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Variational Methods on Elastic Curves

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VARIATIONAL METHODS ON ELASTIC CURVES

by

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(Under the Direction of Scott Kersey)

ABSTRACT

In this thesis we investigate elastic curves. These are curves with minimal bending energy as measured by the total squared curvature functional. We show that these can be computed by evolving curves in the direction of the negative gradient in certain Hilbert space settings. By discretizing the curves and using numerical integration, we compute approximate minimizers and display using computer graphics. We propose a conjecture based on the rotation number of a curve that predicts the critical point curves that minimize bending energy.

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CHAPTER 1
INTRODUCTION

This thesis investigates elastic curves, in particular, what happens when an elastic wire is deformed out of shape, and released. To state this mathematically, it takes a bit more effort. The initial problem is to find the curve of minimal bending energy between two points in the plane. We aim to find the curve with minimum bending energy joining these two points. As such, the main investigation is done using the curvature-squared energy functional over a curve (γ),

$$E(\gamma) = \frac{1}{2}\int_0^L \kappa^2(\ell)\mathrm{d}\ell,$$

where ℓ is the arc length parameter.

This question was first posed in 1638 by Galilieo, and revisited several times throughout history. Most famously, however, is the work done on the problem by the Bernoullis, James in 1691 and Daniel in 1742. Daniel Bernoulli's approach involved taking directional derivatives, and we again use this technique to perfrom key computations for this thesis. Euler in 1744 proposes the treatment of the problem using the energy functional introduced earlier, and makes significant progress. [10] We mainly follow Anders Linnér treatment of the problem in [7] and[8] , in terms of the energy functional and expressing the directional derivatives in terms of gradients.

Among the critical points of this functional we find the curves that minimize the bending energy. We begin the thesis by gaining familiarity with the energy functional. We investigate $E(\gamma)$ to find a representation involving the curve's tangent angle. This allows us deal with the problem in a familiar manner.

The problem is framed in a Hilbert space and allows us to develop an inner product to better understand how different boundary conditions affect the problem. We define both the spaces for each functional, as well as explictly state each solution for varied boundary cases. We then use the Riesz representation theorem to cleverly use gradients to represent the directional derivatives in the inner product space. We use the gradient to travel in the direction of the steepest descent to find the minimizers in the final chapters. By following further work in [7], which establishes use of the Palais-Smale condition and minimax arguments to solve the problem, we are able to use the gradients numerically. The Palais-Smale condition guarentees existence of critical points and is highly useful to the theory. Examples are included at the end of this thesis.

The thesis is intended to clarify much of the work investigated in [8], as much of the proofs and examples are difficult to follow or are incomplete. We provide proofs that were previously omitted, make alterations to notations for clarity, and provide new computational code and results (including graphics). A clear demonstration of the material being presented allows for a proper understanding of a truly elegant problem.

We propose a conjecture in Chapter 7 regarding the rotation number of an elastic curve and the resulting curve after evolving it in the direction of the negative gradient. We provide examples demonstrating the conjecture in Chapter 9.

Several of the references listed deal more explictly with spline theory, where this problem is of high interest (see [1],[2],[3], and [6]). Our problem deals with simply two points, rather than a spline which could include many points.

The thesis is organized in the following manner:

- In Chapter 2, we investigate the **Bending Energy of Planar Curves** by looking at the curvature-squared energy functional and performing analysis.

- In Chapter 3, we introduce **The Variational Problem** that the thesis will be attempting to better understand. Boundary conditions are introduced.

- In Chapter 4, the **Hilbert Spaces and Submanifolds of Constraints** are described to gain a framework for the machinery being used in the thesis. We develop a Hilbert space from a cross product of Hilbert spaces to define a space for our functional to make sense.
- In Chapter 5, we take the directional derivatives of the various functionals used in the paper. The **Calculus of Variations** allows us to use powerful techniques to better analyze the functionals at hand.
- In Chapter 6, we take the directional derivatives and find representers for them as **Gradients** in our Hilbert space. These gradients allow us to later perform computations on elastic curves by travelling in the negative gradient flow towards minmizers.
- In Chapter 7, we introduce **Minimax Theory and the Palais-Smale Condition**. This powerful condition helps us guarentee the existence of minimzers and we discuss some basic examples here as well as give our conjecture.
- In Chapter 8, we perform **Computation of Elastic Curves** using the gradients from the previous sections. A description of our process is given.
- In Chapter 9, we provide **Examples** for the reader to gain a visual understanding of the problem. These are truly remarkable and worth examining.

CHAPTER 2

BENDING ENERGY OF PLANAR CURVES

In this thesis we are interested in understanding what happens when a thin elastic wire is deformed out of shape, and let go. The key idea is that the wire will evolve to a configuration with less **bending energy**. Mathematically, the energy functional is proportional to the total squared curvature functional (see [9]). Therefore, in this chapter we develop the curvature-squared functional that will be used in the remainder of this thesis. We begin with a definition of the main objects of study in this thesis, that being smooth parametric **curves**.

