*. By Case 1, the 3-line graph of T^* is planar. Since the 3-line graph of T is the subgraph of the 3-line graph of T^*, it follows that $L_3(T)$ is planar by Observation 2.2. ■

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4 *If G is a connected graph that contains T_i of Figure 1 as a subtree for some integer i with $1 \leq i \leq 7$, then $L_3(G)$ is nonplanar.*

3 Trees Having an Outerplanar 3-Line Graph

In this section, we determine all of those trees having an outerplanar 3-line graph. First, we present a characterization of outerplanar graphs that is analogous to the characterization of planar graphs given in Theorem 2.1 (see [7]).

Theorem 3.1 *A graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

The following observation is a consequence of Theorem 3.1, which is analogous to Observation 2.2 and will be useful to us.

Observation 3.2 *If G is an outerplanar graph, then every subgraph of G is also outerplanar.*

We now determine all of those trees having an outerplanar 3-line graph by means of the following forbidden subtree characterization.

Theorem 3.3 *A tree T of order at least 3 has an outerplanar 3-line graph if and only if T does not contain any of the trees of Figure 3 as a subtree.*

Proof. First, it can be shown that if a tree T contains any tree T_i, $1 \leq i \leq 5$, as a subtree, then $L_3(T)$ contains either K_4 or $K_{2,3}$ as a subgraph and so $L_3(T)$ is not outerplanar by Theorem 3.1. Next, we show that if T is a tree of order at least 3 that does not contain any such tree T_i as a subtree, then $L_3(T)$ is outerplanar.

We proceed by induction on the order of a tree. The result is clearly true for all trees of order 3, 4 or 5. Assume for an integer $k \geq 5$ that if T' is a tree of order k containing no

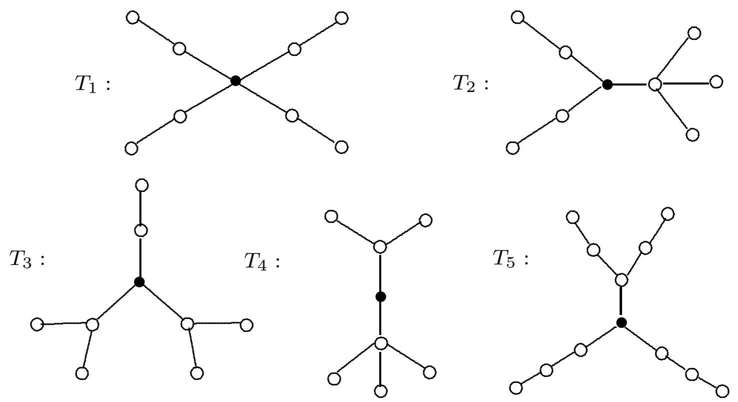

Figure 3: Six trees T_i for $1 \leq i \leq 6$ for which $L_3(T_i)$ is not outerplanar

tree T_i, $1 \leq i \leq 5$, as a subtree, then $L_3(T')$ is outerplanar. Let T be a tree of order $k+1$ containing no tree T_i, $1 \leq i \leq 5$, as a subtree. We show that $L_3(T)$ is outerplanar.

Let v be an end-vertex of T. Express T as a tree rooted at v and let u be the child of v. Suppose that $u_1, u_2, \ldots, u_d$ are the children of u. Since T does not contain T_1 as a subtree, at most three children of u are not end-vertices of T. We consider two cases, according to whether exactly three children of u are not end-vertices of T or at most two children of u are not end-vertices of T.

Case 1. *Exactly three children of u are not end-vertices of T.* We may assume that

$$\deg_T u_1 \geq \deg_T u_2 \geq \deg_T u_3 \geq 2$$

and so $\deg_T u_i = 1$ for $4 \leq i \leq d$. Thus, the set

$$\{(v, u, u_i) : 1 \leq i \leq d\} = V(L_3(T)) - V(L_3(T'))$$

of all 3-paths of T having interior vertex u is a set of independent vertices of $L_3(T)$ and each 3-path (v, u, u_i) is, in fact, an isolated vertex in $L_3(T)$ for $i = 4, 5, \ldots, d$.

