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Generalized Matching Preclusion in Bipartite Graphs

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Abstract

The *matching preclusion number* of a graph with an even number of vertices is the minimum number of edges whose deletion results in a graph that has no perfect matchings. For many interconnection networks, the optimal such sets are precisely sets of edges incident to a single vertex, whose deletion creates an isolated vertex, which is an obstruction to the existence of a perfect matching. The *conditional matching preclusion number* of a graph was introduced to look for obstruction sets beyond these, and it is defined as the minimum number of edges whose deletion results in a graph with neither isolated vertices nor perfect matchings. In this paper we generalize this concept to get a hierarchy of stronger matching preclusion properties in bipartite graphs, and completely characterize such properties of complete bipartite graphs and hypercubes.

Keywords: Interconnection network; perfect matching; bipartite graph

1 Introduction

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. So if a graph has a perfect matching, then it has an even number of vertices. In this paper we only consider graphs with an even number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without perfect matchings. Any such optimal set is called an *optimal matching preclusion set*. We note that $\mathrm{mp}(G) = 0$ if G has no perfect matchings. This concept of matching preclusion was introduced by Brigham et al. [3] and further studied in [9, 6] as a measure of robustness in the event of edge failures in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [3] for details and additional references. The idea of studying the effect of deleting edges on maximum matchings has been considered prior to [3]. For example, Hung et al. [13] studied the most "vital" edges of a matching in a bipartite graph, that is, those edges whose individual removal results in the largest decrease of the objective function value in the corresponding weighted matching problem. It turns out that this problem can be obtained from the dual solution (by linear programming) as observed by Volgenant [23] in reference of [15]. More recently, Zenklusen [25] studied matching interdiction and [21, 26] studied d-blockers of a graph. In particular, a d-blocker is a set of edges whose deletion decreases the cardinality of a maximum matching by at least d. So a 1-blocker corresponds to a matching preclusion set if the underlying graph has a perfect matching. Algorithmic aspects of finding optimal 1-blockers have been considered by Boros et al. [2].

Distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. In certain applications every vertex requires a special partner at any given time, and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [3]. Hence in these interconnection networks it is desirable to have the property that the only optimal matching preclusion sets are those whose elements are incident to a single vertex. Indeed, since one common criterion in interconnection networks is regularity, we are only interested in connected regular graphs. Clearly, deleting all edges incident to a single vertex will give an isolated vertex, so the following result is immediate:

Proposition 1.1. *If G is a graph with an even number of vertices, then $mp(G) \le \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

If $\mathrm{mp}(G) = \delta(G)$, then G is called *maximally matched*. We call an optimal matching preclusion set whose deletion isolates a vertex in G a *trivial optimal matching preclusion set*. As mentioned above, it is desirable for an interconnection network to have only trivial optimal matching preclusion sets, so we call a graph G *super matched* if every optimal matching preclusion set is trivial (and thus $\mathrm{mp}(G) = \delta(G)$, so G is also maximally matched). Note that this condition implies that the graph has a perfect matching unless it has an isolated vertex. Most classes of interconnection networks have too many vertices to consider practical algorithms on these graphs (for example, the hypercube Q_n has 2^n vertices, so one usually cannot find a polynomial time algorithm in terms of n). So instead of finding algorithms to compute a particular measure such as the matching preclusion number of a graph (for which the corresponding decision problem has recently been shown to be NP-complete, see [12]), researchers in this area usually are only interested in graphs that have the best possible value of the given measure, for example, super matched graphs. This naturally leads to the desire of finding necessary and sufficient conditions that are easy to check, and if it is difficult to do so, one wants a "good" sufficient condition. It turns out that a classical result of Plesník [20] gives such a result for graphs to be maximally matched, and [4, 8] extended this result in other contexts.

In the event of random link failures it is unlikely in a distributed system that all edges incident to a single vertex fail simultaneously. Hence it is natural to require that the *faulty graph* (i.e., the graph obtained by deleting the failed edges) has no isolated vertices, and ask what the obstruction sets are for the graph to have a perfect matching. This motivates the following definition given in [7]: The *conditional matching preclusion number* of a super matched graph G, denoted by $\mathrm{mp}_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and no perfect matchings. Any such optimal set is called an *optimal conditional matching preclusion set*. We note that $\mathrm{mp}_1(G) = 0$ if G has no perfect matchings (assuming G has no isolated vertices), and we will leave $\mathrm{mp}_1(G)$ undefined if a conditional matching preclusion set does not exist, that is, we cannot delete edges to satisfy both conditions in the definition. We remark that in our definition we only consider the conditional matching preclusion problem for graphs that are super matched. Although it is not necessary to do so, we believe this approach is more natural in light of the later generalization of these concepts, because it results in a nested hierarchy of stronger and stronger matching preclusion properties.

If we delete edges so that the resulting graph has no isolated vertices, then a basic obstruction to a perfect matching will be the existence of a path u–v–w in the resulting graph where the degree of u and the degree of w are 1. (In other words, the neighborhood of the two vertices u and w consists of only one vertex, v.) To produce such an obstruction set, one can pick any path u–v–w in the original graph and delete all the edges incident to either u or w but not to v. Accordingly we define

$$\nu_e(G) = \min\{d_G(u) + d_G(w) - 2 - y_G(u, w) \colon u \text{ and } w \text{ are ends of a 2-path}\},$$

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where $d_G(u)$ is the degree of vertex u and $y_G(u,w) = 1$ if u and w are adjacent and 0 otherwise. (We will suppress G and simply write $d(u)$ and $y(u,w)$ if it is clear from the context.) So mirroring Proposition 1.1, we have the following easy result.

Proposition 1.2. *If G is a super matched graph with an even number of vertices and $\delta(G) \geq 3$, then $mp_1(G) \leq \nu_e(G)$.*

Note that the degree condition $\delta(G) \geq 3$ in Proposition 1.2 simply ensures that deleting all edges incident to either u or w but not to v of any path u–v–w does not create an isolated vertex (which can happen when vertices u and w have a common neighbor of degree 2 different from v). In fact, it is easy to show that Proposition 1.2 holds even when $\delta(G) = 2$ except for the 4-cycle, for which the conditional matching preclusion number is undefined.

