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Gorenstein Flat Dimension of Complexes

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ABSTRACT. We define a notion of Gorenstein flat dimension for unbounded complexes over left GF-closed rings.
Over Gorenstein rings we introduce a notion of Gorenstein cohomology for complexes; we also define a generalized Tate cohomology for complexes over Gorenstein rings, and we show that there is a close connection between the absolute, the Gorenstein and the generalized Tate cohomology.

1. INTRODUCTION

In 1966 Auslander introduced the notion of G-dimension of a finite R-module over a commutative noetherian local ring. In 1969 Auslander and Bridger extended this notion to two sided noetherian rings. Calling the modules of G-dimension zero Gorenstein projective modules, in 1995 Enochs and Jenda defined Gorenstein projective (whether finitley generated or not) and Gorenstein injective modules over an arbitrary ring. Another extension of the G-dimension is based on Gorenstein flat modules. These modules were introduced by Enochs, Jenda and Torrecillas ([9]).

Gorenstein homological algebra is the relative version of homological algebra that uses the Gorenstein projective (Gorenstein injective, Gorenstein flat respectively) modules instead of the usual projective (injective, flat respectively) modules. The Gorenstein dimensions for modules are defined in a similar manner with the classical homological dimensions, but using Gorenstein projective (Gorenstein injective, Gorenstein flat respectively) resolutions instead of projective (injective, flat respectively) resolutions.

It seems quite likely that there is a version of Gorenstein homological algebra in the category of complexes. For homologically right bounded complexes over commutative rings, Yassemi ([15]) and Christensen ([5]) introduced a Gorenstein projective dimension. Christensen, Frankild

Key words and phrases.

and Holm gave generalizations of the Gorenstein projective, Gorenstein injective and Gorenstein flat dimensions to homologically right bounded complexes ([7]). Veliche ([14]) extended the concept of Gorenstein projective dimension to the setting of unbounded complexes over associative rings. Asadollahi and Salarian ([1]) defined the dual notion, that of Gorenstein injective dimension of complexes over an associative ring.

We define here a notion of Gorenstein flat dimension of unbounded complexes over left GF-closed rings. These are the rings for which the class of left Gorenstein flat modules is closed under extensions. The class of left GF-closed rings includes (strictly) the one of right coherent rings and the one of rings of finite weak dimension (for examples of left GF-closed rings that are neither right coherent nor of finite weak dimension see [4]).

Our definition of Gorenstein flat dimension for complexes is given by means of DG-flat resolutions. We say that the Gorenstein flat dimension of a complex N of left R-modules, $GfdN$, is less than or equal to $g \in Z$, if there exists a DG-flat resolution $F \to N$ with $supH(F) \leq g$ and with $C_j(F)$ Gorenstein flat for all $j \geq g$. If $GfdN \leq g$ for all integers g then $GfdN = -\infty$; if $GfdN \leq g$ does not hold for any g then $GfdN = \infty$. We show that most properties of modules of finite Gorenstein flat dimension are preserved for complexes of finite Gorenstein flat dimension. We also show that for a right homologically bounded complex our definition agrees with [7], Definition 2.7 .

The second part of this paper deals with Gorenstein cohomology and generalized Tate cohomology for complexes over Gorenstein rings. The fact that over such a ring every complex has a special Gorenstein projective precover ([13]) allows us to define a notion of Gorenstein relative cohomology for complexes: if $G \to M$ is a special Gorenstein projective precover of M, then for a complex N we define the nth Gorenstein cohomology group $Ext^n_{\mathcal{G}}(M,N)$ by the equality $Ext^n_{\mathcal{G}}(M,N) = H^n\mathcal{H}om(G,N)$.

Over Gorenstein rings again, we also define a notion of generalized Tate cohomology, $\overline{Ext}^n(M,-)$, by the combined use of a special Gorenstein projective precover and a DG-projective precover of M. We show that there is a close connection between the absolute, the Gorenstein and this generalized Tate cohomology: for each complex N there exists an exact sequence $\ldots \to Ext^n_{\mathcal{G}}(M,N) \to Ext^n_R(M,N) \to \overline{Ext}^n(M,N) \to \ldots$ (where $Ext^n_R(-,-)$ are the absolute cohomology functors). We also prove that for a bounded complex M over a Gorenstein ring,

$\overline{Ext}^n(M,N) \simeq \widehat{Ext}^n(M,N)$ for $n > supM$, for any R-module N (where $\widehat{Ext}^n(-,-)$ are the Tate cohomology functors introduced by Veliche [14]).

2. Preliminaries

Let R be an associative ring with unit. By R-module we mean left R-module.

A (chain) complex C of R-modules is a sequence $C = \ldots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$ of R-modules and R-homomorphisms such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$.

A complex C is exact if for each n, $Ker\partial_n = Im\partial_{n+1}$.

Definition 1. *A module M is Gorenstein projective if there is a $Hom(-,Proj)$ exact exact complex $\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$ of projective modules such that $M = Ker(P_0 \to P_{-1})$.*

The class of Gorenstein projective modules is projectively resolving, i.e. if $0 \to M' \to M \to M" \to 0$ is an exact sequence with $M"$ a Gorenstein projective module, then M' is Gorenstein projective if and only if M is Gorenstein projective ([11], Theorem 2.5).

The dual notion is that of Gorenstein injective module:

Definition 2. *A module G is Gorenstein injective if there is a $Hom(Inj,-)$ exact exact complex $\ldots \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \ldots$ of injective modules such that $G = Ker(E_0 \to E_{-1})$.*

The class of Gorenstein injective modules is injectively resolving: if $0 \to G' \to G \to G" \to 0$ is an exact sequence with G' a Gorenstein injective module, then $G"$ is Gorenstein injective if and only if G is Gorenstein injective ([11], Theorem 2.6)

The Gorenstein flat modules are defined in terms of the tensor product:

Definition 3. *A module N is Gorenstein flat if there is an $Inj \otimes -$ exact exact sequence $\ldots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \ldots$ of flat modules such that $N = Ker(F_0 \to F_{-1})$.*

Enochs and Jenda proved ([8]) that over Gorenstein rings the class of Gorenstein flat modules is projectively resolving. Holm ([11]) showed that if the ring R is right coherent then the class of left Gorenstein flat modules is projectively resolving.

Bennis ([4]) calls a ring R left GF-closed if the class of left Gorenstein flat modules is closed under extensions. Over such a ring, the class of Gorenstein flat modules is projectively resolving ([4], Theorem 2.3).

A Gorenstein projective (flat) resolution of a module M is an exact sequence $\ldots \to P_1 \to P_0 \to M \to 0$, with each P_j Gorenstein projective (Gorenstein flat).

Definition 4. *The Gorenstein projective (flat) dimension of an R-module M,* $Gpd_R M$ *(*$Gfd_R M$ *respectively) is defined as the least integer n such that there is a Gorenstein projective (flat) resolution* $0 \to P_n \to \ldots \to P_0 \to M \to 0$ *. If such an n does not exist then the Gorenstein projective (flat) dimension of M is* ∞*.*

If $GpdM = n$ then any n-syzygy of M is Gorenstein projective.
If the ring is left GF-closed and $GfdM = n$ then any n-flat syzygy of M is Gorenstein flat.

The Gorenstein injective dimension is defined in terms of Gorenstein injective resolutions. A Gorenstein injective resolution of a module M is an exact complex $0 \to M \to G^0 \to G^1 \to \ldots$ with each G^j Gorenstein injective.

Definition 5. *The Gorenstein injective dimension of a module M,* $Gid_R M$*, is the least integer* $n \geq 0$ *such that there is a Gorenstein injective resolution* $0 \to M \to G^0 \to \ldots \to G^n \to 0$*. If such an n does not exist then* $GidM = \infty$*.*

In the category of complexes the projective (injective, flat) dimension were defined by Avramov and Foxby, by means of DG-projective (DG-injective, DG-flat) resolutions.

If X and Y are both complexes of left R-modules then $\mathcal{H}om(X, Y)$ denotes the complex with $\mathcal{H}om(X, Y)_n = \Pi_{q=p+n} Hom_R(X_p, Y_q)$ and with differential given by $\partial(f) = \partial^Y \circ f - (-1)^n f \circ \partial^X$, for $f \in \mathcal{H}om(X, Y)_n$.

Definition 6. *A complex P is DG-projective if each* P_n *is projective and* $\mathcal{H}om(P, E)$ *is exact for any exact complex E.*

For example, every right bounded complex of projective modules
$P = \ldots \to P_{n_0+2} \to P_{n_0+1} \to P_{n_0} \to 0$ is a DG-projective complex ([2], Remark 1.1 P).
The class of DG-projective complexes is projectively resolving: if $0 \to$

$P' \to P \to P" \to 0$ is a short exact sequence of complexes with $P"$ DG-projective, then P' is DG-projective if and only if P is DG-projective. ([10], Remark pp. 31).

A quasi-isomorphism $P \to X$ with P DG-projective is called a DG-projective resolution of X. By ([10], Corollary 3.10), every complex has a surjective DG-projective resolution.

Veliche defined the Gorenstein projective dimension for unbounded complexes over associative rings; her definition uses complete resolutions.

Definition 7. *Let N be a complex of left R-modules. A complete resolution of N is a diagram $T \xrightarrow{u} P \to N$ with $P \to N$ a DG-projective resolution, T a $Hom(-, Proj)$ exact exact complex of projective modules, and u a map of complexes such that u_i is bijective for $i \gg 0$.*

Definition 8. *The Gorenstein projective dimension of a complex N is defined by $GpdN = inf\{n \in Z,\ T \xrightarrow{u} P \to N$ is a complete resolution such that u_i is bijective for each $i \geq n\}$.*

The dual notion of DG-projective complex is that of DG-injective complex.

Definition 9. *A complex I is DG-injective if each I^n is injective and if $\mathcal{H}om(E, I)$ is exact for any exact complex E.*

For example, every left bounded complex of injective modules
$I = 0 \to I_{n_0} \to I_{n_0-1} \to \ldots$ is DG-injective ([2],Remark 1.1 I).
By [10], Remark, pp. 31, the class of DG-injective complexes is injectively resolving.
A DG-injective resolution of X is a quasi-isomorphism $X \to I$ with I DG-injective. By [10], Corollary 3.10, every complex has an injective DG-injective resolution.
The Gorenstein injective dimension for unbounded complexes over associative rings was introduced by Asadollahi and Salarian. Their definition is given in terms of complete coresolutions:

Definition 10. *Let N be a complex of left R-modules. A complete coresolution of N is a diagram $N \to I \xrightarrow{\nu} T$ with $N \to I$ a DG-injective resolution, T a $Hom(Inj, -)$ exact exact complex of injective modules, and ν a map of complexes such that ν_i is bijective for $i \ll 0$.*

Definition 11. *The Gorenstein injective dimension of a complex N is defined by $GidN = inf\{-n \in Z,\ N \to I \xrightarrow{\nu} T$ is a complete coresolution such that ν_i is bijective for each $i \leq n\}$.*

The DG-flat complexes are defined in terms of the tensor product of complexes. The tensor product of a complex of right R-modules X and a complex of left R-modules Y is the complex of Z-modules $X \otimes Y$ with $(X \otimes Y)_n = \oplus_{t \in Z} X_t \otimes Y_{n-t}$ and $\delta(x \otimes y) = \delta_t^X(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^Y(y)$ for $x \in X_t$ and $y \in Y_{n-t}$.

Definition 12. *A complex F of left R-modules is DG-flat if each F_n is flat and for any exact complex E of right R-modules the complex $E \otimes F$ is exact.*

Remark 1. *(*[13]*, page 118) Let E be an exact complex of right R-modules. Since $(E \otimes F)^+ \simeq \mathcal{H}om(E, F^+)$ we have that F is DG-flat if and only if F^+ is DG-injective.*

Thus every right bounded complex of flat modules is DG-flat.

The class of DG-flat complexes is projectively resolving.

A DG-flat resolution of a complex X is a quasi-isomorphism $F \to X$ with F a DG-flat complex. Since every DG-projective complex is DG-flat, every complex has a surjective DG-flat resolution.
We also recall the definition of a projective (injective, flat respectively) complex:

Definition 13. *(*[13]*, Theorem 3.1.3, Theorem 4.1.3) A complex $F = \ldots \to F_{n+1} \xrightarrow{\delta}_{n+1} F_n \xrightarrow{\delta}_n F_{n-1} \to \ldots$ is projective (injective, flat respectively) if F is exact, each F_n is projective (injective, flat respectively) and $Ker\delta_n$ is projective (injective, flat respectively) for all $n \in Z$.*

By [10], Proposition 3.7, a complex is projective (injective, flat respectively) if and only if it is exact and DG-projective (DG-injective, DG-flat respectively).

3. Gorenstein flat dimension for complexes

We recall that a ring R is left GF-closed if the class of Gorenstein flat left R-modules is closed under extensions, i.e. if $0 \to A \to B \to C \to 0$ is an exact sequence with A and C Gorenstein flat modules, then B is also Gorenstein flat.
Bennis proved ([4], Theorem 2.3) that a ring R is left GF-closed if and only if the class of Gorenstein flat left R-modules is projectively resolving.

Throughout this section R denotes a left GF-closed ring.

In [7] the authors define the Gorenstein flat dimension for homologically right-bounded complexes over associative rings:

Definition 14. *The Gorenstein flat dimension, $Gfd_R X$, of a homologically right-bounded complex X is defined as*
$Gfd_R X = inf\{sup\{l \in Z / A_l \neq 0\}, A$ *a right-bounded complex of Gorenstein flat modules such that* $A \simeq X\}$, *where* $\simeq$ *is the equivalence relation induced by quasi-isomorphisms.*

We introduce a notion of Gorenstein flat dimension for unbounded complexes over left GF-closed rings. We show that for homologically right-bounded complexes over left GF-closed rings, our definition agrees with [7], 2.7

We show first that if a complex N has a DG-flat resolution $F \to N$ with $supH(F) \leq g$ and with $C_j(F)$ Gorenstein flat for $j \geq g$, then for every DG-flat resolution $F' \to N$, $supH(F') \leq g$ and $C_j(F')$ is Gorenstein flat for all $j \geq g$. (We recall that for a complex $F = \ldots \to F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \ldots$, $C_j(F)$ denotes the module $coker f_{j+1}$.)

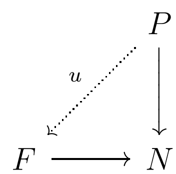
For two complexes, X and Y, we denote by $Hom(X,Y) = Z^0\mathcal{H}om(X,Y)$ the group of maps of complexes from X to Y.
We denote by $Ext(-,-)$ the right derived functors of $Hom(-,-)$. By [10], Proposition 3.5, a complex P is DG-projective if and only if $Ext^1(P,E) = 0$ for any exact complex E.

We begin by proving:

Lemma 1. *Let R be a left GF-closed ring and let N be a complex of left R-modules. If N has a DG-flat resolution $F \to N$ such that $supH(F) \leq g$ and $C_j(F)$ is Gorenstein flat for $j \geq g$, then for any DG-projective resolution $P \to N$, we have $supH(P) \leq g$ and $C_j(P)$ is Gorenstein flat for any $j \geq g$.*

Proof. If $P \to N$ is a DG-projective resolution then $supH(P) = supH(N) = supH(F) \leq g$.
We can assume without loss of generality that $F \to N$ is a surjective DG-flat resolution (if not, let $\overline{F} \to N$ be surjective with $\overline{F}$ a flat complex; then $F \oplus \overline{F} \to N$ is a surjective DG-flat resolution and $C_j(F) \oplus C_j(\overline{F})$ is Gorenstein flat for all $j \geq g$).

Then there is an exact sequence $0 \to U \to F \to N \to 0$ with U exact; this gives an exact sequence $0 \to Hom(P, U) \to Hom(P, F) \to Hom(P, N) \to Ext^1(P, U) = 0$ ([10], Proposition 3.6). So there exists a map of complexes $P \to F$ that makes the diagram



commutative. Since both $P \to N$ and $F \to N$ are quasi-isomorphisms, so is $P \to F$.

We can assume that $P \to F$ is a surjective quasi-isomorphism (if not, let $\overline{P} \to F$ be surjective with $\overline{P}$ a projective complex; then $P \oplus \overline{P} \to F$ is a surjective quasi-isomorphism). Then there exists an exact sequence $0 \to V \to P \to F \to 0$, with V an exact complex. Both F and P are DG-flat complexes, so V is DG-flat. Thus V is exact and DG-flat, so V is flat. (1)

We have an exact sequence $0 \to C_g(V) \to C_g(P) \to C_g(F) \to 0$ with $C_g(F)$ Gorenstein flat and $C_g(V)$ flat (by (1)). It follows that $C_g(P)$ is Gorenstein flat. Since the complex $\ldots \to P_{g+1} \to P_g \to C_g(P) \to 0$ is exact with $C_g(P)$ Gorenstein flat, each P_j flat and the ring is left GF-closed it follows that $C_j(P)$ is Gorenstein flat for any $j \geq g$.

□

We will also use the following result:

Lemma 2. *Let R be a left GF-closed ring. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If A is flat, B is Gorenstein flat and C^+ is Gorenstein injective, then C is Gorenstein flat.*

Proof. Since B is Gorenstein flat there is an exact sequence $0 \to B \to F \to Y \to 0$, with F flat and Y Gorenstein flat. Consider the push out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & = & Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The exact sequence $0 \to C \to X \to Y \to 0$ gives an exact sequence $0 \to Y^+ \to X^+ \to C^+ \to 0$. By hypothesis, C^+ is Gorenstein injective. By [11], Theorem 3.6, Y^+ is Gorenstein injective. Since the class of Gorenstein injective modules is injectively resolving it follows that X^+ is Gorenstein injective. The exact sequence $0 \to A \to F \to X \to 0$ also gives an exact sequence $0 \to X^+ \to F^+ \to A^+ \to 0$. Since A and F are flat, both F^+ and A^+ are injective. So $idX^+ \leq 1$. Since X^+ is Gorenstein injective and has finite injective dimension it follows ([8], Proposition 10.1.2) that X^+ is injective. By [8], Th. 3.2.10, X is flat. Since R is left GF-closed, the sequence $0 \to C \to X \to Y \to 0$ is exact, X is flat and Y is Gorenstein flat, it follows that C is Gorenstein flat. □

And, we will also use:

Lemma 3. *Let R be a left GF-closed ring, and let N be a complex of left R-modules. If N has a DG-flat resolution $F \to N$ with $supH(F) \leq g$ and with $C_j(F)$ Gorenstein flat for $j \geq g$, then $GidN^+ \leq g$.*

Proof. Since $F \to N$ is a DG-flat resolution, it follows that $N^+ \to F^+$ is a DG-injective resolution.
We have $supH(F) \leq g$, so $infH(F^+) \geq -g$.
Let C denote $C_g(F)$. The exact sequence $\ldots \to F_{g+1} \xrightarrow{g_{g+1}} F_g \xrightarrow{\pi} C \to 0$ gives an exact sequence $0 \to C^+ \xrightarrow{\pi^+} F_g^+ \xrightarrow{f_{g+1}^+} F_{g+1}^+ \to \ldots$.

So $kerf_{g+1}^{+} = Im\pi^{+} \simeq C^{+}$ is Gorenstein injective (by [11], Theorem 3.6, since C is Gorenstein flat). Then there exists an exact sequence $\overline{T} = \ldots T_{g+2} \rightarrow T_{g+1} \rightarrow Im\pi^{+} \rightarrow 0$, with each T_j injective and such that the sequence is $Hom(Inj, -)$ exact. Since each F_j^{+} is injective we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{g-2}^{+} & \longrightarrow & F_{g-1}^{+} & \xrightarrow{\pi^{+}} & Im\ \pi^{+} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle u_g} & & \big\downarrow{\scriptstyle u_{g-1}} & & \Big\| & & \\
\cdots & \longrightarrow & T_{-g+2} & \longrightarrow & T_{-g+1} & \longrightarrow & Ker\ f_{g+1}^{+} & \longrightarrow & 0
\end{array}$$

Since $\ldots \rightarrow F_{g+1} \rightarrow F_g \rightarrow C \rightarrow 0$ is exact and $Inj \otimes -$ exact, it follows that $0 \rightarrow Kerf_{g+1}^{+} \rightarrow F_g^{+} \rightarrow F_{g+1}^{+} \rightarrow \ldots$ is an exact complex of right R-modules that is also $Hom(Inj, -)$ exact.

So we have a commutative diagram

$$\begin{array}{ccccccccc}
F^{+} = \cdots & \longrightarrow & F_{g-1}^{+} & \longrightarrow & F_g^{+} & \longrightarrow & F_{g+1}^{+} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle u_{g-1}} & & \Big\| & & \Big\| & & \\
T = \cdots & \longrightarrow & T_{-g+1} & \longrightarrow & F_g^{+} & \longrightarrow & F_{g+1}^{+} & \longrightarrow & \cdots
\end{array}$$

with T an exact complex of injective modules which is also $Hom(Inj, -)$ exact.

Since N^{+} has a complete coresolution $N^{+} \rightarrow F^{+} \xrightarrow{u} T$ with $u_j = 1_{F_j^{+}}$ bijective for all $j \leq -g$ it follows that $GidN^{+} \leq g$ ([1], Theorem 2.3).

□

We can prove now:

Lemma 4. *Let R be a left GF-closed ring. If a complex N has a DG-flat resolution $F \rightarrow N$ such that $supH(F) \leq g$ and $C_j(F)$ is Gorenstein flat for all $j \geq g$, then for any DG-flat resolution $F' \rightarrow N$ we have that $supH(F') \leq g$ and $C_i(F')$ is Gorenstein flat for all $i \geq g$.*

Proof. Let $F' \rightarrow N$ be another DG-flat resolution. Then $supH(F') = supH(N) = supH(F) \leq g$.

Since $F' \to N$ is a DG-flat resolution it follows that $N^+ \to F'^+$ is a DG-injective resolution.
By Lemma 3, $GidN^+ \leq g$. By [1], Theorem 2.3, there exists an injective complete coresolution $N \to F'^+ \xrightarrow{u} T$ with u_j bijective for all $j \leq -g$. Then $Z_j(F'^+) \simeq Z_j(T)$ is Gorenstein injective for all $j \leq -g$, that is $Kerf_j'^+$ is Gorenstein injective for all $j \geq g+1$.

The exact sequence $F'_{g+1} \xrightarrow{f_{g+1}} F'_g \xrightarrow{\pi} C_g(F') \to 0$ gives an exact sequence $0 \to C_g(F')^+ \xrightarrow{\pi^+} F_g'^+ \xrightarrow{f_{g+1}'^+} {F'_{g+1}}^+$. So $C_g(F')^+ \simeq kerf_{g+1}'^+$ is Gorenstein injective.
Let $P \to F'$ be a surjective DG-projective resolution. Then $P \to N$ is a DG-projective resolution, so by Lemma 1, $C_j(P)$ is Gorenstein flat for all $j \geq g$.
There is an exact sequence $0 \to V \to P \to F' \to 0$ with V exact. Since V is also DG-flat (because P and F' are DG-flat), it follows that V is flat. Then $C_j(V)$ is flat, for all j.
We have an exact sequence $0 \to C_g(V) \to C_g(P) \to C_g(F') \to 0$ with $C_g(V)$ flat and $C_g(P)$ Gorenstein flat, and with $C_g(F')^+$ Gorenstein injective. By Lemma 2, $C_g(F')$ is Gorenstein flat.

Since $H_i(F') = 0$ for all $i \geq g+1$ and each F'_n is flat it follows that $C_i(F')$ is Gorenstein flat for all $i \geq g$.

□

Definition 15. *Let R be a left closed GF-ring. Let N be a complex of left R-modules. The Gorenstein flat dimension of N is defined by:*
$GfdN \leq g$ if there is a DG-flat resolution $F \to N$ such that $supH(F) \leq g$ and $C_j(F)$ is Gorenstein flat for any $j \geq g$. If $GfdN \leq g$ but $GfdN \leq g-1$ does not hold then $GfdN = g$.
If $GfdN \leq g$ for any g then $GfdN = -\infty$.
If $GfdN \leq g$ does not hold for any g then $GfdN = \infty$.

Remark 2. *$GfdN = -\infty$ if and only if N is an exact complex.*

Proof. "$\Rightarrow$" If $GfdN = -\infty$ then $H_i(N) = 0$ for any integer i.
"$\Leftarrow$" If N is exact and $F \to N$ is a surjective DG-flat resolution then F is a flat complex. Then $H_j(F) = 0$ and $C_j(F)$ is flat hence Gorenstein flat for any integer j. So $GfdN = -\infty$. □

Theorem 1. *Let N be a complex of R-modules. The following are equivalent:*
1) $GfdN \leq g$;

2) $supH(N) \leq g$ and $C_i(F)$ is Gorenstein flat for any $i \geq g$, for any DG-flat resolution $F \to N$;
3) For any DG-projective resolution $P \to N$ we have $supH(P) \leq g$ and $C_j(P)$ is Gorenstein flat for any $j \geq g$;
4) There exists a DG-projective resolution $P \to N$ such that $H_j(P) = 0$ for any $j \geq g+1$, and $C_j(P)$ is Gorenstein flat for any $j \geq g$.

Proof. 1) $\Rightarrow$ 2) by Lemma 4;
2) $\Rightarrow$ 1) straightforward;
1) $\Rightarrow$ 3) By Lemma 1;
3) $\Rightarrow$ 4) Straightforward;
4) $\Rightarrow$ 1) By definition, since every DG-projective resolution is a DG-flat resolution.

□

Properties of dimensions

We recall that the flat dimension of a complex N, fdN, is defined ([2]) by $fdN \leq g$ if $supH(N) \leq g$ and for any DG-flat complex F, such that $F \simeq N$, $C_j(F)$ is flat for all $j \geq g$ (where $\simeq$ is the equivalence relation generated by quasi-isomorphisms).
If $fdN \leq g$ does not hold for any $g \in Z$, then $fdN = \infty$; if $fdN \leq g$ holds for any $g \in Z$ then $fdN = -\infty$.

Proposition 1. *Let R be a left GF-closed ring. For any complex of R-modules N, we have $GfdN \leq fdN$ with equality if $fdN < \infty$.*

Proof. - Clear if $fdN = \infty$
- If $fdN = -\infty$ then N is exact, so $GfdN = -\infty$
- Let $fdN = g < \infty$. Then for any DG-flat resolution $F \to N$ we have $supH(F) \leq g$ and $C_j(F)$ is flat, hence Gorenstein flat, for all $j \geq g$. By definition, $GfdN \leq g$.
Suppose $GfdN \leq g-1$. Then for any DG-flat resolution $F \to N$, $C_j(F)$ is Gorenstein flat for all $j \geq g-1$, and $H_j(F) = 0$ for all $j \geq g$. The exact sequence $0 \to C_g(F) \to F_{g-1} \to C_{g-1}(F) \to 0$ with $C_g(F)$ and F_{g-1} flat modules gives that $C_{g-1}(F)$ has finite flat dimension. Since $C_{g-1}(F)$ is a Gorenstein flat module of finite flat dimension it follows that $C_{g-1}(F)$ is flat ([8], Corollary 10.3.4).
But then $fdN \leq g-1$. Contradiction.
So $GfdN = fdN$ if $fdN < \infty$.

□

Proposition 2. *Let R be a left GF-closed ring. Let M be an R-module. If $\overline{M}$ is M as a complex at zero then $Gfd\overline{M} = Gfd_R M$.*

Proof. Let $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M. Then $F \to \overline{M}$ is a DG-flat resolution (where $F = \ldots \to F_1 \to F_0 \to 0$).
- Case $Gfd_R M = \infty$.
Suppose $Gfd\overline{M} = l < \infty$. Then $C_j(F)$ is Gorenstein flat for any $j \geq l$. Since $0 \to C_l(F) \to F_{l-1} \to \ldots \to F_0 \to M \to 0$ is exact with $C_l(F)$ Gorenstein flat and each F_j flat, it follows that $Gfd_R M \leq l$. Contradiction. So $Gfd\overline{M} = \infty$.

- Case $Gfd_R M = l < \infty$.
Then $C_l(F)$ is Gorenstein flat; since the ring is GF-closed, $C_j(F)$ is Gorenstein flat for any $j \geq l$. Then $F \to \overline{M}$ is a DG-flat resolution with $C_j(F)$ Gorenstein flat for all $j \geq l$ and with $H_j(F) = 0$ for any $j \geq 1$. By definition, $Gfd\overline{M} \leq l$.
Suppose $Gfd\overline{M} \leq l - 1$. Then $C_{l-1}(F)$ is Gorenstein flat. The exact sequence $0 \to C_{l-1} \to F_{l-2} \to \ldots \to F_0 \to M \to 0$ with C_{l-1} Gorenstein flat and each F_j flat gives that $Gfd_R(M) \leq l - 1$. Contradiction. So $Gfd\overline{M} = l$.

□

Proposition 3. *Let R be a left GF-closed ring and let N be a complex of left R-modules.*
a) If $GfdN \leq g$ then $GidN^+ \leq g$.
b) If R is right coherent then $GfdN \leq g$ if and only if $GidN^+ \leq g$.
c) If R is right coherent then $GfdN = GidN^+$.

Proof. a) By Lemma 3.

b) Let R be right coherent. Let N be a complex of left R-modules with $GidN^+ \leq g$.

If $F \to N$ is a DG-flat resolution then $N^+ \to F^+$ is a DG-injective resolution. Since $GidN^+ \leq g$ it follows that $Z_j(F^+)$ is a Gorenstein injective module for any $j \leq -g$, that is $ker f_j{}^+$ is Gorenstein injective for all $j \geq g+1$, and $H_j(F^+) = 0$ for all $j \leq -g-1$.
The sequence $F_{g+1} \xrightarrow{f_{g+1}} F_g \xrightarrow{\pi} C_g(F) \to 0$ is exact, therefore

$0 \to C_g(F)^+ \xrightarrow{\pi^+} F_g^+ \xrightarrow{f_{g+1}^+} F_{g+1}^+$ is exact. Then $C_g(F)^+ \simeq Ker f_{g+1}^+$ is Gorenstein injective. Since R is right coherent it follows that $C_g(F)$ is Gorenstein flat ([11], Theorem 3.6).

We have $H_j(F^+) = 0$ for any $j \leq -g-1$. Then $H_j(F)^+ \simeq H_j(F^+) = 0$, so $H_j(F) = 0$, for any $j \geq g+1$.
Since $\ldots \to F_{g+1} \to F_{g+1} \to F_g \to C_g(F) \to 0$ is exact, each F_j is flat, $C_g(F)$ is Gorenstein flat and the class of Gorenstein flat modules is closed under kernels of epimorphisms, it follows that $C_j(F)$ is Gorenstein flat for all $j \geq g$. Thus, $GfdN \leq g$.

c) If $GfdN = -\infty$ then N is exact and therefore N^+ is exact, so $GidN^+ = -\infty$;
-If $GfdN = g < \infty$ then by the above $GidN^+ \leq g$. Suppose $GidN^+ \leq g-1$. Since R is right coherent it follows that $GfdN \leq g-1$. Contradiction. So $GidN^+ = g$.
-Case $GfdN = \infty$
Suppose that $GidN^+ = g < \infty$. Then by b), $GfdN \leq g < \infty$. Contradiction.

□

Proposition 4. *Let R be a left GF-closed ring. Let $0 \to N' \to N \to N" \to 0$ be an exact sequence of complexes of R-modules. If two complexes have finite Gorenstein flat dimension then so does the third.*

Proof. By [14], Proposition 1.3.8, there is an exact sequence of complexes $0 \to F' \to F \to F" \to 0$ with $F' \to N'$, $F \to N$, and $F" \to N"$ DG-projective resolutions. If two of the complexes N', N, $N"$ have finite Gorenstein flat dimension then there is $g \in Z$ such that $H_j(F') = H_j(F) = H_j(F") = 0$ for all $j \geq g$.
For each $j \geq g$ we have an exact sequence $0 \to C_j(F') \to C_j(F) \to C_j(F") \to 0$. If $C_j(F")$ is Gorenstein flat, then $C_j(F')$ is Gorenstein flat if and only if $C_j(F)$ is Gorenstein flat. If both $C_j(F')$ and $C_j(F)$ are Gorenstein flat then $Gfd_R C_j(F") \leq 1$.

□

We recall that the finitistic projective dimension of a ring R, $FPD(R)$, is defined as $FPD(R) = sup\{pdM : pdM < \infty\}$.

We also recall [11], Proposition 3.4:

Proposition 5. *If R is right coherent with finite left finitistic projective dimension, then every Gorenstein projective (left) R-module is also Gorenstein flat.*

Proposition 6. *Let R be right coherent of finite left finitistic projective dimension. For any complex of left R-modules N, $GfdN \leq GpdN$.*

Proof. - Obvious if $GpdN = \infty$;
- If $GpdN = g < \infty$ then for any DG-projective resolution $P \to N$, $C_j(P)$ is Gorenstein projective for all $j \geq g$ and sup $H(P) \leq g$. By Proposition 5, $C_j(P)$ is Gorenstein flat. By Theorem 1, $GfdN \leq g$.
- If $GpdN = -\infty$ then N is exact, so $GfdN = -\infty$. □

Proposition 7. *Let R be a left noetherian ring of finite Krull dimension and let N be a complex of R-modules. If $GfdN < \infty$ then $GpdN < \infty$.*

Proof. Let $GfdN = g < \infty$. If $P \to N$ is a DG-projective resolution, then $C_g(P)$ is Gorenstein flat and $H_j(P) = 0$ for all $j \geq g + 1$. By [6], Theorem 29, $GpdC_g(P) = l < \infty$. Then $C_{g+l}(P)$ is Gorenstein projective. By [14], Theorem 3.4, $GpdN \leq g + l$. □

For homologically bounded complexes Christensen, Frankild and Holm defined the Gorenstein flat dimension by

Definition 16. *([7], 2.7) Let X be a homologically right-bounded complex. The Gorenstein flat dimension of X, Gfd_RX, is*
$Gfd_RX = inf\{sup\ \{l \in Z, A_l \neq 0\}$, A a right bounded complex of Gorenstein flat modules that is isomorphic to X in $D(R)\}$
(where $D(R)$ is the derived category)

Remark 3. *Let R be a left GF-closed ring, and let X be a homologically right-bounded complex. Then $Gfd_RX = GfdX$.*

Proof. We can assume $infH(X) = 0$.
By [14], 1.3.4, X has a DG-projective resolution $F \to X$ with $infF = 0$.
If $GfdX \leq g$ then $C_j(F)$ is Gorenstein flat for all $j \geq g$.
Let $\overline{F} = 0 \to C_g(F) \to F_{g-1} \to \ldots \to F_0 \to 0$ and let $X' = 0 \to C_g(X) \to X_{g-1} \to X_{g-2} \to \ldots$. The quasi-isomorphism $F \to X$ gives a quasi-isomorphism $\overline{F} \to X'$. Since $\overline{F} \simeq X'$ and $X \simeq X'$ in $D(R)$, it follows that $\overline{F} \simeq X$ in $D(R)$. Each component of $\overline{F}$ is a Gorenstein flat module, so $Gfd_RX \leq g$.

Let X be a homologically right bounded complex (with $infH(X) = 0$), such that $Gfd_RX \leq g$ where $g < \infty$. We show that $GfdX \leq g$.

Since $Gfd_R X \leq g$ there exists a complex $A = 0 \to A_g \to A_{g-1} \to \ldots \to A_0 \to 0$ of Gorenstein flat modules such that $A \simeq X$.
X is right bounded, so X has a DG-projective resolution $P \to X$ with P right bounded.
Since $P \simeq X \simeq A$ and P is DG-projective it follows that there is a quasi-isomorphism $P \to A$ ([2], 1.4.P).
A is right bounded, so there is a surjective map $F \to A$ with F a right bounded projective complex. Then $P \oplus F \to A$ is a surjective quasi-isomorphism, so we have an exact sequence $0 \to V \to P \oplus F \to A \to 0$ with V exact. Since $P_j \oplus F_j$ and A_j are Gorenstein flat modules for all j, it follows that each V_j is Gorenstein flat. We have $V \subseteq P \oplus F$, so V is a right bounded complex of Gorenstein flat modules; then $C_j(V)$ is Gorenstein flat for each j.
The exact sequence $0 \to C_g(V) \to C_g(P) \oplus C_g(F) \to C_g(A) \to 0$ with both $C_g(A) = A_g$ and $C_g(V)$ Gorenstein flat gives that $C_g(P) \oplus C_g(F)$ is Gorenstein flat. By [4], Corollary 2.6, the class of Gorenstein flat modules over a left GF-closed ring is closed under summands, so $C_g(P)$ is Gorenstein flat. Since $P \to X$ is a DG-projective resolution with $supH(P) \leq g$ and $C_j(P)$ Gorenstein flat for $j \geq g$ it follows that $GfdX \leq g$.
By the above, $Gfd_R X = GfdX$ for $Gfd_R X < \infty$. Also by the above, $Gfd_R X = \infty$ if and only if $GfdX = \infty$.
We have $Gfd_R X = -\infty$ if and only if X is exact if and only if $GfdX = -\infty$. □

We recall ([8], Definition 9.1.1) that a ring is Gorenstein if it is left and right noetherian and has finite self injective dimension on both sides.
A Gorenstein ring with $id_R R$ at most n is called n-Gorenstein. In this case idR_R is also at most n ([8], Proposition 9.1.8)

Gorenstein rings can be characterized in terms of Gorenstein flat dimensions of complexes of their modules.

Theorem 2. *Let R be a left and right noetherian ring. The following are equivalent:*
a) R is n-Gorenstein;
b) For every complex of R-modules N, $GfdN \leq n + supH(N)$.

Proof. a) $\Rightarrow$ b) True if $supH(N) = \infty$.
- Case $supH(N) = l < \infty$
Let $F \to N$ be a DG-flat resolution. Then $H_j(F) = 0$ for any $j > l$. So we have an exact complex $\ldots \to F_{l+1} \to F_l \to C_l(F) \to 0$. Since R is n-Gorenstein, $Gfd_R C_l(F) \leq n$. Thus $C_j(F)$ is Gorenstein flat for

any $j \geq n+l$. Therefore $GfdN \leq n+l$.
b) $\Rightarrow$ a)If N is a left R-module then by Proposition 2, $Gfd_R N = Gfd\overline{N} \leq n$ (where $\overline{N}$ is N as a complex at zero). So every n-flat syzygy of N is Gorenstein flat. Similarly, every n-flat syzygy of any right R-module is Gorenstein flat. It follows that R is n-Gorenstein ([8], Theorem 12.3.1).

□

4. Gorenstein cohomology for complexes; generalized Tate cohomology for complexes

We define the Gorenstein cohomology for complexes over Gorenstein rings. We also define a notion of generalized Tate cohomology for complexes over Gorenstein rings and we show that there is a close connection between the absolute, the Gorenstein, and the generalized Tate cohomology.

Our definition of Gorenstein cohomology for complexes uses Gorenstein projective precovers.

We recall first that a complex G is Gorenstein projective if there exists an exact resolution of complexes:

$\ldots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$
such that each P_i is a projective complex, the sequence remains exact when applying $Hom(-,P)$, for any projective complex P, and $G = Ker(P_0 \to P_{-1})$.

We recall that $Hom(X,Y) = Z^0\mathcal{H}om(X,Y)$ is the group of maps of complexes from X to Y, and $Ext(-,-)$ are the right derived functors of $Hom(-,-)$.

We also recall the definition of a Gorenstein projective precover:

Definition 17. *(*[13]*, Definition 1.2.3) Let M be a complex. A Gorenstein projective precover of M is a map of complexes $\phi : G \to M$ with G Gorenstein projective and with the property that for every Gorenstein projective complex G' the sequence $Hom(G',G) \to Hom(G',M) \to 0$ is exact.*

Throughout this section we work with the projective dimension defined by García-Rozas in [13]:
The projective dimension of a complex M is the least integer $n \geq 0$

such that there exists an exact sequence $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ with each P_j a projective complex; if such an n does not exist, then the projective dimension of M is ∞.

It is shown in [13] that a complex $L = \ldots \to L_{n+1} \to L_n \to L_{n-1} \to \ldots$ has finite projective dimension if and only if L is exact, and for each $n \in Z$, L_n and $Ker(L_n \to L_{n-1})$ are modules of finite projective dimension. The class of complexes of finite projective dimension is denoted $\mathcal{L}$.

Over a Gorenstein ring, García-Rozas gave the following characterization of Gorenstein projective complexes ([13], Theorem 3.3.5):

Theorem 3. *Let R be a Gorenstein ring. The following conditions are equivalent for a complex G:*
1) G is Gorenstein projective;
2) $Ext^1(G, L) = 0$ for all complexes L of finite projective dimension;
3) Each G_n is a Gorenstein projective module.

Theorem 3 gives the following result:

Proposition 8. *The projective dimension of a Gorenstein projective complex is either zero or infinite.*

Proof. Let G be a Gorenstein projective complex. Suppose $pdG = l < \infty$ and let $0 \to P_l \xrightarrow{f_l} P_{l-1} \to \ldots \to P_0 \xrightarrow{f_0} G \to 0$ be a finite projective resolution. Since the class of Gorenstein projectives is projectively resolving, Imf_j is Gorenstein projective for all j. By Theorem 3 (part 2), the resolution is split exact since each P_j is projective. Thus G is projective. □

García-Rozas showed that over a Gorenstein ring every complex M has a special Gorenstein precover, i.e. a Gorenstein projective precover $G \to M$ with $Ker(G \to M)$ a complex of finite projective dimension.

Remark 4. *A special Gorenstein projective precover is unique up to homotopy.*

Proof. Let $\phi : G \to M$ and $\phi' : G' \to M$ be two special Gorenstein projective precovers. Let $u : G \to G'$ and $v : G \to G'$ be maps of complexes induced by 1_M. Then $\phi' u = \phi$ and $\phi' v = \phi$. So $u - v : G \to Ker\phi'$. By hypothesis, $L = Ker\phi'$ is a complex of finite projective dimension.

If $\theta : A \to L$ is a special Gorenstein projective precover then we have an exact sequence $0 \to Ker\theta \to A \xrightarrow{\theta} L \to 0$ with L and $Ker\theta$ of finite projective dimension. Then A is Gorenstein projective of finite projective dimension, hence projective. Since G is Gorenstein projective, and $u - v : G \to L$, there exists $\omega : G \to A$ such that $\theta\omega = u - v$.
By the definition of Gorenstein projective complexes, there is an exact sequence $0 \to G \xrightarrow{j} P \to X \to 0$ with P projective and with X Gorenstein projective. This gives an exact sequence $0 \to Hom(X, A) \to Hom(P, A) \to Hom(G, A) \to Ext^1(X, A) = 0$ (by Theorem 3).
Since $Hom(P, A) \to Hom(G, A) \to 0$ is exact and $\omega \in Hom(G, A)$ there exists $l : P \to A$ such that $\omega = lj$. But P is a projective complex, so l is homotopic to 0. Then $\omega = lj$ is homotopic to zero and therefore $u - v = \theta\omega$ is homotopic to zero.
The identity map 1_M also induces a map of complexes $\alpha : G' \to G$. Then $\alpha u : G \to G$ and 1_G are both induced by 1_M. By the above αu and 1_G are homotopic. Similarly, $u\alpha \sim 1_{G'}$. □

Since over a Gorenstein ring every complex has a special Gorenstein projective precover and such a precover is unique up to homotopy, we can compute right derived functors of $\mathcal{H}om(-,-)$ by means of special Gorenstein projective precovers.

Definition 18. *Let R be a Gorenstein ring and let M be a complex of R-modules. For each complex N, the nth relative Gorenstein cohomology group $Ext^n_{\mathcal{G}}(M, N)$ is defined by the equality $Ext^n_{\mathcal{G}}(M, N) = H^n\mathcal{H}om(G, N)$, where $G \to M$ is a special Gorenstein projective precover of M.*

Remark 5. *If M and N are modules regarded as complexes at zero, then $Ext^n_{\mathcal{G}}(M, N)$ are the usual Gorenstein cohomology groups.*

Proof. Let $\overline{M} = 0 \to M \to 0$. Let $\ldots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \to 0$ be a special Gorenstein projective resolution of M (i.e. $G_0 \xrightarrow{g_0} M$ and $G_i \xrightarrow{g_i} Ker(G_{i-1} \to G_{i-2})$ are Gorenstein projective precovers such that $Ker g_i$ has finite projective dimension). Then $(\overline{G} \to \overline{M})$ is a special Gorenstein projective precover, where $\overline{G} = \ldots \to G_1 \to G_0 \to 0$. Since $\overline{N}$ is the module N at zero, $\mathcal{H}om(\overline{G}, \overline{N})$ is the complex $\ldots \to Hom(G_1, N) \to Hom(G_0, N) \to 0$. Thus $Ext^n_{\mathcal{G}}(M, N) = H^n\mathcal{H}om(\overline{G}, \overline{N})$ are the usual Gorenstein cohomology groups. □

Over a Gorenstein ring we also define generalized Tate cohomology groups $\overline{Ext}^n(M,N)$, by the combined use of a DG-projective resolution and a special Gorenstein projective resolution of M.

Let R be a Gorenstein ring, and let M be a complex of R-modules. Let $P \xrightarrow{\delta} M$ be a surjective DG-projective resolution and let $G \xrightarrow{\phi} M$ be a Gorenstein projective precover. Since P is Gorenstein projective there is a map of complexes $u : P \to G$ such that $\delta = \phi \circ u$.

Remark 6. *If $P \xrightarrow{\delta} M$ is a surjective DG-projective precover, $G \xrightarrow{\varphi} M$ a special Gorenstein projective precover, and α, $\beta : P \to G$ are maps of complexes induced by 1_M then α and β are homotopic.*

Proof. Let $P' \xrightarrow{\theta} G$ be a surjective DG-projective resolution. The exact sequence $0 \to E \to P' \to G \to 0$ with E exact complex gives an exact sequence $0 \to Hom(P,E) \to Hom(P,P') \to Hom(P,G) \to Ext^1(P,E) = 0$ (by [10], Proposition 3.5). Since $Hom(P,P') \to Hom(P,G)$ is surjective, there is a map of complexes $u : P \to P'$ such that $\alpha = \theta u$. Similarly, $\beta = \theta v$ for some $v : P \to P'$.
Since both $P \xrightarrow{\delta} M$ and $P' \xrightarrow{\varphi\theta} M$ are DG-projective resolutions, and $u, v : P \to P'$ are induced by 1_M, we have $u \sim v$. Then $\alpha = \theta u \sim \theta v = \beta$.

□

Definition 19. *(generalized Tate cohomology) Let R be a Gorenstein ring and let M be a complex of R-modules. Let $P \to M$ be a surjective DG-projective resolution, let $G \to M$ be a special Gorenstein projective resolution, and let $u : P \to G$ be a map of complexes induced by 1_M. Let $M(u)$ be the mapping cone of u. For each complex N the nth generalized Tate cohomology group is defined by the equality $\overline{E}xt^n(M,N) = H^{n+1}\mathcal{H}om(M,N)$.*

We show first that the $\overline{Ext}^n(M,N)$ are well defined.

- If $P \to M$ is a surjective DG-projective resolution, $G \to M$ a special Gorenstein projective precover and $u, v : P \to G$ are maps of complexes induced by 1_M then their mapping cones $M(u)$ and $M(v)$ are isomorphic.
Proof: Let $P = \ldots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \to \ldots$, and $G = \ldots \to$

$G_{n+1} \xrightarrow{g_{n+1}} G_n \xrightarrow{g_n} G_{n-1} \to \ldots$.
Since u and v are both induced by 1_M they are homotopic; so for each n there is $s_n \in Hom(P_n, G_{n+1})$ such that $u_n - v_n = g_{n+1}s_n + s_{n-1}f_n$.
There are maps of complexes $\omega : M(u) \to M(v)$, with $\omega_n : G_{n+1} \oplus P_n \to G_{n+1} \oplus P_n$, $\omega_n(x, y) = (x + s_n(y), y)$, and $\psi : M(v) \to M(u)$, with $\psi_n : G_{n+1} \oplus P_n \to G_{n+1} \oplus P_n$ given by $\psi_n(x, y) = (x - s_n(y), y)$. Then $\omega_n\psi_n(x, y) = (x, y)$ and $\psi_n\omega_n(x, y) = (x, y)$ for all $(x, y) \in G_{n+1} \oplus P_n$.

- If $P \xrightarrow{\theta} M$, $P' \xrightarrow{\theta'} M$ are two surjective DG-projective resolutions and $G \xrightarrow{\phi} M$ is a special Gorenstein projective precover then there are maps of complexes $u : P \to G$ and $v : P' \to G$, such that $\phi u = \theta$ and $\phi v = \theta'$. We show that their mapping cones, $M(u)$ and $M(v)$, are homotopically equivalent.

Proof: Since $P \to M$ and $P' \to M$ are DG-projective resolutions there are maps $h : P \to P'$ and $k : P' \to P$ induced by 1_M. Then $vh : P \to G$ and $u : P \to G$ are both induced by 1_M. By the above, $M(u)$ and $M(vh)$ are isomorphic. So it suffices to show that $M(v)$ and $M(vh)$ are homotopically equivalent.

Let $P = \ldots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \to \ldots$, let $P' = \ldots \to P'_{n+1} \xrightarrow{f'_{n+1}} P'_n \xrightarrow{f'_n} P'_{n-1} \to \ldots$, and $G = \ldots \to G_{n+1} \xrightarrow{g_{n+1}} G_n \xrightarrow{g_n} G_{n-1} \to \ldots$.
Since $hk : P' \to P'$ is induced by 1_M, hk is homotopic to $1_{P'}$.
So for each n there exists $s_n \in Hom(P'_n, P'_{n+1})$ such that $1 - h_nk_n = f'_{n+1}s_n + s_{n-1}f'_n$.
Similarly kh is homotopic to 1_P, so for each n there exists $\overline{s}_n \in Hom(P_n, P_{n+1})$, such that $1 - k_nh_n = f_{n+1}\overline{s}_n + \overline{s}_{n-1}f_n$.

Let $M(v)$ denote the mapping cone of v:

$M(v) = \ldots \to G_{n+2} \oplus P'_{n+1} \xrightarrow{\delta'_{n+1}} G_{n+1} \oplus P'_n \xrightarrow{\delta'_n} G_n \oplus P'_{n-1} \to \ldots$, with $\delta'_n = (g_{n+1}(x) + v_n(y), -f'_n(y))$.
Let $M(vh)$ be the mapping cone of vh:

$M(vh) = \ldots \to G_{n+2} \oplus P_{n+1} \xrightarrow{\delta_{n+1}} G_{n+1} \oplus P_n \xrightarrow{\delta_n} G_n \oplus P_{n-1} \to \ldots$, with $\delta_n = (g_{n+1}(x) + v_nh_n(y), -f_n(y))$.

There are maps of complexes: $\alpha : M(v) \to M(vh)$, with $\alpha_n : G_{n+1} \oplus P'_n \to G_{n+1} \oplus P_n$, $\alpha_n(x, y) = (x + v_{n+1}s_n(y), k_n(y))$, and $\beta : M(vh) \to M(v)$, with $\beta_n : G_{n+1} \oplus P_n \to G_{n+1} \oplus P'_n$, given by $\beta_n(x, y) = (x + v_{n+1}h_{n+1}\overline{s}_n(y) - v_{n+1}s_nh_n(y), h_n(y))$.

Then $\alpha_n\beta_n(x,y) = (x + v_{n+1}h_{n+1}\overline{s}_n(y), k_nh_n(y))$.

Let $\eta_n : G_{n+1} \oplus P_n \to G_{n+2} \oplus P_{n+1}$, $\eta_n(x,y) = (0, \overline{s}_n(y))$.
Then $(\delta_{n+1}\eta_n + \eta_{n-1}\delta_n)(x,y) = (v_{n+1}h_{n+1}\overline{s}_n(y), (k_nh_n - 1)(y)) = \alpha_n\beta_n(x,y) - (x,y)$. So $\alpha\beta \sim 1$.

We have $\beta_n\alpha_n(x,y) = (x + v_{n+1}s_n(y) + v_{n+1}h_{n+1}\overline{s}_nk_n(y) - v_{n+1}s_nh_nk_n(y), h_nk_n(y))$.
Let $\mu_n : G_{n+1} \oplus P'_n \to G_{n+2} \oplus P'_{n+1}$, $\mu_n(x,y) = (0, (s_n + h_{n+1}\overline{s}_nk_n - s_nh_nk_n)(y))$.
Then $(\delta'_{n+1}\mu_n + \mu_{n-1}\delta'_n)(x,y) = ((v_{n+1}s_n + v_{n+1}h_{n+1}\overline{s}_nk_n - v_{n+1}s_nh_nk_n)(y), (h_nk_n - 1)(y)) = \beta_n\alpha_n(x,y) - (x,y)$.
Thus $\beta\alpha \sim 1$.
So $M(v) \sim M(vh) \simeq M(u)$. (2)

- Similarly, if $P \to M$ is a surjective DG-projective resolution, $G \to M$ and $G' \to M$ are special Gorenstein projective precovers, and $u : P \to G$ and $v : P \to G'$ are induced by 1_M, then $M(u) \sim M(v)$. (3)

- If $P \to M$, $P' \to M$ are surjective DG-projective resolutions, $G \to M$, $G' \to M$ are special Gorenstein projective precovers, and $u : P \to G$, $v : P' \to G'$ are maps of complexes induced by 1_M, then $M(u) \sim M(v)$.

Proof: There are maps of complexes $h : P \to P'$, $l : G' \to G$, both induced by 1_M.
By (2), $M(lv) \sim M(u)$, and by (3), $M(v) \sim M(lv)$. So $M(u) \sim M(v)$.

We denote by $Ext_R(-,-)$ the right derived functors of $\mathcal{H}om(-,-)$ (the absolute cohomology). We show that over Gorenstein rings there is a close connection between the absolute, the Gorenstein and the generalized Tate cohomology:

Proposition 9. *Let R be a Gorenstein ring, and let M be a complex of R-modules. For each complex N of R-modules there is an exact sequence* $\ldots \to Ext_R^{n-1}(M,N) \to \overline{Ext}^{n-1}(M,N) \to Ext_{\mathcal{G}}^n(M,N) \to Ext_R^n(M,N) \to \overline{Ext}^n(M,N) \to \ldots$.

Proof. Let $P \xrightarrow{\beta} M$ be a surjective DG-projective resolution and let $G \xrightarrow{\alpha} M$ be a special Gorenstein projective precover. P is Gorenstein projective, so there is a map of complexes $u : P \to G$ such that $\beta = \alpha u$.
Since the sequence $0 \to G \to M(u) \to P[1] \to 0$ is split exact in each degree, for each complex N we have an exact sequence $0 \to \mathcal{H}om((P[1], N) \to \mathcal{H}om(M(u), N) \to \mathcal{H}om(G, N) \to 0$. This gives a long exact sequence $\ldots \to H^n\mathcal{H}om(P[1], N) \to H^n\mathcal{H}om(M(u), N) \to H^n\mathcal{H}om(G, N) \to H^{n+1}\mathcal{H}om(P[1], N) \to \ldots$, that is
$\ldots \to Ext_R^{n-1}(M, N) \to \overline{Ext}^{n-1}(M, N) \to Ext_{\mathcal{G}}^n(M, N) \to Ext_R^n(M, N) \to \ldots$.

□

Remark 7. *Let R be a Gorenstein ring. If M and N are R-modules regarded as complexes at zero, then the exact sequence above gives the Avramov-Martsinkovsky exact sequence connecting the absolute, the Gorenstein relative and the Tate cohomology of modules (see* [3]*, Theorem 7.1,* [14]*, Theorem 6.6, or* [12]*,Corollary 1):*
$0 \to Ext_{\mathcal{G}}^1(M, N) \to Ext_R^1(M, N) \to \widehat{Ext}^1(M, N) \to \ldots$

Proof. Let $\ldots \to P_1 \to P_0 \to M \to 0$ be a projective resolution and let $\ldots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \to M \to 0$ be a special Gorenstein projective resolution of M (i.e. $G_0 \to M$, $G_i \to Ker g_{i-1}$ are Gorenstein projective precovers and $Ker g_i$ has finite projective dimension for each $i \geq 0$). Let $P = \ldots \to P_1 \to P_0 \to 0$ and $G = \ldots \to G_1 \to G_0 \to 0$. Then $P \to (0 \to M \to 0)$ is a surjective DG-projective resolution, and $G \to (0 \to M \to 0)$ is a special Gorenstein projective resolution. If $u : P \to G$ is induced by 1_M then by [12] Proposition 1, the cohomology modules $H^n\mathcal{H}om(M(u), N)$ are the usual Tate cohomology modules $\widehat{Ext}_R^n(M, N)$ for $n > 0$; by Remark 5, $H^n\mathcal{H}om(G, N)$ are the usual Gorenstein cohomology groups. □

We show that if R is a Gorenstein ring then a $Hom(GorProj, -)$ exact sequence of complexes $0 \to M' \to M \to M" \to 0$ gives a long exact sequence $\ldots \to \overline{Ext}^n(M", -) \to \overline{Ext}^n(M, -) \to \overline{Ext}^n(M', -) \to \overline{Ext}^{n+1}(M", -) \to \ldots$.

We will use the following result (the Horseshoe lemma for Gorenstein precovers).

Lemma 5. *(Horseshoe Lemma) Let R be a Gorenstein ring. If $0 \to M' \xrightarrow{l} M \xrightarrow{h} M" \to 0$ is a $Hom(GorProj, -)$ exact sequence of complexes then there is an exact sequence $0 \to F' \to F' \oplus F" \to F" \to 0$ with $F' \to N'$, $F' \oplus F" \to N$ and $F" \to N"$ special Gorenstein projective precovers.*

Proof. Let $F' \xrightarrow{\phi'} M'$ and $F" \xrightarrow{\phi"} M"$ be special Gorenstein projective precovers. By hypothesis the sequence $Hom(F", M) \to Hom(F", M") \to 0$ is exact. So there exists $u \in Hom(F", M)$ such that $\phi" = hu$.
Let $\phi : F' \oplus F" \to M$ be given by $\phi_n(x, y) = l_n\phi'_n(x) + u_n(y)$.
We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \longrightarrow & F' \oplus F'' & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\
0 & \longrightarrow & M' & \xrightarrow{l} & M & \xrightarrow{h} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exact sequence $0 \to Ker\phi' \to Ker\phi \to Ker\phi" \to 0$ with both $Ker\phi'$ and $Ker\phi"$ of finite projective dimension gives that $Ker\phi$ has finite projective dimension. So $F' \oplus F" \xrightarrow{\phi} M$ is a special Gorenstein projective precover.

□

Proposition 10. *Let $0 \to M' \xrightarrow{l} M \xrightarrow{h} M" \to 0$ be a $Hom(GorProj, -)$ exact sequence. Then for each complex N we have an exact sequence $\ldots \to Ext^n_{\mathcal{G}}(M", N) \to Ext^n_{\mathcal{G}}(M, N) \to Ext^n_{\mathcal{G}}(M', N) \to Ext^{n+1}_{\mathcal{G}}(M", N) \to \ldots$*

Proof. By the Horseshoe Lemma there exists an exact sequence $0 \to F' \to F' \oplus F" \to F" \to 0$ with $F' \to M'$, $F' \oplus F" \to M$ and $F" \to M"$ special Gorenstein projective precovers. For each complex N we have an exact sequence $0 \to \mathcal{H}om(F", N) \to \mathcal{H}om(F' \oplus F", N) \to \mathcal{H}om(F', N) \to 0$ and therefore an exact sequence: $\ldots \to H^n\mathcal{H}om(F", N) \to H^n\mathcal{H}om(F' \oplus F", N) \to H^n\mathcal{H}om(F', N) \to \ldots$.

□

The same argument as in Lemma 5 gives:

Lemma 6. *If* $0 \to M' \to M \to M" \to 0$ *is a* $Hom(DGProj, -)$ *exact sequence then there exists an exact sequence* $0 \to P' \to P' \oplus P" \to P" \to 0$ *with* $P' \to M'$, $P' \oplus P" \to M$, *and* $P" \to M"$ *surjective DG-projective resolutions.*

Proposition 11. *If* $0 \to M' \to M \to M" \to 0$ *is a* $Hom(GorProj, -)$ *exact sequence then for each complex* N *we have an exact sequence:* $\ldots \to \overline{Ext}^n(M", N) \to \overline{Ext}^n(M, N) \to \overline{Ext}^n(M', N) \to \overline{Ext}^{n+1}(M", N) \to \ldots$.

Proof. By Lemma 5 and Lemma 6 we have commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{j} & P' \oplus P'' & \xrightarrow{\pi} & P'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\
0 & \longrightarrow & M' & \xrightarrow{l} & M & \xrightarrow{h} & M'' & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G' & \xrightarrow{i} & G' \oplus G'' & \xrightarrow{p} & G'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{l} & M & \xrightarrow{h} & M'' & \longrightarrow & 0
\end{array}$$

with $P' \xrightarrow{\phi'} M'$, $P' \oplus P" \xrightarrow{\phi} M$, $P" \xrightarrow{\phi"} M"$ surjective DG-projective resolutions, and with $G' \xrightarrow{\tau'} M'$, $G' \oplus G" \xrightarrow{\tau} M$ and $G" \xrightarrow{\tau"} M"$ special Gorenstein projective precovers.
Since P' is Gorenstein projective there exists a map of complexes $u : P' \to G'$ such that $\tau' u = \phi'$. Let $\alpha : P" \to M$ be given by $\alpha_j(y) = \phi_j(0, y)$. Since $P"$ is Gorenstein projective and $G' \oplus G" \xrightarrow{\tau} M$ is a Gorenstein projective precover there exists $\beta : P" \to G' \oplus G"$ such that $\tau\beta = \alpha$.
Let $\omega : P' \oplus P" \to G' \oplus G"$ be defined by $\omega_j(x, y) = (u_j(x), 0) + \beta_j(y)$.

We have an exact sequence of complexes $0 \to M(u) \xrightarrow{(i,j)} M(\omega) \xrightarrow{(p,\pi)} M(p\beta) \to 0$ (where $p : G' \oplus G" \to G"$, $p_n(x, y) = y$, $\pi : P' \oplus P" \to P"$, $\pi_n(z, t) = t$, $i : G' \to G' \oplus G"$, $i_n(x) = (x, 0)$, $j : P' \to P' \oplus P"$,

$j_n(y) = (y, 0))$.
The sequence is split exact in each degree, so for each complex N there is an exact sequence $0 \to \mathcal{H}om(M(p\beta), N) \to \mathcal{H}om(M(\omega), N) \to \mathcal{H}om(M(u), N) \to 0$, and therefore an exact sequence
$\ldots \to \overline{Ext}^n(M", N) \to \overline{Ext}^n(M, N)\overline{Ext}^n(M', N') \to \overline{Ext}^{n+1}(M, N) \to \ldots$.

∎

In [14] Veliche defined Tate cohomology functors for complexes of finite Gorenstein projective dimension over arbitrary rings.

Definition 20. *([14], Definition 4.1) Let M be a complex of finite Gorenstein projective dimension, let $T \to P \to M$ be a complete resolution of M and let N be an arbitrary complex. For each integer n, the nth Tate cohomology group is defined by $\widehat{Ext}^n_R(M, N) = H^n \mathcal{H}om(T, N)$.*

Let R be a Gorenstein ring. We show that for a bounded complex $M : 0 \to M_n \to \ldots \to M_0 \to 0$, we have $\overline{Ext}^j(M, N) \simeq \widehat{Ext}^j(M, N)$ for any $j > n$, for any module N.

We show first

Lemma 7. *Let R be a Gorenstein ring. A bounded complex $M(n) = 0 \to M_n \to \ldots \to M_0 \to 0$ has a Gorenstein projective precover $G(n) \to M(n)$ with $G(n) = 0 \to G_k \xrightarrow{g_k} \ldots \to G_0 \to 0$, such that G_i is projective for $i > n$, such that $L^n = Ker(G(n) \to M(n))$ is a complex of finite projective dimension, and $L^{n+1}_k = L^n_k$ for $0 \le k \le n$.*

Proof. Proof by induction on n:

Case $n = 0$. Then $M = 0 \to M_0 \to 0$
Let $0 \to G_k \to G_{k-1} \xrightarrow{g_k} \ldots \to G_0 \xrightarrow{g_0} M_0 \to 0$ be a special Gorenstein projective resolution of M_0, and let $G = \ldots \to G_1 \to G_0 \to 0$.

Then $G \to M$ is a special Gorenstein projective precover and $Ker(G \to M)$ is a complex of finite projective dimension.

Case $n \to n + 1$. Let $M = 0 \to M_{n+1} \xrightarrow{l_{n+1}} M_n \to \ldots M_1 \xrightarrow{l_1} M_0 \to 0$.
There is a map of complexes:

$$\begin{array}{cccccccccccc}
M' : & & 0 & \longrightarrow & M_{n+1} & \longrightarrow & 0 & & & & & \\
\downarrow l & & & & \downarrow l_{n+1} & & & & & & & \\
\overline{M} : & & 0 & \longrightarrow & M_n & \xrightarrow{l_n} & \cdots & \longrightarrow & M_1 & \xrightarrow{l_1} & M_0 & \longrightarrow 0
\end{array}$$

By induction hypothesis there is a special Gorenstein projective precover $\overline{G} \xrightarrow{\alpha} \overline{M}$ where $\overline{G} = \ldots \xrightarrow{g_{n+1}} G_n \xrightarrow{g_n} G_{n-1} \to \ldots \xrightarrow{g_0} G_0 \to 0$ is a bounded complex with G_i projective for $i \geq n+1$.

Let $G' = \ldots \to G'_0 \xrightarrow{g'_0} M_{n+1} \to 0$ be a special Gorenstein projective resolution of M_{n+1}. Then G_i is projective for all $k \geq 1$. The map $M' \xrightarrow{l} \overline{M}$ induces a map of complexes $u : G' \to \overline{G}$.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & G'_1 & \longrightarrow & G'_0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow u_1 & & \downarrow u_0 & & & & & & \\
\cdots & \longrightarrow & G_{n+1} & \longrightarrow & G_n & \xrightarrow{g_n} & G_{n-1} & \longrightarrow & G_{n-2} & \longrightarrow & \cdots
\end{array}$$

such that the diagram

$$\begin{array}{ccc}
G' & \xrightarrow{g} & M' \\
\downarrow u & & \downarrow l \\
\overline{G} & \xrightarrow{\alpha} & \overline{M}
\end{array}$$

is commutative. In particular, $\alpha_n u_0 = l_{n+1} g'_0$.

Let G be the mapping cone of u:

$G = \ldots \to G_{n+1} \oplus G'_0 \xrightarrow{\overline{g}_{n+1}} G_n \xrightarrow{g_n} G_{n-1} \to \ldots \to G_0 \to 0$ with $\overline{g}_{n+1}(x,y) = g_{n+1}(x) + u_0(y)$ and $\overline{g}_{n+k}(x,y) = (g_{n+k}(x) + u_k(y), -g'_k(y))$ for $k \geq 1$.

We show that $G \to M$ is a special Gorenstein projective precover (with G_i projective for $i > n+1$).

Let A be a Gorenstein projective complex and let $\omega : A \to M$; this gives a map of complexes $\omega : \overline{A} \to \overline{M}$ where $\overline{A} = 0 \to A_n \to \ldots \to A_0 \to 0$.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega_n} & & \downarrow{\scriptstyle \omega_{n-1}} & & & & \downarrow{\scriptstyle \omega_1} & & \downarrow{\scriptstyle \omega_0} & & \\
0 & \longrightarrow & M_n & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & 0
\end{array}$$

Since $\overline{G} \xrightarrow{\alpha} \overline{M}$ is a Gorenstein projective precover there exists a map $\gamma : \overline{A} \to \overline{G}$ such that $\alpha \circ \gamma = \omega$.

Since $G'_0 \xrightarrow{g'_0} M_{n+1}$ is a Gorenstein projective precover and A_{n+1} is a Gorenstein projective module, there exists $t_0 : A_{n+1} \to G'_0$ such that $g'_0 t_0 = \omega_{n+1}$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+2} & \xrightarrow{a_{n+2}} & A_{n+1} & \xrightarrow{a_{n+1}} & A_n & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle (0,t_0)} & & \downarrow{\scriptstyle \gamma_n} & & \\
\cdots & \longrightarrow & G_{n+2} \oplus G'_1 & \xrightarrow{\overline{g}_{n+2}} & G_{n+1} \oplus G'_0 & \xrightarrow{\overline{g}_{n+1}} & G_n & \xrightarrow{g_n} & \cdots \\
 & & & & \downarrow{\scriptstyle (0,g'_0)} & & \downarrow{\scriptstyle \alpha_n} & & \\
 & & 0 & \longrightarrow & M_{n+1} & \xrightarrow{l_{n+1}} & M_n & \longrightarrow & \cdots
\end{array}$$

Then $(0, g'_0)(0, t_0) = \omega_{n+1}$ and $\alpha_n \overline{g}_{n+1} = l_{n+1}(0, g'_0)$.

Both $\overline{G}$ and G' are bounded, so G is also a bounded complex. Since $H_j(\overline{G}) = 0 = H_j(G')$ for any $j \geq n+1$ it follows that $H_j(G) = 0$ for $j \geq n+1$. Then the complex $\ldots \to G_{n+3} \oplus G'_2 \xrightarrow{\overline{g}_{n+3}} G_{n+2} \oplus G'_1 \xrightarrow{\overline{g}_{n+2}} Im\overline{g}_{n+2} \to 0$ is exact, bounded, with $G_{n+k} \oplus G'_{k-1}$ projective for $k \geq 2$. It follows that $Im\overline{g}_j$ has finite projective dimension for any $j > n+1$.

Since $g_n(\gamma_n a_{n+1} - \overline{g}_{n+1}(0, t_0)) = 0$ and also, $\alpha_n(\gamma_n a_{n+1} - \overline{g}_{n+1}(0, t_0)) = \omega_n a_{n+1} - l_{n+1}(0, g'_0)(0, t_0) = 0$, we have $\gamma_n a_{n+1} - \overline{g}_{n+1}(0, t_0) : A_{n+1} \to Ker g_n \bigcap Ker \alpha_n = Ker g_n \bigcap L^n_n$. (4)

By induction hypothesis $L^n = \ldots L^n_{n+1} \xrightarrow{g_{n+1}} L^n_n \xrightarrow{g_n} L^n_{n-1} \to \ldots$ has finite projective dimension. Since A_{n+1} is Gorenstein projective, the complex $Hom(A_{n+1}, L^n)$ is exact. By (4), $\gamma_n a_{n+1} - \overline{g}_{n+1}(0, t_0) = g_{n+1}h$ for some $h : A_{n+1} \to L_{n+1} \subset G_{n+1}$. Thus $\gamma_n a_{n+1} = \overline{g}_{n+1}\gamma_{n+1}$ with $\gamma_{n+1} : A_{n+1} \to G_{n+1} \oplus G'_0$, $\gamma_{n+1} = (h, t_0)$.

We have $\gamma_{n+1}a_{n+2} : A_{n+2} \to Ker\overline{g}_{n+1} = Im\overline{g}_{n+2}$. Since the complex:
$\ldots \to G_{n+k} \oplus G'_{k-1} \to \ldots \to G_{n+2} \oplus G'_1 \xrightarrow{\overline{g}_{n+2}} Im\overline{g}_{n+2} \to 0$
has finite projective dimension and A_{n+2} is Gorenstein projective, it follows that $\ldots \to Hom(A_{n+2}, G_{n+2} \oplus G'_1) \to Hom(A_{n+2}, Img_{n+2}) \to 0$ is exact. Then $\gamma_{n+1}a_{n+2} = \overline{g}_{n+2}\gamma_{n+2}$ for some $\gamma_{n+2} \in Hom(A_{n+2}, G_{n+2} \oplus G'_1)$. Similarly there exists γ_{n+k} such that $\gamma_{n+k}a_{n+k+1} = \overline{g}_{n+k+1}\gamma_{n+k+1}$ for $k \geq 2$.

Since $L^{n+1} = \ldots G_{n+2} \oplus G'_1 \xrightarrow{\overline{g}_{n+2}} G_{n+1} \oplus Kerg'_0 \to Ker\alpha_n \to \ldots \to Ker\alpha_0 \to 0$ and $L^n = \ldots \to G_{n+2} \to G_{n+1} \to Ker\alpha_n \to \ldots \to Ker\alpha_0 \to 0$ we have $L^n_k = L^{n+1}_k$ for $0 \leq k \leq n$.

Since $\frac{L^{n+1}}{L^n} \simeq \ldots \to G'_2 \xrightarrow{g'_2} G'_1 \xrightarrow{g'_1} Kerg'_0 \to 0$, the module $Kerg'_0$ has finite projective dimension and G'_j is projective for each $j \geq 1$, it follows that $\frac{L^{n+1}}{L^n}$ has finite projective dimension. By induction hypothesis L^n has finite projective dimension. The exact sequence $0 \to L^n \to L^{n+1} \to \frac{L^{n+1}}{L^n} \to 0$ gives that L^{n+1} has finite projective dimension. □

Proposition 12. *Let R be a Gorenstein ring. If $M = 0 \to M_n \to M_{n-1} \to \ldots \to M_1 \to M_0 \to 0$ is a bounded complex then $\overline{Ext}^j(M, N) \simeq \widehat{Ext}^j(M, N)$ for $j > n$, for any module N.*

Proof. By Lemma 7 there is a special Gorenstein projective precover $G \to M$ with $G = 0 \to G_g \to \ldots \to G_0 \to 0$ a bounded complex and such that G_j is projective for all $j \geq n+1$. By [14], 1.3.4, there exists a surjective DG-projective resolution $P \to M$ with $P = \ldots \to P_1 \to P_0 \to 0$. Since P is Gorenstein projective there is a map of complexes $u : P \to G$ such that the diagram

$$\begin{array}{ccccccccccccc}
P = \ldots & \longrightarrow & P_{g+1} & \longrightarrow & P_g & \xrightarrow{f_g} & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
\downarrow & & & & \downarrow{\scriptstyle u_g} & & & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \\
G = \ldots & \longrightarrow & 0 & \longrightarrow & G_g & \xrightarrow[g_g]{} & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 0
\end{array}$$

is commutative.

Let $M(u) = \ldots \rightarrow P_g \xrightarrow{f_g} G_g \oplus P_{g-1} \xrightarrow{\delta_{g-1}} \ldots \rightarrow G_1 \oplus P_0 \xrightarrow{\delta_0} G_0 \rightarrow 0$ be the mapping cone.

The sequence $0 \rightarrow G \rightarrow M(u) \rightarrow P[1] \rightarrow 0$ is exact, $H_n(G) = H_n(M) = H_n(P)$ for all n, so $M(u)$ is exact. Since $M(u)$ is a right bounded exact complex of Gorenstein projective modules it follows that $Ker\delta_j$ is Gorenstein projective for all j.
Let $\overline{M} = \ldots \rightarrow G_{n+2} \oplus P_{n+1} \rightarrow G_{n+1} \oplus P_n \rightarrow Im\delta_n \rightarrow 0$. Each P_l is projective and G_j is projective for $j > n$, so $G_j \oplus P_{j-1}$ is projective for any $j > n$.
Since $Im\delta_n$ is a Gorenstein projective module, there is a $Hom(-, Proj)$ exact exact sequence $0 \rightarrow Im\delta_n \rightarrow U_{n-1} \rightarrow U_{n-2} \rightarrow \ldots$ with each U_j a projective module..
Let $M' = \ldots G_{n+1} \oplus P_n \rightarrow U_{n-1} \rightarrow U_{n-2} \rightarrow \ldots$ Then M' is an exact complex of projective modules that is also $Hom(-, Proj)$ exact.

The map of complexes $M(u) \rightarrow P[1]$ gives an R-homomorphism $\omega : Im\delta_n \rightarrow Imf_n$. Since $0 \rightarrow Im\delta_n \rightarrow U_{n-1} \rightarrow U_{n-2} \rightarrow \ldots$ is a $Hom(-, Proj)$ exact sequence there are homomorphisms $\omega_j : U_j \rightarrow P_j$ for all $j \leq n-1$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Im\ \delta_n & \xrightarrow{i} & U_{n-1} & \xrightarrow{u_n} & U_{n-2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega_{n-1}} & & \downarrow{\scriptstyle \omega_{n-2}} & & \\
0 & \longrightarrow & Im\ f_n & \xrightarrow{j} & P_{n-1} & \xrightarrow[f_{n-1}]{} & P_{n-2} & \longrightarrow & \cdots
\end{array}$$

is commutative.

So there exists an exact complex M' of projective modules that is $Hom(-, Proj)$ exact, and a map of complexes $M' \xrightarrow{\pi} P[1]$, with $\pi_j = \omega_j$ for $j \leq n-1$, $\pi_l(x, y) = y$ for $l \geq n$ and with $\pi_j = 1_j$ for all $j \geq g+1$ (because $G_j = 0$ for $j \geq g+1$). Then $M'[-1] \rightarrow P \rightarrow M$ is a complete resolution.

Since $M'_j = M(u)_j$ for $j \geq n+1$ it follows that $\widehat{Ext}^j(M, N) \simeq \overline{Ext}^j(M, N)$ for any $j \geq n+1$, for any module N.

$\square$

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