Definition 2.1. *A planar* **curve** *on* $[a, b]$ *is a map*

$$\gamma : [a, b] \mapsto \mathbb{R}^2 : t \mapsto \gamma(t) = (x(t), y(t)).$$

- *The curve is* **k-smooth**, *i.e.,* $\gamma \in C^k(I \to \mathbb{R}^2)$, *if* $x, y \in C^k(I)$.
- *The curve is* **regular** *if the magnitude of the tangent vector* $|\gamma'(t)|$ *is nonvanishing.*
- *The* **unit tangent** *vector of* γ *is*

$$T(t) := \frac{\gamma'(t)}{|\gamma'(t)|}.$$

- *The* **signed curvature** $\kappa := \kappa(\gamma) \in \mathbb{R} : t \mapsto (x(t), y(t))$ *is defined by*

$$\frac{dT}{d\ell} = \kappa N,$$

  where N *is the unit normal vector, and* ℓ *is the arc-length parameter. This may be positive or negative.*

The following result is well-known in differential geometry:

Lemma 2.2. *The signed curvature of a planar curve $\gamma : [a, b] \to \mathbb{R}^2$ is given by*

$$\kappa(t) = \frac{\gamma' \times \gamma''}{|\gamma'(t)|^3} = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

In this thesis we assume that the curves are smooth and regular, and for such curves it is well-known that they can be parametrized by arc length. However, in this thesis we choose a different parametrization. First, we restrict our planar curves to $[a, b] = I := [0, 1]$. Then, we define:

Definition 2.3. *Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve, and let L be the length of the curve. We say that the curve is* **parametrized proportional to arc length** *if $|\gamma'(t)| = L$ for all t.*

We use the parameter "s" to indicate this, i.e., $\gamma(s)$, and we use the notation $\dot{\gamma}(s)$ to be derivative of the curve parametrized proportional to arc length. Note that this definition reduces to the arc-length parametrization when $|\gamma'(t)| \equiv 1$ for all t (i.e., $L = 1$). We assume from here on that our curves are maps on $I = [0, 1]$ that are parametrized proportional to arc length, into the plane $\mathbb{R}^2$.

It is common to define the tangent indicatrix to be the unit tangent curve. This is a curve on the unit circle in our case. We parametrize the indicatrix by $\theta(s)$, the angle to the unit tangent vector as a function of s. This defnition allows us to include angles greater than 2π. Hence, we define the indicatrix as following:

Definition 2.4. *Let $\gamma(s) = \big(x(s), y(s)\big)$ be a planar curve parametrized proportional to arc length. The* **tangent indicatrix** *of a smooth curve $\gamma : I \to \mathbb{R}^2$ is the function $\theta : I \to \mathbb{R}$ given by $\dot{\gamma}(s) = \big(\dot{x}(s), \dot{y}(s)\big) = Le^{i\theta(s)} := L\Big(\cos\big(\theta(s)\big), \sin\big(\theta(s)\big)\Big)$, with $\theta(0) \in [0, 2\pi)$.*

Proposition 2.5.

1. *A curve is uniquely characterized by its length L, tangent indicatrix $\theta(s)$, and starting point $\gamma(0)$ on the curve. Specifically, $\gamma(s) = \gamma(0) + L\int_0^s e^{i\theta(s)}\mathrm{d}s$.*

2. *The* **end-point gap** *$\gamma(1) - \gamma(0)$ equals $L\int_0^1 e^{i\theta(s)}\mathrm{d}s$.*

3. *The signed curvature of $\gamma(s)$ is $\kappa(s) = \frac{1}{L}\dot{\theta}(s)$.*

Proof. Part (1) follows by the fundamental theorem of calculus and definition of the indicatrix,

$$\gamma(s) = \gamma(0) + \int_0^s \dot{\gamma}(s)\mathrm{d}s = \gamma(0) + \int_0^s Le^{i\theta(s)}\mathrm{d}s.$$

For part (2), we can use part (1) to get

$$\gamma(1) - \gamma(0) = \left(\gamma(0) + L\int_0^1 e^{i\theta(s)}\mathrm{d}s\right) - \left(\gamma(0) + L\int_0^0 e^{i\theta(s)}\mathrm{d}s\right) = L\int_0^1 e^{i\theta(s)}\mathrm{d}s.$$

It remains to prove (3). It is clear to see $|\dot{\gamma}|$ is L. Taking the cross product of $\dot{\gamma}$ and $\ddot{\gamma}$,

$$\begin{aligned}
\dot{\gamma} \times \ddot{\gamma} &= L\Big(\cos(\theta(s)), \sin(\theta(s))\Big) \times L\dot{\theta}\Big(-\sin(\theta(s)), \cos(\theta(s))\Big) \\
&= \Big(L\cos\big(\theta(s)\big), L\sin\big(\theta(s)\big)\Big) \times \Big(-L\sin\big(\theta(s)\big)\ \dot{\theta}, L\cos\big(\theta(s)\big)\ \dot{\theta}\Big) \\
&= L^2\cos^2\big(\theta(s)\big)\,\dot{\theta}(s) + L^2\sin^2\big(\theta(s)\big)\,\dot{\theta}(s) \\
&= L^2\dot{\theta}(s).
\end{aligned}$$

Therefore,

$$\kappa(s) = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma}\|^3} = \frac{L^2\dot{\theta}(s)}{L^3} = \frac{\dot{\theta}(s)}{L} = \frac{1}{L}\dot{\theta}(s).$$

□

Now that we have some tools to work with, we can define the total squared curvature functional.

Definition 2.6. *Let γ be a planar curve of length L with arc length parameter ℓ. The* **total squared curvature functional** *is*

$$E(\gamma) := \frac{1}{2}\int_0^L \kappa^2(\ell)\, d\ell.$$

The **length penalized total squared curvature functional** *is*

$$E_\lambda(\gamma) := E(\gamma) + \lambda L,$$

with λ some fixed positive constant.