- ⋆ If u_1 has three or more children, then T contains T_2 as a subtree, a contradiction. Thus, u_1 has at most two children. Again, we may assume that u_1 has exactly two children w_1 and w_2. (If u_1 has only one child, then the 3-line graph is the subgraph of the 3-line graph when u_1 has exactly two children.)

- ⋆ If u_2 has two or more children, then T contains T_3 as a subtree, a contradiction. Thus, u_2 has at most one child. Similarly, u_3 has at most one child. We may assume that each of u_2 and u_3 has exactly one child. Let x and y be the child of u_2 and u_3, respectively.

- ⋆ If w_1 or w_2 has two or more children, then T contains T_4 as a subtree, a contradiction. Thus, either w_1 or w_2 has at most one child. If each of w_1 and w_2 has exactly one child, then T contains T_2 as a subtree, a contradiction. Hence, at most one of w_1 and w_2 has at most one child. We may assume, without loss of generality, that w_1 has exactly one child w_1' and w_2 is an end-vertex of T.

⋆ If one of x and y has two or more children, then T contains T_4 as a subtree, a contradiction. Thus, each of x and y has at most one child. Again, we may assume that each of x and y has exactly one child. Let x_1 a be the child of x and y_1 the child of y.

Thus, T contains the subtree shown in Figure 4.

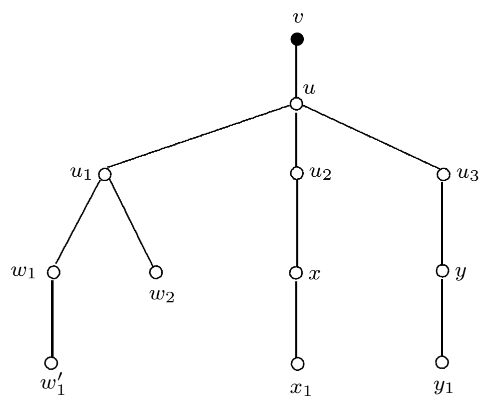

Figure 4: The subtree in T

By the induction hypothesis, the tree $T' = T - v$ has an outerplanar 3-line graph. Let there be given a planar embedding of $L_3(T')$ such that every vertex of $L_3(T')$ lies on the boundary of the exterior region R. Next, we extend this planar embedding to a planar embedding of $L_3(T)$ so that every vertex of $L_3(T)$ lies on the boundary of R. For convenience, let $P_i = (v, u, u_i)$ for $i = 1, 2, 3$, where all other 3-paths having interior vertex u (if any exists) are isolated vertices of $L_3(T)$. Then P_1, P_2, P_3 are three independent vertices of $L_3(T)$. Observe that P_1 is adjacent only to $Q_1 = (u_1, w_1, w_1')$, P_2 is adjacent only to $Q_2 = (u_2, x, x_1)$ and P_3 is adjacent only to $Q_3 = (u_3, y, y_1)$. We can place P_1, P_2, P_3 in the exterior region R and join P_i to Q_i for $i = 1, 2, 3$ to produce a planar embedding of $L_3(T)$ such that every vertex of $L_3(T)$ lies on the boundary of the exterior region R. Therefore, $L_3(T)$ is outerplanar.

Case 2. *At most two children of* u *are not end-vertices of* T. Then T is a subgraph of a tree T^* (as described in Case 1) having exactly three children of u are not end-vertices of T^*. By Case 1, the 3-line graph of T^* is outerplanar. Since the 3-line graph of T is the subgraph of the 3-line graph of T^*, it follows that $L_3(T)$ is outerplanar by Observation 3.2. ■

The following is an immediate consequence of Theorem 3.3.

Corollary 3.4 *If* G *is a connected graph that contains* T_i *of Figure* 3 *as a subgraph for some integer* i *with* $1 \leq i \leq 5$, *then* $L_3(G)$ *is not outerplanar.*

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