If G is super matched and $\mathrm{mp}_1(G) = \nu_e(G)$, then G is called *conditionally maximally matched*, and an optimal conditional matching preclusion set whose deletion creates two vertices whose neighborhood is one vertex (as in the definition of $\nu_e(G)$) is called a *trivial optimal conditional matching preclusion set*. As mentioned earlier, the matching preclusion number measures the robustness of this requirement in the event of link failures, so it is desirable for an interconnection network to be super matched. Similarly, it is desirable to have the property that all optimal conditional matching preclusion sets are trivial as well. Such an interconnection network is called *conditionally super matched*. This concept was introduced in [7], where the conditional matching preclusion problem was considered for a number of basic networks including hypercubes, and it was proved that they have this desired property. We have remarked earlier that we only consider the conditional matching preclusion problem for super matched graphs. One question is what happens without this restriction: Can we find a graph that is not super matched but satisfies the other conditions of conditionally super matched? The answer is yes: The 6-cycle C_6 is such an example. Since the purpose of considering the conditional matching preclusion problem is to determine which super matched graphs are comparatively more resilient, it is natural to exclude such examples and only consider super matched graphs as we have done here. More importantly, this way we develop a nested hierarchy of successively stronger matching preclusion properties.

Since the introduction of the concepts of matching preclusion and conditional matching preclusion, a lot of research has been done in this area for various classes of interconnection networks in [9, 6, 10, 11, 18, 19, 17, 14, 5] and sufficient conditions have been given for general graphs in [8]. In this paper we generalize this concept to get a hierarchy of stronger matching preclusion properties in bipartite graphs, and determine such properties of the complete bipartite graphs and the hypercubes. In Section 2 we introduce the necessary concepts, in Section 3 we examine complete bipartite graphs, and in Section 4 we prove our main results about hypercubes.

2 Generalized matching preclusion

A 2-regular connected bipartite graph is an even cycle, so we consider only graphs that are at least 3-regular. We use standard graph theory terminology. In bipartite graphs there is one obvious impediment to finding perfect matchings. Given a bipartite graph H with bipartition (V_1, V_2), consider a subset of vertices W from one part. Let $N_H(W)$ be the set of vertices that are adjacent to a vertex in W. (We suppress H if it is clear from the context.) If $|N_H(W)| < |W|$, then there is clearly no possibility of forming a perfect matching, since every matching would have to omit at least one vertex in W. Such a set W is called an *obstruction set.* (Note that a matching preclusion set is a set of edges in the original graph, while an obstruction set is a set of vertices showing the non-existence of a perfect matching after a matching preclusion set has been deleted.) A well-known result regarding obstruction sets is the following corollary of Hall's Theorem [16]:

Theorem 2.1. *A bipartite graph H has a perfect matching if and only if there are no obstruction sets in it.*

Another easy well-known corollary to Hall's Theorem is that the edges of a k-regular bipartite graph can be partitioned into k perfect matchings. This fact immediately implies the following: If G is a k-regular bipartite graph, then G is maximally matched, that is, $\mathrm{mp}(G) = k$.

Let H be a bipartite graph. If W is an obstruction set in H, we call W a $(|W|, |N_H(W)|)$-*obstruction set.*

So deleting edges of a trivial matching preclusion set creates a $(1, 0)$-obstruction set, and deleting a trivial conditional matching preclusion set creates a $(2, 1)$-obstruction set without creating $(1, 0)$-obstruction sets in the resulting graph.

We will now look at the matching preclusion problem from a different perspective. We want to delete a minimum sized set of edges from a bipartite graph G with bipartition (V_1, V_2) to destroy all perfect matchings. By Theorem 2.1, we need to create an obstruction set, i.e., choose (W, U) with $|W| > |U|$ such that either $W \subseteq V_1, U \subseteq V_2$ or $W \subseteq V_2, U \subseteq V_1$, and choose the set F of edges to be deleted to be $F = \delta_G(W) - \delta_G(W, U)$, where $\delta_G(W, U)$ is the set of edges between W and U, and $\delta_G(W)$ is the notational simplification of $\delta_G(W, \overline{W})$ ($\overline{W}$ is the complement of W with respect to the set of vertices). Note that this should cause no confusion with the use of $\delta(G)$ to mean the minimum degree of G.

We will need the following proposition:

Proposition 2.1. *Let $G = (V, E)$ be a bipartite graph, and let $F \subseteq E$ be a matching preclusion set in G. If $G - F$ has an (a, b)-obstruction set (where $a > b$), then there exists $F' \subseteq F$ such that $G - F'$ has a $(b + 1, b)$-obstruction set.*

Proof. Let G have bipartition (V_1, V_2), and assume that $G - F$ contains an (a, b)-obstruction set W, so $|W| = a > b = |N_{G-F}(W)|$. By symmetry we may assume that $W \subseteq V_1$. Clearly $\delta_G(W, V_2 - N_{G-F}(W)) \subseteq F$. For each vertex in $N_{G-F}(W)$ pick a vertex adjacent to it in W, then pick more vertices of W until we have picked $b + 1$ vertices in total. Let W' be the set of these chosen vertices, and let $F' = \delta_G(W', V_2 - N_{G-F}(W)) \subseteq F$. Clearly $N_{G-F'}(W') = N_{G-F}(W)$, so W' is a $(b+1, b)$-obstruction set in $G - F'$, and $F' \subseteq F$, proving the claim. □

Proposition 2.1 implies that if we want to find $F \subseteq E$ of smallest size in a bipartite graph G such that $G-F$ has no perfect matchings, we only need to consider creating $(a, a-1)$-obstruction sets. This observation motivates us to introduce the following concepts. Let $G = (V, E)$ be a bipartite graph and $a \geq 1$ an integer. A set $F \subseteq E$ is a *matching preclusion set of order a* if $G-F$ has no perfect matchings and it does not have any $(b, b-1)$-obstruction sets for all positive b such that $b \leq a-1$ (it is possible that it has no $(a, a-1)$-obstruction set either, only a $(c, c-1)$-obstruction set for some $c > a$). The *matching preclusion problem of order* a is to find a smallest matching preclusion set of order a; the size of such a set is the *matching preclusion number of order* a. Ideally, we would want a smallest matching preclusion set of order a to have the property that the deletion of its edges does create an $(a, a-1)$-obstruction set. Such a matching preclusion set will be called *trivial*. We call G *maximally matched of order* a if the matching preclusion problem of order a is solved by a trivial matching preclusion set of order a. Graph G is *super matched of order* a if, in addition, every optimal solution to the matching preclusion problem of order a is a trivial matching preclusion set of order a. These definitions, as given here, allow a graph to be not super matched of order $a-1$ but still maximally matched of order a. (For example, a 6-cycle is not super matched of order 1 but it is maximally matched of order 2 and super matched of order 2.) Since it is desirable to have these to be successively stronger properties, we only want to investigate whether a graph is maximally matched of order a if it is also super matched of order $1, 2, \ldots, a-1$. So we call a graph *strongly maximally matched of order* a if it is super matched of order $1, 2, \ldots, a-1$ and maximally matched of order a. Similarly we call a graph *strongly super matched of order* a if it is super matched of order $1, 2, \ldots, a-1, a$. Thus strongly maximally matched of order a and strongly super matched of order a for $a = 1, 2, \ldots$ will give a nested hierarchy of successively stronger matching preclusion properties, as we wanted.

Note that the earlier notion of maximally matched is the same as maximally matched of order 1, and similarly, super matched is the same as super matched of order 1, conditionally maximally matched is the same as strongly maximally matched of order 2, and conditionally super matched is the same as strongly super matched of order 2. We will study this problem first for complete bipartite graphs in Section 3 and then for hypercubes in Section 4, and give a complete characterization of when they are strongly maximally matched of order a and strongly super matched of order a.

3 Complete bipartite graphs

Although the complete bipartite graph $K_{n,n}$ is not a suitable interconnection network, it is still illuminative for us to solve the generalized matching preclusion problem for this graph to gain understanding. Let $g_K(n,k)$ be the minimum number of edges whose deletion creates a $(k, k-1)$-obstruction set in $K_{n,n}$. We have the following easy result:

Proposition 3.1. *Let* $1 \leq k \leq n$. *Then* $g_K(n,k) = g_K(n, n-k+1) = k(n-k+1)$. *Moreover, if* F *is a minimum set of edges in* $K_{n,n}$ *such that the deletion of* F *creates a* $(k, k-1)$-*obstruction set, then* $K_{n,n} - F$ *contains exactly one* $(k, k-1)$-*obstruction set and exactly one* $(n-k+1, n-k)$-*obstruction set, but no other obstruction sets.*

Proof. Let the bipartition of $K_{n,n}$ be (V_1, V_2), and assume that F is a minimum set of edges in $K_{n,n}$ such that the deletion of F creates a $(k, k-1)$-obstruction set W. Thus $|W| = k$ and $|N_{K_{n,n}-F}(W)| = k-1$. By symmetry we may assume that $W \subseteq V_1$ and $N_{K_{n,n}-F}(W) \subseteq V_2$. Clearly F must contain all edges between the k vertices of W and the $n-(k-1)$ vertices in $V_2 - N_{K_{n,n}-F}(W)$ to create a $(k, k-1)$-obstruction set, so if it is minimum, it must contain exactly these edges. Therefore $g_K(n,k) = k(n-k+1)$. Notice that the neighborhood of $V_2 - N_{K_{n,n}-F}(W)$ in $K_{n,n} - F$ is exactly $V_1 - W$, so $V_2 - N_{K_{n,n}-F}(W)$ is an $(n-k+1, n-k)$-obstruction set in $K_{n,n} - F$, and it is clearly minimum, hence $g_K(n,k) = g_K(n, n-k+1)$. It remains to show that these are the only obstruction sets that are created by the deletion of F. Let Y be any obstruction set in $K_{n,n} - F$. First assume $\emptyset \neq Y \subseteq V_1$. If Y contains a vertex of $V_1 - W$, then $N_{K_{n,n}-F}(Y) = V_2$, so Y is not an obstruction set. If $Y \subseteq W$, then $|N_{K_{n,n}-F}(Y)| = |N_{K_{n,n}-F}(W)| = k-1$, so the only way for Y to be an obstruction set is if $Y = W$.

If $\emptyset \neq Y \subseteq V_2$, we can show by a similar argument that we must have $Y = V_2 - N_{K_{n,n}-F}(W)$, finishing the proof. □

Next we have the following result about how to determine whether $K_{n,n}$ is maximally or super matched of order a.

Proposition 3.2. *Let $n \geq 2$ and $a \leq \lceil \frac{n}{2} \rceil$. The graph $K_{n,n}$ is maximally matched of order a if and only if $g_K(n,a) \leq g_K(n,k)$ for all k with $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$; moreover, $K_{n,n}$ is super matched of order a if and only if $g_K(n,a) < g_K(n,k)$ for all k with $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$.*

Proof. First note that as we have seen in the proof of Proposition 3.1, deleting $g_K(n,k)$ edges appropriately to create a $(k, k-1)$-obstruction set will also create an $(n-k+1, n-k)$-obstruction set when viewed from the other side, but it will not create any other obstruction set, in particular, any $(b, b-1)$-obstruction set with $b < \min(k, n-k+1)$. This limits the possible values of a that we can consider. If n is even, say $n = 2r$, then $K_{n,n}$ cannot be maximally or super matched of order $r+1$ or higher. Indeed, deleting $g_K(n, r+1)$ appropriate edges will produce both an $(r+1, r)$-obstruction set and an $(r, r-1)$-obstruction set, so there are no matching preclusion sets of order $r+1$ (or higher) at all in $K_{n,n}$. Similarly, if n is odd, say $n = 2r+1$, then $K_{n,n}$ cannot be maximally or super matched of order $r+2$ or higher. This is why we have $a \leq \lceil \frac{n}{2} \rceil$ in the proposition. In addition, one may be tempted to think that for a fixed a, the graph $K_{n,n}$ being maximally matched of order a is equivalent to $g_K(n,a) \leq g_K(n,k)$ for all k such that $a+1 \leq k \leq n$ (with strict inequality for g_K instead for super matched of order a). The above observation also shows that we need to ignore values of $g_K(n,k)$ when $k > n-a+1$, and then because of the symmetry of the function $g_K(n,k)$, we can have $k \leq \lceil \frac{n}{2} \rceil$ in the claim.

First assume that $g_K(n,a) \leq g_K(n,k)$ for all k with $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$, and let F be a minimum matching preclusion set of order a in $K_{n,n}$. Then the deletion of F in $K_{n,n}$ must create a $(k, k-1)$-obstruction set for some $k \geq a$. Proposition 3.1 then implies that $|F| = g_K(n,k)$, and there are exactly two obstruction sets in $K_{n,n} - F$: a $(k, k-1)$-obstruction set and an $(n-k+1, n-k)$-obstruction set. From the definition of a matching preclusion set of order a, we must have $n-k+1 \geq a$. If $a \leq k \leq \lceil \frac{n}{2} \rceil$, then immediately we get $|F| = g_K(n,k) \geq g_K(n,a)$. If, on the other hand, $k > \lceil \frac{n}{2} \rceil$, then $\lceil \frac{n}{2} \rceil > n-k+1 \geq a$, so again $|F| = g_K(n, n-k+1) \geq g_K(n,a)$. Now let F be a minimum set of edges whose deletion creates an $(a, a-1)$-obstruction set in $K_{n,n}$. Then $|F| = g_K(n,a)$, and there are exactly two obstruction sets in $K_{n,n} - F$: an $(a, a-1)$-obstruction set and an $(n-a+1, n-a)$-obstruction set. Since $a \leq \lceil \frac{n}{2} \rceil$, we have $n-a+1 \geq a$, so F is a trivial matching preclusion set of order a. Since every minimum matching preclusion set of order a in $K_{n,n}$ has size at least $g_K(n,k) = |F|$, the graph $K_{n,n}$ is maximally matched of order a.

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A similar argument shows that if $g_K(n,a) < g_K(n,k)$ for all k with $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$, then $K_{n,n}$ is super matched of order a.

Now fix $a \leq \lceil \frac{n}{2} \rceil$ and assume that $K_{n,n}$ is maximally matched of order a. Let F be a set of edges in $K_{n,n}$ whose deletion produces a $(k,k-1)$-obstruction set such that $|F| = g_K(n,k)$ and $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$. (This is unique up to isomorphism.) Then $K_{n,n} - F$ contains only a $(k,k-1)$ obstruction set and an $(n-k+1, n-k)$-obstruction set. Notice that $n-k+1 \geq \lceil \frac{n}{2} \rceil \geq a$ whenever $k \leq \lceil \frac{n}{2} \rceil$, so $K_{n,n} - F$ has no $(b,b-1)$-obstruction sets for any $b < a$. Therefore F is a matching preclusion set of order a, so we must have $g_K(n,a) \leq g_K(n,k)$ by the definition of maximally matched of order a.

By a similar argument we get $g_K(n,a) < g_K(n,k)$ for all k with $a+1 \leq k \leq \lceil \frac{n}{2} \rceil$ if instead we assume that $K_{n,n}$ is super matched of order a.

□

Note that when a reaches its upper bound in Proposition 3.2, there is an interesting scenario. If $n = 2r$, Proposition 3.2 states that $K_{2r,2r}$ is maximally matched of order r if and only if $g_K(n,r) \leq g_K(n,k)$ for all $r+1 \leq k \leq r$, which is vacuously true (every matching preclusion set of order r is trivial). However, the ultimate goal is to consider the strong version, that is, we are only interested in this case if $K_{2r,2r}$ is already super matched of order $1, 2, \ldots, r-1$, and these statements have substance. We are now ready to show our result for the complete bipartite graph $K_{n,n}$.

Theorem 3.1. *Let $n \geq 2$. The graph $K_{n,n}$ is super matched of order a whenever $1 \leq a \leq \lceil \frac{n}{2} \rceil$. Moreover, it is strongly super matched of order a whenever $1 \leq a \leq \lceil \frac{n}{2} \rceil$.*

Proof. Both statements follow directly from Proposition 3.2 and the fact that $g_K(n,k) = k(n-k+1)$ (where n is a parameter) is a quadratic function in k with global maximum at $k = \frac{n+1}{2}$, so it is strictly increasing from 1 to $\frac{n+1}{2}$. □

Note that Theorem 3.1 shows that $K_{n,n}$ has the best possible behavior that a regular bipartite graph can have regarding these properties, because it is strongly super matched of order a for every value of a for which it possibly could.

4 The Hypercube

In this section we consider the generalized matching preclusion problem for the hypercube. The n-dimensional hypercube Q_n, where $n \geq 1$, is defined as follows: The vertex set is the set of binary strings of length n and two vertices are adjacent if and only if they differ in exactly one bit position. The hypercube has many well-known properties and for space considerations, we will not explicitly state them. Define $g_Q(n,k)$ as the minimum number of edges whose deletion produces a $(k,k-1)$-obstruction set in Q_n. As in the previous section, we want a formula for $g_Q(n,k)$. A function related to $g_Q(n,k)$ was considered by Yang and Lin [24], who studied a strong edge-connectivity concept. So instead of deriving $g_Q(n,k)$ from first principles here, we will use their relevant results. Given a positive integer m, they studied the maximum number of edges in a subgraph of Q_n containing m vertices. This number was shown to be related to the binary representation of m. Let $t_0 = \lfloor \log_2 m \rfloor$ and

$t_i = \left\lfloor \log_2(m - \sum_{r=0}^{i-1} 2^{t_r}) \right\rfloor$ for $i \geq 1$. For example, if $m = 26$, then $t_0 = 4, t_1 = 3, t_2 = 1$ and $26 = 2^4 + 2^3 + 2^1$. More compactly, we can simply define the t_i's as the exponents from the binary representation $m = 2^{t_0} + 2^{t_1} + \cdots + 2^{t_s}$ where $t_i > t_{i+1}$ for all $0 \leq i \leq s-1$. We note that this definition is well-defined because m has a unique binary representation.

Theorem 4.1 (Yang and Lin [24]). *Let $1 \leq m \leq 2^n$. The maximum number of edges in a subgraph of Q_n containing exactly m vertices is $f(m)/2$, where*

$$f(m) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} i 2^{t_i+1}$$

is defined by the binary representation of m: $m = 2^{t_0} + 2^{t_1} + \cdots + 2^{t_s}$ such that $t_i > t_{i+1}$ for all $0 \leq i \leq s-1$.

We consider three special cases to illustrate this theorem. If $m = 1$, then $s = 0$ and $t_0 = 0$, thus $f(1) = 0$. If $m = 2^n$, then $s = 0$ and $t_0 = n$, thus $f(2^n) = n2^n$. If $m = 2^p + 2^r$ such that $n > p > r$, then $s = 1$ and $t_0 = p$ and $t_1 = r$, thus $f(2^p + 2^r) = p2^p + r2^r + 2^{r+1}$. Note that $f(m)$ is independent of n.

Fortunately, a maximizer to Theorem 4.1 is precisely of the form that we are interested in. We now describe the maximizer (attaining $f(m)/2$ edges) as given in [24]. Given m, we write $m = 2^{t_0} + 2^{t_1} + \cdots + 2^{t_s}$ where $t_i > t_{i+1}$ for $0 \leq i \leq s-1$. The objective is to use $s+1$ subcubes of dimension $t_0, t_1, \ldots, t_s$ to obtain the desired subgraph. Here is the procedure: Let A_0 be the subgraph of Q_n induced by vertices of the form $x_1 x_2 \ldots x_{t_0} 0^{n-t_0}$; let A_1 be the subgraph of Q_n induced by vertices of the form $x_1 x_2 \ldots x_{t_1} 0^{t_0-t_1} 1 0^{n-t_0-1}$; let A_2 be the subgraph of Q_n induced by vertices of the form $x_1 x_2 \ldots x_{t_2} 0^{t_1-t_2} 1 0^{t_0-t_1-1} 1 0^{n-t_0-1}$; and so on (i.e., in A_i the first t_i bits are unrestricted, the next $t_{i-1} - t_i$ bits are zero, and the remaining bits are just like in A_{i-1}, except the first bit is 1 to make it disjoint from the previous subcubes). Notice that every vertex in A_1 has a unique neighbor in A_0, every vertex in A_2 has a unique neighbor in both A_0 and A_1, etc. Now let $G_{n,m}$ be the subgraph of Q_n induced by the vertices of $A_0, A_1, \ldots, A_s$. Note that $G_{n,m}$ is isomorphic to $G_{n+1,m}$, which follows immediately from the structure of $G_{n,m}$ as given. Hence we will often abbreviate $G_{n,m}$ to G_m for convenience.

Proposition 4.1 (Yang and Lin [24]). *$G_{n,m}$ has m vertices and $f(m)/2$ edges whenever $1 \leq m \leq 2^n$.*

Theorem 4.1 and Proposition 4.1 are precisely the pieces of information that we need to deduce the formula for $g_Q(n,k)$:

Proposition 4.2. *$g_Q(n,k) = nk - \frac{1}{2} f(2k-1)$ whenever k and n are integers such that $1 \leq 2k-1 \leq 2^n$.*

Proof. Let $m = 2k-1$, and consider the subgraph G_m in Q_n. Since m is odd, we have $t_s = 0$, thus exactly one of the smaller hypercubes that constitute G_m has exactly one vertex (A_s), and the other hypercubes have an even number of vertices. Since G_m is bipartite, its two partite sets must differ in size by 1. Let W and U denote these partite sets such that $|W| = k$ and $|U| = k-1$, and define $F = \delta_{Q_n}(W) - \delta_{Q_n}(W, U)$. After deleting the edges of F

in Q_n, all neighbors of the vertices in W will be in U, so (W, U) is a $(k, k-1)$-obstruction set in $Q_n - F$. Since Q_n is n-regular and G_m has $f(2k-1)/2$ edges, it is easy to count that $|F| = nk - f(2k-1)/2$. By Theorem 4.1, G_m maximizes the number of edges in a subgraph of Q_n with m vertices, so choosing its larger partite set to form the larger part in a $(k, k-1)$-obstruction set minimizes the number of edges needed to be deleted, hence $g_Q(n, k) = |F| = nk - f(2k-1)/2$. □

Figure 1 provides the graph of g_Q when $n = 13$.

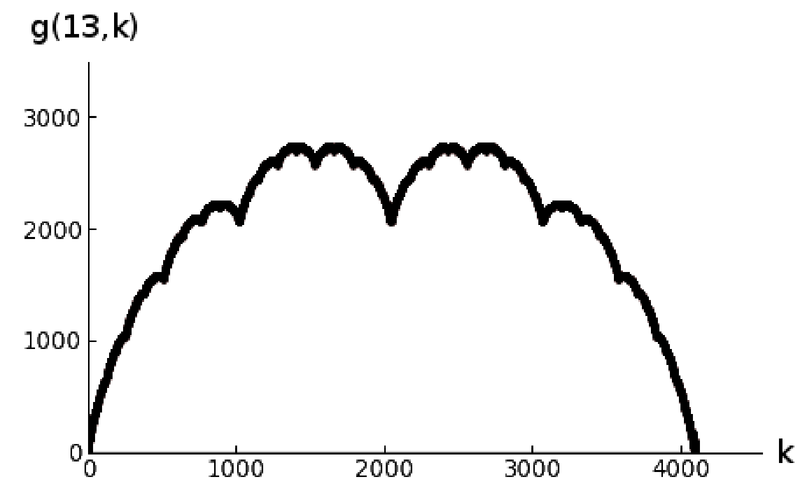

Figure 1: Plot of g_Q for $n = 13$

As before, deleting edges that create a $(k, k-1)$-obstruction set will also create a $(2^{n-1} - k + 1, 2^{n-1} - k)$-obstruction set in Q_n. So Q_n cannot be maximally matched of order k for $k > 2^{n-2}$: Indeed, this would imply by definition the existence of an obstruction set of size k, and hence there would also be a smaller one (because $k \geq 2^{n-2}+1$ implies $2^{n-1}-k+1 \leq 2^{n-2}$), which would contradict the existence of a matching preclusion set of order k. Moreover, this also shows that the graph of $g_Q(n, k)$ for fixed n is symmetric about the line $k = 2^{n-2} + \frac{1}{2}$ (see also Figure 1):

Proposition 4.3. $g_Q(n, k) = g_Q(n, 2^{n-1} - k + 1)$ *whenever* $1 \leq k \leq 2^{n-2}$.

Care must be taken regarding F in the proof of Proposition 4.2. It is tempting to say that F is a matching preclusion set of order k, but this claim requires one to check that $Q_n - F$ does not contain any $(a, a-1)$-obstruction set for every $a < k$. (In Proposition 3.1, the justification is easy for $K_{n,n}$.)

Theorem 4.2. *Let* $m = 2k - 1$ *such that* $1 \leq m \leq 2^{n-1}$, *and let* W *and* U *denote the two partite sets of* G_m *such that* $|W| > |U|$. *If* $F = \delta_{Q_n}(W) - \delta_{Q_n}(W, U)$, *then* $Q_n - F$ *does not contain any* $(a, a-1)$*-obstruction set for all* $a < k$.

Proof. From the proof of Proposition 4.2 we have $|W| = k$ and $|U| = k-1$, so F is a $(k, k-1)$-obstruction set. Note that here we need $k \le 2^{n-2}$, so $m \le 2^{n-1}$, because we must consider the smaller obstruction set of the two that is created in Q_n.

We first claim that for any $w \in W$, the subgraph H induced by $(W - \{w\}) \cup U$ has a perfect matching. First note that G_m is constructed from the subgraphs $A_0, A_1, \ldots, A_s$, the subgraph A_s contains a single vertex that belongs to W, and this singleton has a vertex adjacent to it in each of $A_0, \ldots, A_{s-1}$. Moreover, A_i is a hypercube for each $i \neq s$. Now the claim follows from the following two facts. First, every hypercube has a perfect matching, and second, every hypercube is Hamiltonian laceable, that is, there is a Hamiltonian path in it between any two vertices from different partite sets. Both facts are well-known and easy to prove (in fact, the same is true even after deleting up to $n-2$ edges in Q_n, see [22]). Note that the second fact implies that if we delete one vertex from each partite set of a hypercube, the resulting graph still has a Hamiltonian path, and hence a perfect matching. To see how these facts establish the claim, proceed as follows: if w is the unique vertex in A_s, then the claim follows from the first fact; if w is a vertex in A_i such that $0 \le i \le s-1$, then match the unique vertex in A_s to a vertex v in A_i ($v \neq w$, since v belongs to U), and use the second fact to conclude that there is a perfect matching in $A_i - \{v, w\}$, and use the first fact for the other A_i's.

Since $G_m - w$ has a perfect matching for every $w \in W$, it follows from Hall's Theorem that $|J| \le |N_{G_m}(J)|$ for every $J \subsetneq W$ and $|J| \le |N_{G_m}(J)|$ for every $J \subseteq U$. In particular, G_m has no $(a, a-1)$-obstruction sets for every $a < k$. By switching the roles of 0 and 1 in the definition of G_m, it is easy to see that $Q_n - V(G_m)$ is isomorphic to G_{2^n-m} (for simplicity we will refer to it as G_{2^n-m}). Let the partite sets of Q_n be X and Y such that $W \subseteq X$ and $U \subseteq Y$. The argument given above applies to G_{2^n-m} as well, so we have $|J| \le |N_{G_{2^n-m}}(J)|$ for every $J \subsetneq Y - U$, and $|J| \le |N_{G_{2^n-m}}(J)|$ for every $J \subseteq X - W$. In particular, G_{2^n-m} has no $(a, a-1)$-obstruction set for every $a < (2^n - m + 1)/2$. Note that $(2^n - m + 1)/2 \ge k$ as $m \le 2^{n-1}$. Thus G_{2^n-m} has no $(a, a-1)$-obstruction set for every $a < k$.

We now show that $Q_n - F$ contains no $(a, a-1)$-obstruction sets for every $a < k$. Let $A \subseteq X$ such that $|A| = a < k$. Since $a < k$, we have $A \cap W \subsetneq W$, thus $|A \cap W| \le |N_{G_m}(A \cap W)|$. Since $A \cap (X - W) \subseteq X - W$, we get $|A \cap (X - W)| \le |N_{G_{2^n-m}}(A \cap (X - W))|$. Thus $|A| = |A \cap W| + |A \cap (X - W)| \le |N_{G_m}(A \cap W)| + |N_{G_{2^n-m}}(A \cap (X - W))| \le |N_{Q_n - F}(A)|$ because of the results in the previous paragraph and that both G_m and G_{2^n-m} are subgraphs of $Q_n - F$. (The last inequality is not tight in general because of the edges between $X - W$ and U.) Thus A does not induce an $(a, a-1)$-obstruction set in $Q_n - F$. A similar argument holds if $A \subseteq Y$, and the proof is complete. $\square$

Now we have the following claim mirroring Proposition 3.2 about how to determine whether Q_n is maximally or super matched of order λ. Similarly as in Proposition 3.2, we limit the possible values of λ up to a quarter of the number of vertices in Q_n, because, as we noted earlier, Q_n cannot be maximally or super matched of order $2^{n-2} + 1$ or higher.

Proposition 4.4. *Let $n \ge 3$ and $\lambda \le 2^{n-2}$. The graph Q_n is maximally matched of order λ if and only if $g_Q(n, \lambda) \le g_Q(n, k)$ for all k with $\lambda + 1 \le k \le 2^{n-2}$; moreover, Q_n is super matched of order λ if and only if $g_Q(n, \lambda) < g_Q(n, k)$ for all k with $\lambda + 1 \le k \le 2^{n-2}$.*

Proof. The claims can be proven using an argument similar to the one used in Proposition 3.2

since Proposition 4.3 and Theorem 4.2 show that the function g_Q is symmetric about $2^{n-2}+\frac{1}{2}$ and that the minimizers create no smaller obstruction set. $\square$

Similarly to the case of the complete bipartite graph, one may hope that the best possible scenario happens here as well, that is, Q_n is strongly super matched of order 2^{n-2}. However, we will soon find out that Q_n is strongly super matched of order up to about $2^{\lceil \frac{n}{2} \rceil}$ except for very small n. This can also be inferred from Figure 1.

Before proceeding further, we state several identities of the function f that we will need in later proofs.

Lemma 4.3. *Let x and y be positive integers. We have*

1. $f(2^x) = x2^x$;
2. $f(2^x - 1) = x2^x - 2x$;
3. $f(2^x + y) = x2^x + 2y + f(y)$ *if* $1 \leq y \leq 2^x - 1$;
4. $f(2^x + y) = 2^{x+1} + x2^x + f(y)$ *if* $2^x \leq y \leq 2^{x+1} - 1$; *and*
5. $f(y+1) - f(y)$ *is twice the number of* 1*s in the binary representation of* y,

where f is the function defined in Theorem 4.1.

Proof. We use the notation as defined in Theorem 4.1. Identity (1) is obvious as $t_0 = x$ and $s = 0$. For identity (2), observe that $2^x - 1 = \sum_{i=0}^{x-1} 2^i$, hence $t_i = x - 1 - i$ for $0 \leq i \leq x - 1$. Thus

$$\begin{aligned} f(2^x - 1) &= \sum_{i=0}^{x-1} (x-1-i)2^{x-1-i} + \sum_{i=0}^{x-1} i2^{(x-1-i)+1} \\ &= \sum_{i=0}^{x-2} (x-1-i)2^{x-1-i} + \sum_{i=0}^{x-2} (i+1)2^{x-1-i} \\ &= \sum_{i=0}^{x-2} x2^{x-1-i} = x(2^x - 2) = x2^x - 2x. \end{aligned}$$

For identity (3), let $y = \sum_{i=0}^{s} 2^{t_i}$ such that $y < 2^x$ and $t_i > t_{i+1}$ for all $0 \leq i \leq s-1$. Then $2^x + y = \sum_{i=0}^{s+1} 2^{\tau_i}$, where $\tau_0 = x$ and $\tau_{i+1} = t_i$ for $0 \leq i \leq s$, therefore

$$\begin{aligned} f(2^x + y) &= \sum_{i=0}^{s+1} \tau_i 2^{\tau_i} + \sum_{i=0}^{s+1} i2^{\tau_i+1} = \tau_0 2^{\tau_0} + 2\sum_{i=0}^{s} 2^{t_i} + \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} i2^{t_i+1} \\ &= x2^x + 2y + f(y). \end{aligned}$$

Note that the case $y = 2^x - 1$ is consistent with identity (2), because (3) gives $f(2^x + 2^x - 1) = x2^x + 2(2^x - 1) + f(2^x - 1) = x2^x + 2(2^x - 1) + x2^x - 2x = (x+1)2^{x+1} - 2(x+1)$, which equals $f(2^{x+1} - 1)$ in identity (2).

For identity (4), let $y = \sum_{i=0}^{s} 2^{t_i}$ as before. Note that $t_0 = x$, so $2^x + y = \sum_{i=0}^{s} 2^{\tau_i}$ with $\tau_0 = t_0 + 1 = x + 1$ and $\tau_i = t_i$ for $1 \le i \le s$. Therefore

$$\begin{aligned} f(2^x + y) &= \sum_{i=0}^{s} \tau_i 2^{\tau_i} + \sum_{i=0}^{s} i 2^{\tau_i + 1} = (t_0 + 1) 2^{t_0+1} + \sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} i 2^{t_i+1} \\ &= 2^{t_0+1} + t_0 2^{t_0} + \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} i 2^{t_i+1} \\ &= 2^{x+1} + x 2^x + f(y). \end{aligned}$$

Note that the case $y = 2^x$ is consistent with identity (1), because (4) gives $f(2^x + 2^x) = 2^{x+1} + x2^x + f(2^x) = 2^{x+1} + x2^x + x2^x = (x+1)2^{x+1}$, which equals $f(2^{x+1})$ in identity (1).

Finally, for identity (5) note that if $y = 2^x - 1$, then the claim follows from identities (1) and (2), since $2^x - 1$ has x 1s in its binary representation. Similarly, $f(2^x+1) - f(2^x) = 2$ by identity (3), and 2^x has only one 1. For the remaining values note that any positive integer can be written as either $2^x - 1$, or 2^x, or $2^x + p$ where $1 \le p < 2^x - 1$, so use identity (3) when $1 \le y < 2^x - 1$ to get

$$\begin{aligned} f(2^x + y + 1) - f(2^x + y) &= \big(x2^x + 2(y+1) + f(y+1)\big) - \big(x2^x + 2y + f(y)\big) \\ &= 2 + f(y+1) - f(y), \end{aligned}$$

and with $f(2) - f(1) = 2$ the identity follows by induction on the number of 1s in the binary representation of y. $\square$

Next we derive properties for the function g_Q.

Lemma 4.4. *Let $n \ge 3$ and $1 \le k < 2^{n-1}$. The difference $g_Q(n, k+1) - g_Q(n, k)$ is equal to n minus the sum of the number of 0s k ends with and twice the number of 1s in k, both in its binary representation.*

Proof. Using Proposition 4.2 we get

$$\begin{aligned} g_Q(n, k+1) - g_Q(n, k) &= \left(n(k+1) - \frac{f(2k+1)}{2}\right) - \left(nk - \frac{f(2k-1)}{2}\right) \\ &= n - \frac{1}{2}\big(f(2k+1) - f(2k-1)\big) \\ &= n - \frac{1}{2}\big(f(2k+1) - f(2k)\big) - \frac{1}{2}\big(f(2k) - f(2k-1)\big). \end{aligned}$$

By identity (5) in Lemma 4.3, the term $\frac{1}{2}\big(f(2k+1) - f(2k)\big)$ is the number of 1s in the binary representation of $2k$, which is the same as the number of 1s in k. Similarly, the term $\frac{1}{2}\big(f(2k) - f(2k-1)\big)$ is the number of 1s in the binary representation of $2k-1$, and notice that this is equal to the number of 1s in k plus the number of 0s k ends with in its binary representation. This is because $2k$ ends with one more 0s than k, all of which will become 1s in the binary representation of $2k-1$, but the 1 right before the ending 0s in $2k$ turns into 0. Combining these results gives the claim. $\square$

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Corollary 4.5. *Let $n \geq 1$ be a fixed integer. The function $g_Q(n,k)$ is strictly increasing for k in the interval $[1, 2^{\frac{n}{2}} - 1]$ when n is even, and in the interval $[1, 2^{\frac{n+1}{2}} - 2]$ when n is odd.*

Proof. When n is even, Lemma 4.4 implies that the smallest positive integer k for which $g_Q(n, k+1) \leq g_Q(n,k)$ happens when k consists of $n/2$ 1s, i.e., when $k = 2^{n/2} - 1$. When n is odd, it is similarly easy to see that $g_Q(n, k+1) \leq g_Q(n,k)$ holds first when k consists of $(n-1)/2$ 1s and ends with one 0, i.e., $k = 2^{(n+1)/2} - 2$. □

We need two additional lemmas on the behavior of g_Q.

Lemma 4.6. *Let $n \geq 3$. We have $g_Q(n, 2^x) < g_Q(n, 2^y)$ for all integers x and y such that $0 \leq x < y \leq n-2$.*

Proof. From Proposition 4.2 and identity (2) in Lemma 4.3 we get

$$\begin{aligned} g_Q(n, 2^y) - g_Q(n, 2^x) &= n2^y - n2^x - \frac{1}{2}\big(f(2^{y+1} - 1) - f(2^{x+1} - 1)\big) \\ &= (n-1-y)2^y - (n-1-x)2^x + (y-x). \end{aligned}$$

The function $(n-1-x)2^x$ is increasing for x in $[0, n-2]$, so this term is maximized when $x = y - 1$, thus the previous expression is no less than

$$(n-1-y)2^y - (n-y)2^{y-1} + (y-x) = (n-2-y)2^{y-1} + (y-x) > 0,$$

since $x < y \leq n-2$. □

Lemma 4.7. *Let $n \geq 4$. We have $g_Q(n, 2^x) < g_Q(n, 2^x + y)$ for all integers x and y such that $1 \leq x \leq n-3$ and $1 \leq y < 2^x$.*

Proof. Using Proposition 4.2 and identities (2) and (3) in Lemma 4.3 we get

$$\begin{aligned} g_Q(n, 2^x + y) - g_Q(n, 2^x) &= ny - \frac{1}{2}\big(f(2^{x+1} + 2y - 1) - f(2^{x+1} - 1)\big) \\ &= (n-2)y - \frac{1}{2}f(2y-1) - x \\ &= g_Q(n-2, y) - x. \end{aligned}$$

Because Q_{n-2} is super matched of order 1 with $g_Q(n-2, 1) = n-2$ (see [3]), we get $g_Q(n-2, y) \geq g_Q(n-2, 1) = n-2 > x$, finishing the proof. □

Note that by Lemma 4.4 we have $g_Q(n, 2^x+1) - g_Q(n, 2^x) = n - (x+2)$, so the computation in Lemma 4.7 also shows that $g_Q(n, 2^x + y) - g_Q(n, 2^x + 1) = g_Q(n-2, y) - (n-2)$, so the graph of $g_Q(n,k)$ in the interval $[2^x + 1, 2^{x+1}]$ is obtained from the graph of $g_Q(n-2, k)$ on the interval $[1, 2^x]$ by a linear shift for each x, giving the graph of g_Q a fractal-like structure shown in Figure 1.

We are now ready to present our main result in this section.

Theorem 4.8. *Let $n \geq 4$. The graph Q_n is strongly maximally matched of order λ for λ in the interval $[1, 2^{\frac{n}{2}} - 1]$ when n is even, and in the interval $[1, 2^{\frac{n+1}{2}} - 3]$ when n is odd. Moreover, Q_n is strongly super matched of order λ for λ in the interval $[1, 2^{\frac{n}{2}} - 2]$ when n is even, and in the interval $[1, 2^{\frac{n+1}{2}} - 4]$ when n is odd. All these results are sharp, so Q_n is not strongly maximally or super matched of order λ for larger values of λ.*

Proof. By Proposition 4.4 all we need to do is to check the behavior of the function g_Q. Note that when $n = 3$, we have $g_Q(3, 1) = g_Q(3, 4) = 3$ and $g_Q(3, 2) = g_Q(3, 3) = 4$, and since $2^{n-2} = 2$, the graph Q_3 is strongly super matched of order 1 and of order 2. Thus we need the condition $n \geq 4$ in the theorem.

Now consider $n \geq 4$. Assume first that n is fixed and odd. From Corollary 4.5 we have that $g_Q(n, k)$ is strictly increasing when k is in the interval $[1, 2^{\frac{n+1}{2}} - 2]$ and by Lemma 4.4 we can easily compute that

$$\begin{aligned} g_Q(n, 2^{\frac{n+1}{2}} - 3) &= g_Q(n, 2^{\frac{n+1}{2}} - 4) + 1 \\ g_Q(n, 2^{\frac{n+1}{2}} - 2) &= g_Q(n, 2^{\frac{n+1}{2}} - 4) + 2 \\ g_Q(n, 2^{\frac{n+1}{2}} - 1) &= g_Q(n, 2^{\frac{n+1}{2}} - 4) + 2 \\ g_Q(n, 2^{\frac{n+1}{2}}) &= g_Q(n, 2^{\frac{n+1}{2}} - 4) + 1. \end{aligned}$$

Moreover, from Lemmas 4.6 and 4.7 we get that $g_Q(n, k) > g_Q(n, 2^{\frac{n+1}{2}}) = g_Q(n, 2^{\frac{n+1}{2}} - 3)$ for all k such that $2^{\frac{n+1}{2}} < k \leq 2^{n-2}$. Thus by Proposition 4.4, Q_n is strongly super matched of order λ for λ in $[1, 2^{\frac{n+1}{2}} - 4]$, but not for larger values of λ, and Q_n is strongly maximally matched of order λ for λ in $[1, 2^{\frac{n+1}{2}} - 3]$, but not for larger values of λ.

Similarly, when n is fixed and even, we have that $g_Q(n, k)$ is strictly increasing when k is in the interval $[1, 2^{\frac{n}{2}} - 1]$, and we can compute that $g_Q(n, 2^{\frac{n}{2}} - 1) = g_Q(n, 2^{\frac{n}{2}}) = g_Q(n, 2^{\frac{n}{2}} - 2) + 1$, and get that $g_Q(n, k) > g_Q(n, 2^{\frac{n}{2}}) = g_Q(n, 2^{\frac{n}{2}} - 1)$ for all k such that $2^{\frac{n}{2}} < k \leq 2^{n-2}$. Thus by Proposition 4.4, Q_n is strongly super matched of order λ for λ in $[1, 2^{\frac{n}{2}} - 2]$, but not for larger values of λ, and Q_n is strongly maximally matched of order λ for λ in $[1, 2^{\frac{n}{2}} - 1]$, but not for larger values of λ. $\square$

Theorem 4.8 tells us that Q_{10}, for instance, is strongly super matched for up to order $2^{\frac{10}{2}} - 2 = 30$, and Q_{11} is strongly super matched for up to order $2^{\frac{12}{2}} - 4 = 60$. We can use Theorem 4.8 to also obtain a result that fixes λ and varies n, that is, given a fixed λ, the graph Q_n is strongly super matched for large enough n.

Corollary 4.9. *Let λ and n be positive integers. The graph Q_n is strongly super matched of order λ if either n is even and $n \geq 2\log_2(\lambda + 2)$ or if n is odd and $n \geq 2\log_2(\lambda + 4) - 1$.*

In particular, Corollary 4.9 implies that Q_n is strongly super matched of order 1 for $n \geq 4$, it is strongly super matched of order 2 for $n \geq 4$ and strongly super matched of order 3 for $n \geq 5$, which were proved respectively in [3], [7] and [1].

5 Conclusion

In this paper, we introduced generalized versions of matching preclusion for bipartite graphs via definitions that tie directly to the classical Hall's Theorem. We studied this problem for the complete bipartite graph and the hypercube and found sharp results for the generalized matching preclusions of these graphs. It is interesting to note that in our approach, once we obtained a formula for $g_Q(n,k)$, the argument involves minimal graph theory.

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