The factor $\frac{1}{2}$ is included to simplify the variations of E and E_λ later in this thesis. The affine term is served as a penalty for lengthy curves (to guarantee existence in certain cases). This is discussed later in this thesis. Using the indicatrix notation, we can write the total squared curvature functionals as follows:

Lemma 2.7. *Let $\gamma(s)$ be a planar curve of length L with indicatrix θ parametrized proportional to arc length. Then,*

$$E(\theta, L) = \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)\,\mathrm{d}s$$

and

$$E_\lambda(\theta, L) = E(\theta, L) + \lambda L.$$

Proof. We use a change of variables to obtain our desired result. We wish to change from the variable ℓ to s. Observing the maps, $\ell : 0 \to L$ and $s : 0 \to 1$, we are able to perform the change of variables by,

$$\ell = Ls$$

$$\mathrm{d}\ell = L\mathrm{d}s.$$

Therefore, together with the fact $\kappa = \frac{1}{L}\dot{\theta}(s)$, we show

$$\begin{aligned}
E(\gamma) &= \frac{1}{2}\int_0^L \kappa^2(\ell)\mathrm{d}\ell = \frac{1}{2}\int_0^1 \kappa^2(s)L\mathrm{d}s \\
&= \frac{1}{2}\int_0^1 \big(\frac{1}{L}\dot{\theta}(s)\big)^2 L\mathrm{d}s \\
&= \frac{1}{2}\int_0^1 \big(\frac{1}{L^2}\dot{\theta}^2(s)\big)L\mathrm{d}s = \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)\mathrm{d}s \\
&= E(\theta, L).
\end{aligned}$$

The length penalized total squared curvature functional follows as above, with the added term outside of the intergal being unaffected by the change of variables, yielding

$$\begin{aligned}
E_\lambda(\theta, L) &= \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)\,\mathrm{d}s + \lambda L \\
&= E(\theta, L) + \lambda L.
\end{aligned}$$

$\square$

CHAPTER 3

THE VARIATIONAL PROBLEM

In this thesis we are studying variational problems connected to the bending energy functional. In particular, we are looking for minimizers and critical points, when they exist. These depend on certain constraints (such prescribed boundary conditions) on the curves. The following are the particular boundary conditions that we will investigate, in terms of the tangent indicatrix:

I Set the endpoints, $\gamma(0) = p_0$, $\gamma(1) = p_1$

II Use (I) and set tangent directions, $\gamma'(0) = v_0$, $\gamma'(1) = v_1$

III Special case of (II): Closed Curves, where $p_0 = p_1, v_0 = v_1$

The variational problems we are considering are:

Definition 3.1.

- *Minimize E and E_λ over curves with constraints above.*
- *Find critical points of E and E_λ over curves with constraints above.*

A cautionary result is the following:

Theorem 3.2. *The existence of minimizers of $E(\theta, L)$ with length free to vary for curves does not exist except in the case of a straight line segment.*

Proof. If a straight line path exists, then $E(\gamma) = 0$ and is a minimizer.
If not, then suppose there exists a point, p, such that $p := \gamma(0) = \gamma(1)$. Informally,

any nontrivial curve traversing a path from $\gamma(0)$ to $\gamma(1)$ must have nonzero energy, in this particular case. That is to say, the energy functional, $E(\gamma) > 0$. Therefore, if such a minimizer exists, then the functional must also be nonzero.

Consider the sequence $\gamma_k(t) = k\big(\cos(t), \sin(t)\big) + p - (k, 0)$, one loop of circles of radius k ranging from $0 \le t \le 2\pi$. Thus the length of each circle is $2\pi k = L$. The curvature of each circle is $\kappa = \frac{1}{k}$. Then,

$$\begin{aligned}
E(\gamma_k) &= \frac{1}{2}\int_0^{2\pi k} \kappa^2(\ell)\mathrm{d}\ell = \frac{1}{2}\int_0^1 \kappa^2(s)\ L\mathrm{d}s \\
&= \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)\mathrm{d}s = \frac{1}{2L}\int_0^1 \big(L\ \kappa(s)\big)^2\mathrm{d}s \\
&= \frac{1}{2L}\int_0^1 L^2\kappa(s)^2\mathrm{d}s \\
&= \frac{L^2}{2L}\int_0^1 \frac{1}{k^2}\mathrm{d}s \\
&= \frac{L}{2k^2} = \frac{2\pi k}{2k^2} = \frac{\pi}{k}
\end{aligned}$$

As $k \to \infty$ it is clear to see $E(\gamma_k) = \frac{\pi}{k} \to 0$. This is a contradiction, as our sequence converges to a solution with zero energy. However, as indicated above, no nontrivial curve has zero energy. Thus there is no minimizer except in the case of a straight line segment.

A remark needs to be made. Suppose we force the sequence of ever expanding circles through two particular points, say p_0 and p_1, by translation of the circles. By similar logic, the energy functional computations would yield the same result, yielding no minimizer for a circle forced through two particular points. $\square$

The following result is known for the length penalized functional:

Theorem 3.3. *The existence of minimizers of $E_\lambda(\gamma)$ for curves with constraints of first or second type exists.*

The proof of existence can be found in [4].

CHAPTER 4

HILBERT SPACES AND SUBMANIFOLDS OF CONSTRAINTS

Definition 4.1.

1. *A nonempty set X together with a bilinear map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ is an* **inner product space (ips)** *over $\mathbb{F}$ if it satisfies:*
   - $\langle x, x \rangle \geq 0$ *with equality iff* $x = 0$, *(positive definite)*
   - $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, *(linear)*
   - $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ *for all* $\lambda \in \mathbb{R}$, *(linear)*
   - $\langle x, y \rangle = \overline{\langle y, x \rangle}$. *(Hermitian)*
2. *The* **norm** *on X is defined as* $||x|| := \sqrt{\langle x, x \rangle}$.
3. *A* **Hilbert space** *is a complete ips (i.e., every Cauchy sequence converges).*

Lemma 4.2. *Suppose that $H_1, \ldots, H_n$ are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{H_1}, \ldots, \langle \cdot, \cdot \rangle_{H_n}$. Then, $H := H_1 \times \cdots \times H_n$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H := \sum_{i=1}^n \langle \cdot, \cdot \rangle_{H_i}$.*

Proof. It is straight forward to establish that $\langle \cdot, \cdot \rangle_H$ is an inner product, as follows. Let $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$ and $h = (h_1, \ldots, h_n)$ be in H. Then, we prove the four axioms of an inner product space. We show

- $\langle f, f \rangle_H = \sum_{i=0}^n \langle f_i, f_i \rangle_{H_i} \geq \sum_{i=0}^n 0 = 0$.
- $\langle f+g, h \rangle_H = \sum_{i=0}^n \langle f_i+g_i, h_i \rangle_{H_i} = \sum_{i=0}^n \langle f_i, h_i \rangle_{H_i} + \langle g_i, h_i \rangle_{H_i} = \langle f, h \rangle_H + \langle g, h \rangle_H$.
- $\langle \lambda f, h \rangle_H = \sum_{i=0}^n \langle \lambda f_i, h_i \rangle_{H_i} = \sum_{i=0}^n \lambda \langle f_i, h_i \rangle_{H_i} = \lambda \sum_{i=0}^n \langle f_i, h_i \rangle_{H_i} = \lambda \langle f, h \rangle_H$.
- $\langle f, g \rangle_H = \sum_{i=0}^n \langle f_i, g_i \rangle_{H_i} = \sum_{i=0}^n \overline{\langle g_i, f_i \rangle_{H_i}} = \overline{\sum_{i=0}^n \langle g_i, f_i \rangle_{H_i}} = \overline{\langle g, f \rangle}_H$.

The first axiom yields equality if and only if $f \equiv 0$. It remains to show H is complete. Now, suppose $\{f^k = (f_1^k, \dots, f_n^k) : k = 1, 2, \dots\}$ is a Cauchy sequence in H. This implies $\{f_i^k\}$ are Cauchy in the H_i, for $i = 1 : n$. Since H_i are Hilbert spaces, they are complete by definition. Therefore, for each i, $\exists f_i \in H_i$ such that, on passing to a subsequence, we may assume $\lim_{k\to\infty} f_i^k \to f_i$. This immediately implies

$$f^k \to f := (f_1, \dots, f_n) \in H.$$

Therefore, we have shown for an arbitrary Cauchy sequences we can extract a convergence subsequence. Therefore, H is complete. $\square$

The following are well-known:

Theorem 4.3.

1. *Parallelogram Law:* $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$*.*
2. *Cauchy-Schwarz:* $|\langle x, y\rangle| \le ||x||\ ||y||$*.*
3. *(Riesz Representation) For every bounded linear functional λ on a Hilbert Space H, there is a unique $z \in H$ such that $\lambda(x) = \langle x, z\rangle$ for all $x \in H$. Moreover, $||\lambda|| = ||z||$.*

Now, in our thesis we require $\dot{\theta} \in \mathrm{L}_2(0, 1)$. This is simply due to the fact the signed curvature κ of γ is given $\kappa(s) = \frac{\dot{\theta}}{L}$. The energy functional can we written using this subsitution as $E_\lambda(\gamma) = \frac{1}{2L}\int_0^1 \dot{\theta}(s)^2 + \lambda L$. Thus, the largest possible space for $E_\lambda(\gamma)$ is the space of all absolutely continuious functions with Lebesgue square integrable derivatives ($\mathrm{L}_2(0, 1)$) and the set of positive reals ($\lambda \in \mathbb{R}^+$). We formally define these spaces below. Let

- $H := \{\gamma : [0, 1] \to \mathbb{R}^2, \gamma \text{ abs. cont } \textit{and} \int_0^1 \dot{\theta}^2(s)\mathrm{d}s < \infty\}$.

- $H_\lambda := \{(\theta, L) \in H \times \mathbb{R}\}$.

- $\Lambda : H \times \mathbb{R}^+ \to \mathbb{C} : (\theta, L) \mapsto L \int_0^1 e^{i\theta(s)} \mathrm{d}s - (\gamma(1) - \gamma(0))$

The functional $E(\gamma)$ is defined on H. The functional $E_\lambda(\gamma)$ is defined on $H \times \mathbb{R}^+ = H_\lambda$.

CHAPTER 5

CALCULUS OF VARIATIONS

In elementary calculus, we learn to take the derivative of functions of one or more real variables, and show how to identify the extrema (minima and maxima) of functions, and critical points. Suppose now that we have a function over objects other than real variables, such as curves. The calculus of variations tells us how to identify critical points and min/max of such functions (or better, **functionals**).

To understand how the calculus of variations works, we first need the concept of a directional derivative. Let $F : f \mapsto \mathbb{R}$ be a (smooth) functional on functions in a linear vector space. Then, the **directional derivative** of F at f in the direction v is

$$D_v F(f) := \frac{d}{d\alpha}\Big|_{\alpha=0} F(f + \alpha v).$$

The variations v are themselves functions in the vector space (since the vector space is linear). In general, when the varations are over a set (surface, manifold) of functions S, then the variations v are functions in the **tangent space** to S at f.

The space H is a Hilbert space, as is the cross product $H_\lambda = H \times \mathbb{R}$. For our application we need the variationals of the functionals

$$E(\theta, L) = \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)\,\mathrm{d}s$$
$$E_\lambda(\theta, L) = E(\theta) + \lambda L.$$

Let

$$\Phi : H \times \mathbb{R}_+ \to R^2 : (\theta, L) \mapsto L\int_I e^{i\theta(s)} = \big(L\int \cos\theta(s)\mathrm{d}s, L\int \sin\theta(s)\mathrm{d}s\big)$$
$$\Lambda : H \times \mathbb{R}_+ \to \mathbb{R} : (\theta, L) \mapsto L\int_I \big(\alpha_1\cos\theta(s) + \alpha_2\sin\theta(s)\mathrm{d}s\big) = \alpha_1 a + \alpha_2 b.$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}_+$. Then, both constraints I and II require $\Phi(\theta, L) = (a, b)$ for fixed end points. The second constraint requires fixed tangents as well.

Proposition 5.1. *Let (v_θ, v_L) be an admissible (linear) variation for the functionals $E(\theta, L)$ and $E_\lambda(\theta, L)$ in the space of the constraints. Then,*

$$D_{(v_\theta,v_L)}E(\theta,L) = \frac{1}{L}\int_I \dot\theta(s)\dot v_\theta(s)\mathrm{d}s - \frac{v_L}{2L^2}\int_I \dot\theta(s)^2\mathrm{d}s,$$

$$D_{(v_\theta,v_L)}E_\lambda(\theta,L) = D_{(v_\theta,v_L)}E(\theta,L) + \lambda v_L,$$

$$\begin{aligned} D_{(v_\theta,v_L)}\Phi(\theta,L) &= \int_I (v_L + iLv_\theta(s))e^{i\theta(s)}\mathrm{d}s \\ &= (\int_I (v_L\cos\theta(s) - v_\theta(s)L\sin\theta(s))\mathrm{d}s,\ \int_I (v_L\sin\theta(s) + v_\theta(s)L\cos\theta(s))\mathrm{d}s \end{aligned}$$

$$D_{(v_\theta,v_L)}\Lambda(\theta,L) = v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} + L\int_I(-\alpha_1\sin\theta(s)v_\theta(s) + \alpha_2\cos\theta(s)v_\theta(s))\mathrm{d}s.$$

Proof. Let (v_θ, v_L) be a variation in $H \times \mathbb{R}$, the tangent space to $H \times \mathbb{R}_+$. The directional derivative of E_λ at (θ, L) in the direction (v_θ, v_L) is:

$$\begin{aligned} D_{(v_\theta,v_L)}E_\lambda(\theta,L) &= \frac{d}{d\alpha}\Big|_{\alpha=0} E_\lambda(\theta+\alpha v_\theta, L+\alpha v_L) \\ &= \frac{d}{d\alpha}\Big|_{\alpha=0} \frac{1}{2(L+\alpha v_L)}\int_I(\dot\theta+\alpha\dot v_\theta)^2\mathrm{d}s + \lambda(L+\alpha v_L) \\ &= \Big(\frac{1}{2(L+\alpha v_L)}\int_I 2(\dot\theta+\alpha\dot v_\theta)\dot v_\theta\ \mathrm{d}s \\ &\quad - \frac{v_L}{2(L+\alpha v_L)^2}\int_I(\dot\theta+\alpha\dot v_\theta)^2\ \mathrm{d}s + \lambda v_L\Big)\Big|_{\alpha=0} \\ &= \frac{1}{L}\int_I\dot\theta(s)\dot v_\theta(s)\mathrm{d}s - \frac{v_L}{2L^2}\int_I\dot\theta(s)^2\mathrm{d}s + \lambda v_L. \end{aligned}$$

The directional derivative of $E(\theta, L)$ is the same without the last term.

For Φ, the directional derivative in the direction of (v_θ, v_L) is

$$\begin{aligned} D_{(v_\theta,v_L)}\Phi(\theta,L) &= \frac{d}{d\alpha}\Big|_{\alpha=0} \Phi(\theta+\alpha v_\theta, L+\alpha v_L) \\ &= \frac{d}{d\alpha}\Big|_{\alpha=0} (L+\alpha v_L)\int_I e^{i(\theta(s)+\alpha v_\theta(s))}\mathrm{d}s \\ &= \Big(v_L\int_I e^{i(\theta(s)+\alpha v_\theta(s))}\mathrm{d}s + (L+\alpha v_L)\int_I iv_\theta(s)e^{i(\theta(s)+\alpha v_\theta(s))}\mathrm{d}s\Big)\Big|_{\alpha=0} \\ &= v_L\int_I e^{i\theta(s)}\mathrm{d}s + iL\int_I v_\theta(s)e^{i\theta(s)}\mathrm{d}s \\ &= \int_I(v_L + iLv_\theta(s))e^{i\theta(s)}\mathrm{d}s. \end{aligned}$$

Then, with $e^{i\theta(s)} = \cos\theta(s) + i\sin\theta(s)$, we get

$$\begin{aligned} D_{(v_\theta, v_L)}\Phi(\theta, L) &= \int_I (v_L + iLv_\theta(s))(\cos\theta(s) + i\sin\theta(s)) \\ &= \int_I (v_L\cos\theta(s) - v_\theta(s)L\sin\theta(s)) + i(v_L\sin\theta(s) + v_\theta(s)L\cos\theta(s))\mathrm{d}s. \end{aligned}$$

This gives the stated result.

Finally, we find the directional derivative of Λ in the direction of (v_θ, v_L) yields,

$$\begin{aligned} D_{(v_\theta, v_L)}\Lambda(\theta, L) &= \frac{d}{d\beta}\Big|_{\beta=0} \Lambda(\theta + \beta v_\theta, L + \beta v_L) \\ &= \frac{d}{d\beta}\Big|_{\beta=0} (L + \beta v_L)\int_I \alpha_1\cos(\theta + \beta v_\theta) + \alpha_2\sin(\theta + \beta v_\theta)\mathrm{d}s \\ &= v_L\int_I (\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\mathrm{d}s + L\int_I (-\alpha_1 sin(\theta)v_\theta + \alpha_2\cos(\theta)v_\theta)\mathrm{d}s \\ &= v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} + L\int_I (-\alpha_1\sin\theta(s)v_\theta(s) + \alpha_2\cos\theta(s)v_\theta(s))\mathrm{d}s. \end{aligned}$$

□

The connection between the functions Φ and Λ is as follows:

Lemma 5.2. *If $\Phi(\theta, L) = (a, b)$, then $D_{(v_\theta, v_L)}\Phi(\theta, L) = 0$ if and only if $D_{(v_\theta, v_L)}\Lambda(\theta, L) = 0$ for all α_1 , $\alpha_2 \in \mathbb{R}$.*

Proof. From above,

$$\begin{aligned} D_{(v_\theta, v_L)}\Phi(\theta, L) &= (\int_I (v_L\cos\theta(s) - v_\theta(s)L\sin\theta(s))\mathrm{d}s, \int_I (v_L\sin\theta(s) + v_\theta(s)L\cos\theta(s))\mathrm{d}s \\ D_{(v_\theta, v_L)}\Lambda(\theta, L) &= v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} + L\int_I (-\alpha_1\sin\theta(s)v_\theta(s) + \alpha_2\cos\theta(s)v_\theta(s))\mathrm{d}s. \end{aligned}$$

We show " $\Rightarrow$ ". We use our assumption $D_{(v_\theta, v_L)}\Phi(\theta, L) = 0$, which implies

$$\begin{aligned} \frac{v_L a}{L} &= L\int_I \sin(\theta)\mathrm{d}s \\ \frac{v_L b}{L} &= -L\int_I \cos(\theta)\mathrm{d}s. \end{aligned}$$

Hence,

$$\begin{aligned} D_{(v_\theta,v_L)}\Lambda(\theta,L) &= v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} - v_L\frac{\alpha_1 a}{L} - v_L\frac{\alpha_2 b}{L} \\ &= \frac{v_L}{L}\big(\alpha_1 a + \alpha_2 b - \alpha_1 a - \alpha_2 b\big) \\ &= 0 \end{aligned}$$

Now we show " $\Leftarrow$ ". Suppose $D_{(v_\theta,v_L)}\Lambda(\theta,L) = 0\ \forall \alpha_1,\alpha_2 \in \mathbb{R}_+$. Then,

$$D_{(v_\theta,v_L)}\Lambda(\theta,L) = v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} + L\int_I(-\alpha_1\sin(\theta)v_\theta + \alpha_2\cos(\theta)v_\theta)\mathrm{d}s = 0.$$

Let $\alpha_1 = 1$, $\alpha_2 = 0$. Then,

$$\frac{v_L a}{L} - L\int_I \sin(\theta)v_\theta \mathrm{d}s = 0.$$

Let $\alpha_1 = 0$, $\alpha_2 = 1$. Then,

$$\frac{v_L b}{L} + L\int_I \cos(\theta)v_\theta \mathrm{d}s = 0.$$

These statements yield that $D_{(v_\theta,v_L)}\Phi(\theta,L) = 0$. $\square$

We require a theorem to gain more information regarding the surjectivity of functionals. Recall from chapter 5

$$\Phi(\theta,L) : H\times\mathbb{R}_+ \to \mathbb{R}^2 : (\theta,L) \mapsto (L\int\cos(\theta)\mathrm{d}s, L\int\sin(\theta)\mathrm{d}s),$$

and that

$$\begin{aligned} &D\Phi(\theta,L) : H\times\mathbb{R}_+ \to \mathbb{R}^2 : (v_\theta, v_L) \\ &\qquad \mapsto (v_L\int\cos\theta(s)\mathrm{d}s - L\int\sin\theta(s)v_\theta\mathrm{d}s,\ v_L\int\sin\theta(s)\mathrm{d}s + L\int\cos\theta(s)v_\theta\mathrm{d}s). \end{aligned}$$

Theorem 5.3. *(Submersion Theorem) Suppose that for all* $(\theta,L) \in M := \Phi^{-1}(a,b)$ *the map* $D\Phi(\theta,L) : H\times\mathbb{R}_+ \to \mathbb{R}^2$ *is onto. Then, the set* M *is a closed submanifold with tangent space* $T_{(\theta,L)}M = \ker D\Phi(\theta,L)$.

Lemma 5.4. *$M := \Phi^{-1}(a,b)$ is a closed submanifold with tangent space $T_{(\theta,L)}M =$ $\ker D\Phi(\theta, L)$.*

Proof. Let $(\theta, L) \in \Phi^{-1}(a,b)$. Assume boundary case I. We aim to show $\mathrm{D}\Phi(\theta, L)$ is onto.

Let $(z_1, z_2) \in \mathbb{R}^2$. We need to find a v_θ, v_L such that

$$D\Phi(\theta, L) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

That is, we have

$$\frac{v_L a}{L} - L\int_I v_\theta \sin(\theta(s))\mathrm{d}s = z_1$$
$$\frac{v_L b}{L} + L\int_I v_\theta \cos(\theta(s))\mathrm{d}s = z_2$$

We need to fine a v_θ such that

$$\int_I v_\theta \sin(\theta(s))\mathrm{d}s = u_1$$
$$\int_I v_\theta \cos(\theta(s))\mathrm{d}s = u_2$$

To find how to choose these v_θ see Linners work in [8]. □

Lemma 5.5. *$\nabla\Lambda(\theta, L) \perp T_{(\theta,L)}M$ for all α_1 , α_2.*

This is shown in [8] as well.

CHAPTER 6

GRADIENTS

Now, suppose that $F(\gamma) = ||\gamma||^2 = \langle \gamma, \gamma \rangle$ in an inner product space. Then,

$$D_v F(f) = \frac{d}{d\alpha}\bigg|_{\alpha=0} \langle \gamma + \alpha v, \gamma + \alpha v \rangle = 2\langle v, \gamma \rangle.$$

This is the Euler-Lagrange equation.

Moreover, this can be written

$$D_v F(f) = 2\langle v, \gamma \rangle = \langle \nabla F(f), v \rangle$$

for some gradient $\nabla F(f)$. That is $\nabla F(f)$ is a representation for the derivative $DF(f)$. By the **Riesz Representation Theorem**, any linear functional on the inner product space can be represented by such a gradient.

For our problem, we identify an inner product on $H \times \mathbb{R}$ by

$$\langle (w_\theta, w_L), (v_\theta, v_L) \rangle = v_\theta(0)\, w_\theta(0) + \textstyle\int_I \dot{v}_\theta \dot{w}_\theta + v_L w_L.$$

We then define the gradient of a real valued function F defined on an inner product space by

$$\langle \nabla F(x), v \rangle = D_v F(x).$$

Thus, if we can find $DE_\lambda(\gamma)$ then we can find a representer for the gradient of E_λ. That is, recall that if $F : X \to R$ is a functional on a Reimannian Manifold then the gradient $\nabla F(x)$ is defined using the Reisz representation theorem ([7]). That is,

$$D_v F(x) = \langle \nabla F(x), v \rangle \ \ \forall x \in T_x X.$$

Lemma 6.1. *The* **duBois-Reymond lemma** *states if* $f : I \to \mathbb{R}$ *is continuous and* $\int_I f(s)\dot{v}(s)\mathrm{d}s = 0\ \forall v : I \to \mathbb{R}$ *such that* $\dot{v}$ *is continuous and* $v(0) = v(1) = 0$, *then* f *is constant.*

Proposition 6.2. *The gradient of $D_{(v_\theta,v_L)}E_\lambda$ in $H\times\mathbb{R}$ is*

$$\nabla E_\lambda\left(v_\theta,v_L\right)=\Big(\frac{1}{L}\theta(s)+Cs+D,\lambda-\frac{1}{2L^2}\int_I\dot{\theta}^2\Big).$$

For boundary case I we have,

$$\begin{aligned}C&=0\\ D&=\frac{-1}{L}\theta(0).\end{aligned}$$

For boundary case II we have,

$$\begin{aligned}C&=\frac{1}{L}\Big(\theta(1)-\theta(0)\Big)\\ D&=\frac{-1}{L}\theta(0).\end{aligned}$$

Proof. Let (v_θ,v_L) be a variation in $H\times\mathbb{R}$, the tangent space to $H\times\mathbb{R}_+$. Recall that the directional derivative of E_λ at (θ,L) in the direction (v_θ,v_L) is:

$$D_{(v_\theta,v_L)}E_\lambda(\theta,L)=\frac{1}{L}\int_I\dot{\theta}(s)\dot{v}_\theta(s)\mathrm{d}s-\Big(\frac{1}{2L^2}\int_I\dot{\theta}(s)^2\mathrm{d}s+\lambda\Big)v_L.$$

From here it is clear to see $w_L=\lambda-\frac{1}{2L^2}\int_I(\dot{\theta})^2$.

Now if we look in the direction of $(v_\theta,0)\in H\times\mathbb{R}$ and use the inner product on H_λ where $\nabla E_\lambda=(w_\theta,w_L)$ we find,

$$\int_I\dot{w}_\theta(s)\dot{v}(s)\mathrm{d}s+w_\theta(0)v_\theta(0)=\frac{1}{L}\int_I\dot{\theta}(s)\dot{v}_\theta\mathrm{d}s.$$

Assume $\dot{v}$ is continuous and $v_\theta(0)=v_\theta(1)=0$. We perform computations on the above, then apply the duBois-Reymond lemma followed by integration, yielding

$$\begin{aligned}
&\int_I \dot{w}_\theta(s)\dot{v}_\theta(s)\mathrm{d}s + w_\theta(0)v_\theta(0) = \frac{1}{L}\int_I \dot{\theta}(s)\dot{v}_\theta \mathrm{d}s \\
&\int_I (\dot{w}_\theta(s) - \frac{\dot{\theta}(s)}{L})\dot{v}_\theta(s)\mathrm{d}s = 0 \\
\Rightarrow &\dot{w}_\theta(s) - \frac{\dot{\theta}(s)}{L} = C \quad \text{by} \quad \text{lemma} \\
\Rightarrow &w_\theta(s) - \frac{\theta(s)}{L} = Cs + D \quad \text{by} \quad \text{integration} \\
&w_\theta(s) = \frac{\theta(s)}{L} + Cs + D,
\end{aligned}$$

where $C, D \in \mathbb{R}$.

Thus, we have shown $\nabla E_\lambda(\theta, L) = (w_\theta, w_L)$. We know use the inner product once again to show

$$\begin{aligned}
\int_I \big(\frac{\dot{\theta}(s)}{L} + C\big)\dot{v}_\theta(s)\mathrm{d}s + w_\theta(0)v_\theta(0) &= \frac{1}{L}\int_I \dot{\theta}(s)\dot{v}_\theta(s)\mathrm{d}s \\
C\int_I \dot{v}_\theta(s)\mathrm{d}s + w_\theta(0)v_\theta(0) &= 0 \\
C(v_\theta(1) - v_\theta(0)) + w_\theta(0)v_\theta(0) &= 0 \\
C(v_\theta(1) - v_\theta(0)) + \big(\frac{\theta(0)}{L} + D)v_\theta(0) &= 0
\end{aligned}$$

Now by choosing v_θ consistent with the various boundary conditions we can determine the constants C, D. We solve for boundary condition (I), where the endpoints are fixed, that is $\gamma(0) = p_0,\ \gamma(1) = p_1$. Then $v_\theta(0) \neq 0$ yields

$$w_\theta = C(v_\theta(1) - v_\theta(0)) + \big(\frac{\theta(0)}{L} + D)v_\theta(0) = 0,$$

which forces $D = \frac{-\theta(0)}{L},\ C = 0$. resulting in $w_\theta = \frac{1}{L}\big(\theta(s) - \theta(0)\big)$ for the first boundary case.

In the second case, the tangent directions are given $\dot{\gamma}(0) = v_0$, $\dot{\gamma}(1) = v_1$ as well as case (I). Then we have,

$$w_\theta = C(v_\theta(1) - v_\theta(0)) + \big(\frac{\theta(0)}{L} + D)v_\theta(0) = 0,$$

with $v_\theta(0) = \theta(0)$, $v_\theta(1) = \theta(1)$ which forces $D = \frac{-\theta(0)}{L}$, $C = \frac{1}{L}\big(\theta(1) - \theta(0)\big)$. Thus we have $w_\theta = \frac{1}{L}(\theta(s) - Cs - \theta(0))$.

$\square$

Proposition 6.3. *The gradient of $D_{(v_\theta, v_L)}\Lambda(\theta, L)$ in $H \times \mathbb{R}$ is the following for each boundary case:*

$$I \ \nabla\Lambda(\theta, L) = \Big(A(s) + \big(\alpha_2 a - \alpha_1 b\big)\big(s+1\big), \tfrac{1}{L}\big(\alpha_1 a + \alpha_2\big)\Big)$$

$$II \ \nabla\Lambda(\theta, L) = \Big(A(s) - A(1)s, \tfrac{1}{L}\big(\alpha_1 a + \alpha_2 b\big)\Big)$$

where

$$A(s) = \int_0^s \Big(\alpha_1\big(y(t) - y(0)\big) - \alpha_2\big(x(t) - x(0)\big)\Big)\mathrm{d}t.$$

Proof. Let (v_θ, v_L) be a variation in $H \times \mathbb{R}$, the tangent space to $H \times \mathbb{R}_+$. Recall that the directional derivative of Λ at (θ, L) in the direction (v_θ, v_L) is:

$$\begin{aligned}
D_{(v_\theta, v_L)}\Lambda(\theta, L) &= v_L\frac{\alpha_1 a}{L} + v_L\frac{\alpha_2 b}{L} + L\int_I(-\alpha_1 \sin\theta(s)v_\theta(s) + \alpha_2\cos\theta(s)v_\theta(s))\mathrm{d}s. \\
&= \frac{v_L}{L}(\alpha_1 a + \alpha_2 b) + L\int_I(-\alpha_1\sin(\theta(s)) + \alpha_2\cos(\theta(s))v_\theta \mathrm{d}s \\
&= \frac{v_L}{L}(\alpha_1 a + \alpha_2 b) + v_\theta(1)(\alpha_2 a - \alpha_1 b) \\
&+ \int_I \Big(\alpha_1(y(s) - y(0)) - \alpha_2(x(s) - x(0))\Big)\dot{v}_\theta \mathrm{d}s.
\end{aligned}$$

It is clear to see $\beta_L = \frac{1}{L}(\alpha_1 a + \alpha_2 b)$.

Now if we look in the direction of $(v_\theta, 0) \in H \times \mathbb{R}$ and use the inner product on H_λ where $\nabla\Lambda = (\beta_\theta, \beta_L)$ we find,

$$\int_I \dot{\beta}_\theta(s)\dot{v}_\theta(s)\mathrm{d}s + \beta_\theta(0)v_\theta(0) = v_\theta(1)(\alpha_2 a - \alpha_1 b) + \int_I \Big(\alpha_1(y(s) - y(0)) - \alpha_2(x(s) - x(0))\Big)\dot{v}_\theta \mathrm{d}s.$$

Now assume $\dot{v}_\theta$ is continuous and $v_\theta(0) = v_\theta(1) = 0$ and apply the duBois-Reymond lemma and integration to yield, with $Y, Z \in \mathbb{R}$,

$$\int_I \dot{\beta}_\theta(s)\dot{v}_\theta(s)\mathrm{d}s + \beta_\theta(0)v_\theta(0) = v_\theta(1)(\alpha_2 a - \alpha_1 b) + \int_I \Big(\alpha_1(y(s) - y(0)) - \alpha_2(x(s) - x(0))\Big)\dot{v}_\theta \mathrm{d}s.$$

$$\int_I (\dot{\beta}_\theta(s) - A(s))\dot{v}_\theta \mathrm{d}s = 0$$

$$\dot{\beta}_\theta - A(s) = Y \quad \text{by lemma}$$

$$\beta_\theta = A(s) + Ys + Z \quad \text{by integration}$$

Thus we have shown $\nabla\Lambda(\theta, L) = (\beta_\theta, \beta_L)$. We now use the different boundary conditions and the equations defining the gradients to find,

$$\int_I \Big(A(s) + Y\Big)\dot{v}_\theta \mathrm{d}s + \beta_\theta(0)v_\theta(0) = v_\theta(1)(\alpha_2 a - \alpha_1 b) - A(s)$$

$$Y\int_I \dot{v}_\theta(s)\mathrm{d}s + \beta_\theta(0)v_\theta(0) = v_\theta(1)(\alpha_2 a - \alpha_1 b)$$

$$Y(v_\theta(1) - v_\theta(0)) + \beta_\theta(0)v_\theta(0) - v_\theta(1)(\alpha_2 a - \alpha_1 b) = 0$$

$$Y(v_\theta(1) - v_\theta(0)) + Zv_\theta(0) - v_\theta(1)(\alpha_2 a - \alpha_1 b) = 0$$

Now we choose v_θ consistent with the various boundary conditions to find for condition I, with $v_\theta(0) = 0,\ v_\theta(1) \neq 0$,

$$Y(v_\theta(1) - v_\theta(0)) + Zv_\theta(0) - v_\theta(1)(\alpha_2 a - \alpha_1 b) = 0$$

With the above conditions, this forces $Y = (\alpha_2 a - \alpha_1 b)$. