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A General Lower Bound on Gallai-Ramsey Numbers for Non-Bipartite Graphs

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Abstract

Given a graph $H$ and a positive integer $k$, the $k$-color Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum number of vertices $n$ for which any $k$-coloring of the complete graph $K_n$ contains either a rainbow triangle or a monochromatic copy of $H$. The behavior of these numbers is rather well understood when $H$ is bipartite but when $H$ is not bipartite, this behavior is a bit more complicated. In this short note, we improve upon existing lower bounds for non-bipartite graphs $H$ to a value that we conjecture to be sharp up to a constant multiple.

Keywords: Gallai-Ramsey numbers, non-bipartite graphs, rainbow triangle

1 Introduction

The structure of edge-colored complete graphs containing no rainbow triangle is well understood through the following fundamental result.

Theorem 1 ([1, 7, 10]). In any colored complete graph containing no rainbow triangle, there exists a partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on edges between each pair of parts.

In honor of this result, colored complete graphs with no rainbow triangle are called Gallai colorings (or G-colorings for short) and for simplicity, the Gallai partition is often called a G-partition. Given a G-coloring with a corresponding G-partition $\mathcal{P}$, the reduced graph $Q = Q(G, \mathcal{P})$ of this partition is constructed by arbitrarily removing all but one vertex from each part of the partition. By Theorem 1, the reduced graph is a 2-colored complete graph.

Given two graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the minimum number of vertices $n$ such that any red-blue coloring of $K_n$ contains either a red copy of $G$ or a blue copy of $H$. Given a graph $H$ and a positive integer $k$, the $k$-color Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum number of vertices $n$ for which any $k$-coloring of $K_n$ contains either a rainbow triangle or a monochromatic copy of $H$. Since every 2-colored complete graph clearly contains no rainbow triangle, we immediately get $gr_2(K_3 : H) = R(H, H)$.

The general behavior of the Gallai-Ramsey numbers, as a function of $k$, is given by the following result.

Theorem 2 ([9]). Let $H$ be a fixed graph with no isolated vertices. If $H$ is not bipartite, then $gr_k(K_3 : H)$ is exponential in $k$. If $H$ is bipartite, then $gr_k(K_3 : H)$ is linear in $k$.

For bipartite graphs, there is a lower bound that is conjectured to be sharp (see Conjecture 6). For this result, let $s(H)$ denote the order of the smaller part of the bipartite graph $H$.

Theorem 3 ([13]). Given a positive integer $k \geq 2$ and a connected bipartite graph $H$ with Ramsey number $R(H, H) = R$, we have

$$gr_k(K_3 : H) \geq R + (s(H) - 1)(k - 2).$$

If $H$ is a non-bipartite graph, by Theorem 2, we know $gr_k(K_3 : H)$ is an exponential function of $k$ but the specifics of this are not yet known in general. The goal of this work is to determine the base of this exponential.
2 Lower bound on Gallai-Ramsey numbers

In this section, for any given non-bipartite graph \( H \), we produce a lower bound on the Gallai-Ramsey number \( gr_k(K_3 : H) \), the main result being Theorem 4. We begin with some discussion about colorings of large \( G \)-colored complete graphs containing no monochromatic copy of \( H \) and present some definitions.

Since, by Theorem 1, every \( G \)-coloring of a complete graph has a partition of the vertices that forms a blow-up of a 2-coloring, it is important to consider colorings that avoid a monochromatic copy of \( H \) while still forming a blow-up of a 2-coloring. A very natural approach would be to consider a blow up of the sharpness example for \( R(H, H) \), an example of which displayed in Figure 1. Here we assume we have constructed a coloring \( G_{k-2} \) on some number of vertices using \( k-2 \) colors, then make 5 copies of \( G_{k-2} \) and insert these copies into a blow-up of the sharpness example for \( R(K_3, K_3) \) using two new colors, to produce a new graph \( G_k \) on \( k \) colors.

![Figure 1: An example of this construction](image)

Given a graph \( H \), call a graph \( H' \) a reduction of \( H \) if \( H' \) can be obtained from \( H \) by identifying sets of non-adjacent vertices (and removing any resulting repeated edges). Let \( \mathcal{H} \) be the set of all possible reductions of \( H \). For the sake of the following main definition, let \( R_2(\mathcal{H}) \) be the minimum integer \( n \) such that every 2-coloring of \( K_n \) contains a monochromatic copy of some graph in the set \( \mathcal{H} \). Since this quantity is bounded above by the Ramsey number \( R(H, H) \), its existence is obvious. Now the main definition of this work.

**Definition 1.** If \( \mathcal{H} \) is the set of all reductions of a given graph \( H \), define the function \( m(H) \) to be

\[
m(H) = R_2(\mathcal{H}).
\]

For example, if \( H = K_n \), then the only reduction of \( H \) is \( H \) itself so \( m(K_n) = R_2(\mathcal{H}) = R(K_n, K_n) \). As a slightly less trivial example, consider the complete graph minus one edge \( H = K_n - e \), say with \( e = uv \). Then the only nontrivial reduction of \( K_n - e \) is \( K_{n-1} \) so \( \mathcal{H} = \{K_{n-1}, K_n - e\} \). Since \( K_{n-1} \subseteq K_n - e \), it is clear that

\[
R_2(K_{n-1}, K_n - e) = R(K_{n-1}, K_{n-1})
\]
so \( m(K_n - e) = R(K_{n-1}, K_{n-1}) \).

First an easy general fact about the value of \( m(H) \).

**Fact 1.** For every graph \( H \),

\[
m(H) \leq R(K_{\chi(H)}, K_{\chi(H)}).\]

**Proof.** Certainly \( K_{\chi(H)} \) is a reduction of \( H \), so \( K_{\chi(H)} \in \mathcal{H} \). Then we get

\[
m(H) = R_2(\mathcal{H}) \leq R(K_{\chi(H)}, K_{\chi(H)}),\]

as claimed. \( \square \)

We now present our main result, a general lower bound on the Gallai-Ramsey number for any non-bipartite graph \( H \).

**Theorem 4.** For a connected non-bipartite graph \( H \) and an integer \( k \geq 2 \), we have that \( \text{gr}_k(K_3 : H) \) is at least

\[
\begin{cases} 
(R(H, H) - 1) \cdot (m(H) - 1)^{(k-2)/2} + 1 & \text{if } k \text{ is even}, \\
(\chi(H) - 1) \cdot (R(H, H) - 1) \cdot (m(H) - 1)^{(k-3)/2} + 1 & \text{if } k \text{ is odd}.
\end{cases}
\]

**Proof.** This result is proven by an inductive construction. For the base of the induction, let \( G_2 \) be a 2-colored complete graph on \( n_2 \) vertices, where \( n_2 = R(H, H) - 1 \), containing no monochromatic copy of \( H \). Such a coloring exists by the definition of the Ramsey number.

We first consider the case when \( k \) is even. Suppose that \( 2i < k \) and there is a \( 2i \)-coloring \( G_{2i} \) of \( K_{n_{2i}} \) on

\[
n_{2i} = (R(H, H) - 1) \cdot (m(H) - 1)^{(2i-2)/2}
\]

vertices containing no rainbow triangle and no monochromatic copy of \( H \).

Let \( D \) be a 2-coloring of \( K_{m(H)-1} \) using colors \( i+1 \) and \( i+2 \) which contains no monochromatic copy of any graph in \( \mathcal{H} \). Blow-up \( D \) by making \( n_{2i} \) copies of each vertex (also copying all edges with their colors) and inserting a copy of \( G_{2i} \) into each independent set, the set of copies of each vertex. See Figure 1 for an example of this construction. Since \( D \) contains no monochromatic copy of any graph in \( \mathcal{H} \), this blow-up of \( D \) contains no monochromatic copy of \( H \). This means that the resulting graph, \( G_{2i+2} \) is a \( (2i + 2) \)-coloring of \( K_{n_{2i+2}} \) on

\[
n_{2i+2} = (R(H, H) - 1) \cdot (m(H) - 1)^{(2i-2)/2} \cdot (m(H) - 1)
\]

vertices containing no rainbow triangle and no monochromatic copy of \( H \). By induction, this proves the desired result for even values of \( k \).

Finally suppose \( k \) is odd. In this case, construct \( G_k \) by making \( \chi(H) - 1 \) copies of \( G_{k-1} \) (note that \( k - 1 \) is even) and inserting all edges between the copies in color \( k \). Any subgraph of the graph induced on the edges of color \( k \) has chromatic number at most \( \chi(H) - 1 \) so there is no copy of \( H \) in color \( k \). This means that the resulting graph \( G_k \) is a \( k \)-coloring of \( K_{n_k} \) on

\[
n_k = (\chi(H) - 1) \cdot (R(H, H) - 1) \cdot (m(H) - 1)^{(2i-2)/2}
\]

vertices containing no rainbow triangle and no monochromatic copy of \( H \), completing the proof of Theorem 4. \( \square \)
3 Finding $m(H)$

Given a graph $H$ with a pair of nonadjacent vertices $u$ and $v$, let $H_{uv}$ be the reduction of $H$ obtained from $H$ by identifying $u$ and $v$ to a single vertex (removing any multiple copies of edges that were created in the process). A very natural question about a relationship between this reduction operation and Ramsey numbers was suggested by Graham, Rothschild, and Spencer.

**Question 1 ([8]).** Is it true that

$$R(H,H) \geq R(H_{uv}, H_{uv})?$$

In particular, if $c = \chi(H)$, then is $R(H,H) \geq R(K_c, K_c)$?

By Fact 1, if the answer to Question 1 was “yes”, then for any graph $H$ with chromatic number $c$, $m(H)$ would essentially equal $R(K_c, K_c)$. As observed above, the answer to this question is clearly yes for a complete graph and for a complete graph minus an edge, i.e. $H = K_n - e$, since $c = n - 1$ and $K_c \subseteq H$. Unfortunately, the answer to Question 1 is “no” in general since the wheel on 6 vertices provides a counterexample. Let $H = W_6$, the wheel $W_6 = C_5 + v$. Then $\chi(H) = 4$ so set $c = 4$.

**Fact 2.**

$$R(W_6, W_6) = 17 < 18 = R(K_4, K_4).$$

There are, however, many graphs that yield an affirmative answer to Question 1. Recall a graph $H$ is called *perfect* if for every induced subgraph $H' \subseteq H$, the clique number $\omega(H')$ equals the chromatic number $\chi(H')$.

**Proposition 1.** If $H$ is a perfect graph with $\omega(H) = \chi(H) = c$, then $m(H) = R(K_c, K_c)$.

**Proof.** Suppose $H$ is a perfect graph with $\omega(H) = \chi(H) = c$. By Fact 1, we have $m(H) \leq R(K_c, K_c)$ so it suffices to prove that $m(H) \geq R(K_c, K_c)$. Since $K_c \subseteq H$, it is clear that $K_c \subseteq H'$ for every reduction $H' \in \mathcal{H}$. This means $R_2(\mathcal{H}) \geq R(K_c, K_c)$ so $m(H) = R(K_c, K_c)$, as desired.

Proposition 1 immediately determines $m(H)$ for several classes of graphs.

**Corollary 5.** The following hold:

- If $B_n$ is the book $B_n = K_2 + \overline{K_n}$, then $m(B_n) = 6$.

- If $F_n$ is the fan $F_n = (mK_2) + \{v\}$, then $m(F_n) = 6$.

- If $K_n^-$ is the complete graph minus an edge, then $m(K_n^-) = R(K_{n-1}, K_{n-1})$.

- If $H$ is a complete multipartite graph with chromatic number $c = \chi(G)$, then $m(H) = R(K_c, K_c)$.

On the other hand, there are certainly non-perfect graphs for which the conclusion of Proposition 1 is false. Consider the cycle $C_5$ for example. Up to isomorphism, the only proper reductions of $C_5$ are $K_3^+$ (the triangle with the addition of a pendant edge), and the triangle. Although $R(K_3, K_3) = R(C_5, C_5) = 6$, the unique sharpness example for $R(K_3, K_3)$ on 5 vertices contains monochromatic copies of $C_5$. This means that

$$m(C_5) = R_2(\{C_5, K_3^+, K_3\}) \leq 5 < 6 = R(K_3, K_3) = R(K_{\chi(H)}, K_{\chi(H)}).$$
4 Conclusion

In light of Theorem 4, a natural question is whether or not this bound might be sharp in some sense.

Conjecture 1. For a connected non-bipartite graph $H$ and an integer $k \geq 3$, there exist constants $c_1$ and $c_2$ such that

$$
gr_k(K_3 : H) = \begin{cases} 
c_1 \cdot (m(H) - 1)^{k/2} + 1 & \text{if } k \text{ is even}, 
c_2 \cdot (m(H) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd}.
\end{cases}$$

For all known sharp Gallai-Ramsey numbers of non-bipartite graphs, the answer to this question is “yes”.

In particular, Conjecture 1 is a generalization of the following recent conjecture about the complete graphs.

Conjecture 2 ([3]). For $k \geq 1$ and $p \geq 3$,

$$
gr_k(K_3 : K_p) = \begin{cases} 
(R(K_p, K_p) - 1)^{k/2} + 1 & \text{if } k \text{ is even}, 
(p - 1)(R(K_p, K_p) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd}.
\end{cases}$$

Other related results can be found in [2, 4, 11, 12]. We refer the interested reader to [5] for a survey of Gallai-Ramsey numbers, with a dynamic version available at [6]. We also state the corresponding conjecture for bipartite graphs.

Theorem 6 ([13]). Given a positive integer $k \geq 2$ and a connected bipartite graph $H$ with Ramsey number $R(H, H) = R$, we have

$$
gr_k(K_3 : H) = R + (s(H) - 1)(k - 2).$$

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References


