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A General Lower Bound on Gallai-Ramsey Numbers for Non-Bipartite Graphs

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Abstract

Given a graph H and a positive integer k, the k-color Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum number of vertices n for which any k-coloring of the complete graph K_n contains either a rainbow triangle or a monochromatic copy of H. The behavior of these numbers is rather well understood when H is bipartite but when H is not bipartite, this behavior is a bit more complicated. In this short note, we improve upon existing lower bounds for non-bipartite graphs H to a value that we conjecture to be sharp up to a constant multiple.

Keywords: Gallai-Ramsey numbers, non-bipartite graphs, rainbow triangle

1 Introduction

The structure of edge-colored complete graphs containing no rainbow triangle is well understood through the following fundamental result.

Theorem 1 ([1, 7, 10]). *In any colored complete graph containing no rainbow triangle, there exists a partition of the vertices (called a* Gallai partition*) such that there are at most two colors on the edges between the parts and only one color on edges between each pair of parts.*

In honor of this result, colored complete graphs with no rainbow triangle are called *Gallai colorings* (or G-colorings for short) and for simplicity, the Gallai partition is often called a G-partition. Given a G-coloring with a corresponding G-partition $\mathscr{P}$, the reduced graph $Q = Q(G, \mathscr{P})$ of this partition is constructed by arbitrarily removing all but one vertex from each part of the partition. By Theorem 1, the reduced graph is a 2-colored complete graph.

Given two graphs G and H, the Ramsey number $R(G, H)$ is the minimum number of vertices n such that any red-blue coloring of K_n contains either a red copy of G or a blue copy of H. Given a graph H and a positive integer k, the k-color Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum number of vertices n for which any k-coloring of K_n contains either a rainbow triangle or a monochromatic copy of H. Since every 2-colored complete graph clearly contains no rainbow triangle, we immediately get $gr_2(K_3 : H) = R(H, H)$.

The general behavior of the Gallai-Ramsey numbers, as a function of k, is given by the following result.

Theorem 2 ([9]). *Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k. If H is bipartite, then $gr_k(K_3 : H)$ is linear in k.*

For bipartite graphs, there is a lower bound that is conjectured to be sharp (see Conjecture 6). For this result, let $s(H)$ denote the order of the smaller part of the bipartite graph H.

Theorem 3 ([13]). *Given a positive integer $k \geq 2$ and a connected bipartite graph H with Ramsey number $R(H, H) = R$, we have*

$$gr_k(K_3 : H) \geq R + (s(H) - 1)(k - 2).$$

If H is a non-bipartite graph, by Theorem 2, we know $gr_k(K_3 : H)$ is an exponential function of k but the specifics of this are not yet known in general. The goal of this work is to determine the base of this exponential.

2 Lower bound on Gallai-Ramsey numbers

In this section, for any given non-bipartite graph H, we produce a lower bound on the Gallai-Ramsey number $gr_k(K_3 : H)$, the main result being Theorem 4. We begin with some discussion about colorings of large G-colored complete graphs containing no monochromatic copy of H and present some definitions.

Since, by Theorem 1, every G-coloring of a complete graph has a partition of the vertices that forms a blow-up of a 2-coloring, it is important to consider colorings that avoid a monochromatic copy of H while still forming a blow-up of a 2-coloring. A very natural approach would be to consider a blow up of the sharpness example for $R(H, H)$, an example of which displayed in Figure 1. Here we assume we have constructed a coloring G_{k-2} on some number of vertices using $k - 2$ colors, then make 5 copies of G_{k-2} and insert these copies into a blow-up of the sharpness example for $R(K_3, K_3)$ using two new colors, to produce a new graph G_k on k colors.

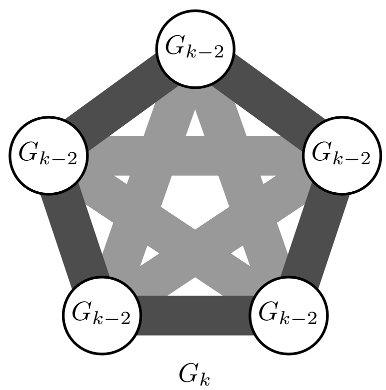

Figure 1: An example of this construction

Given a graph H, call a graph H' a *reduction* of H if H' can be obtained from H by identifying sets of non-adjacent vertices (and removing any resulting repeated edges). Let $\mathscr{H}$ be the set of all possible reductions of H. For the sake of the following main definition, let $R_2(\mathscr{H})$ be the minimum integer n such that every 2-coloring of K_n contains a monochromatic copy of some graph in the set $\mathscr{H}$. Since this quantity is bounded above by the Ramsey number $R(H, H)$, its existence is obvious. Now the main definition of this work.

Definition 1. *If $\mathscr{H}$ is the set of all reductions of a given graph H, define the function $m(H)$ to be*

$$m(H) = R_2(\mathscr{H}).$$

For example, if $H = K_n$, then the only reduction of H is H itself so $m(K_n) = R_2(\mathscr{H}) = R(K_n, K_n)$. As a slightly less trivial example, consider the complete graph minus one edge $H = K_n - e$, say with $e = uv$. Then the only nontrivial reduction of $K_n - e$ is K_{n-1} so $\mathscr{H} = \{K_{n-1}, K_n - e\}$. Since $K_{n-1} \subseteq K_n - e$, it is clear that

$$R_2(\{K_{n-1}, K_n - e\}) = R(K_{n-1}, K_{n-1})$$

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so $m(K_n - e) = R(K_{n-1}, K_{n-1})$.

First an easy general fact about the value of $m(H)$.

Fact 1. *For every graph H,*

$$m(H) \leq R(K_{\chi(H)}, K_{\chi(H)}).$$

Proof. Certainly $K_{\chi(H)}$ is a reduction of H, so $K_{\chi(H)} \in \mathscr{H}$. Then we get

$$m(H) = R_2(\mathscr{H}) \leq R(K_{\chi(H)}, K_{\chi(H)}),$$

as claimed. □

We now present our main result, a general lower bound on the Gallai-Ramsey number for any non-bipartite graph H.

Theorem 4. *For a connected non-bipartite graph H and an integer $k \geq 2$, we have that $gr_k(K_3 : H)$ is at least*

$$\begin{cases} (R(H,H)-1)\cdot(m(H)-1)^{(k-2)/2}+1 & \text{if } k \text{ is even,} \\ (\chi(H)-1)\cdot(R(H,H)-1)\cdot(m(H)-1)^{(k-3)/2}+1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. This result is proven by an inductive construction. For the base of the induction, let G_2 be a 2-colored complete graph on n_2 vertices, where $n_2 = R(H, H) - 1$, containing no monochromatic copy of H. Such a coloring exists by the definition of the Ramsey number.

We first consider the case when k is even. Suppose that $2i < k$ and there is a $2i$-coloring G_{2i} of $K_{n_{2i}}$ on

$$n_{2i} = (R(H,H)-1)\cdot(m(H)-1)^{(2i-2)/2}$$

vertices containing no rainbow triangle and no monochromatic copy of H.

Let D be a 2-coloring of $K_{m(H)-1}$ using colors $i+1$ and $i+2$ which contains no monochromatic copy of any graph in $\mathscr{H}$. Blow-up D by making n_{2i} copies of each vertex (also copying all edges with their colors) and inserting a copy of G_{2i} into each independent set, the set of copies of each vertex. See Figure 1 for an example of this construction. Since D contains no monochromatic copy of any graph in $\mathscr{H}$, this blow-up of D contains no monochromatic copy of H. This means that the resulting graph, G_{2i+2} is a $(2i+2)$-coloring of $K_{n_{2i+2}}$ on

$$\begin{aligned} n_{2i+2} &= (R(H,H)-1)\cdot(m(H)-1)^{(2i-2)/2}\cdot(m(H)-1) \\ &= (R(H,H)-1)\cdot(m(H)-1)^{(2i)/2} \end{aligned}$$

vertices containing no rainbow triangle and no monochromatic copy of H. By induction, this proves the desired result for even values of k.

Finally suppose k is odd. In this case, construct G_k by making $\chi(H) - 1$ copies of G_{k-1} (note that $k-1$ is even) and inserting all edges between the copies in color k. Any subgraph of the graph induced on the edges of color k has chromatic number at most $\chi(H) - 1$ so there is no copy of H in color k. This means that the resulting graph G_k is a k-coloring of K_{n_k} on

$$n_k = (\chi(H)-1)\cdot(R(H,H)-1)\cdot(m(H)-1)^{(2i-2)/2}$$

vertices containing no rainbow triangle and no monochromatic copy of H, completing the proof of Theorem 4. □

3 Finding $m(H)$

Given a graph H with a pair of nonadjacent vertices u and v, let H_{uv} be the reduction of H obtained from H by identifying u and v to a single vertex (removing any multiple copies of edges that were created in the process). A very natural question about a relationship between this reduction operation and Ramsey numbers was suggested by Graham, Rothschild, and Spencer.

Question 1 ([8])**.** *Is it true that*

$$R(H,H) \geq R(H_{uv}, H_{uv})?$$

In particular, if $c = \chi(H)$, then is $R(H,H) \geq R(K_c, K_c)$?

By Fact 1, if the answer to Question 1 was "yes", then for any graph H with chromatic number c, $m(H)$ would essentially equal $R(K_c, K_c)$. As observed above, the answer to this question is clearly yes for a complete graph and for a complete graph minus an edge, i.e. $H = K_n - e$, since $c = n - 1$ and $K_c \subseteq H$. Unfortunately, the answer to Question 1 is "no" in general since the wheel on 6 vertices provides a counterexample. Let $H = W_6$, the wheel $W_6 = C_5 + v$. Then $\chi(H) = 4$ so set $c = 4$.

Fact 2.

$$R(W_6, W_6) = 17 < 18 = R(K_4, K_4).$$

There are, however, many graphs that yield an affirmative answer to Question 1. Recall a graph H is called *perfect* if for every induced subgraph $H' \subseteq H$, the clique number $\omega(H')$ equals the chromatic number $\chi(H')$.

Proposition 1. *If H is a perfect graph with $\omega(H) = \chi(H) = c$, then $m(H) = R(K_c, K_c)$.*

Proof. Suppose H is a perfect graph with $\omega(H) = \chi(H) = c$. By Fact 1, we have $m(H) \leq R(K_c, K_c)$ so it suffices to prove that $m(H) \geq R(K_c, K_c)$. Since $K_c \subseteq H$, it is clear that $K_c \subseteq H'$ for every reduction $H' \in \mathscr{H}$. This means $R_2(\mathscr{H}) \geq R(K_c, K_c)$ so $m(H) = R(K_c, K_c)$, as desired. □

Proposition 1 immediately determines $m(H)$ for several classes of graphs.

Corollary 5. *The following hold:*

- *If B_n is the book $B_n = K_2 + \overline{K_n}$, then $m(B_n) = 6$.*
- *If F_n is the fan $F_n = (nK_2) + \{v\}$, then $m(F_n) = 6$.*
- *If K_n^- is the complete graph minus an edge, then $m(K_n^-) = R(K_{n-1}, K_{n-1})$.*
- *If H is a complete multipartite graph with chromatic number $c = \chi(G)$, then $m(H) = R(K_c, K_c)$.*

On the other hand, there are certainly non-perfect graphs for which the conclusion of Proposition 1 is false. Consider the cycle C_5 for example. Up to isomorphism, the only proper reductions of C_5 are K_3^+ (the triangle with the addition of a pendant edge), and the triangle. Although $R(K_3, K_3) = R(C_5, C_5) = 6$, the unique sharpness example for $R(K_3, K_3)$ on 5 vertices contains monochromatic copies of C_5. This means that

$$m(C_5) = R_2(\{C_5, K_3^+, K_3\}) \leq 5 < 6 = R(K_3, K_3) = R(K_{\chi(H)}, K_{\chi(H)}).$$

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4 Conclusion

In light of Theorem 4, a natural question is whether or not this bound might be sharp in some sense.

Conjecture 1. *For a connected non-bipartite graph H and an integer $k \geq 3$, there exist constants c_1 and c_2 such that*

$$gr_k(K_3 : H) = \begin{cases} c_1 \cdot (m(H)-1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ c_2 \cdot (m(H)-1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

For all known sharp Gallai-Ramsey numbers of non-bipartite graphs, the answer to this question is "yes".

In particular, Conjecture 1 is a generalization of the following recent conjecture about the complete graphs.

Conjecture 2 ([3])**.** *For $k \geq 1$ and $p \geq 3$,*

$$gr_k(K_3 : K_p) = \begin{cases} (R(K_p, K_p)-1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (p-1)(R(K_p, K_p)-1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Other related results can be found in [2, 4, 11, 12]. We refer the interested reader to [5] for a survey of Gallai-Ramsey numbers, with a dynamic version available at [6]. We also state the corresponding conjecture for bipartite graphs.

Theorem 6 ([13])**.** *Given a positive integer $k \geq 2$ and a connected bipartite graph H with Ramsey number $R(H, H) = R$, we have*

$$gr_k(K_3 : H) = R + (s(H) - 1)(k - 2).$$

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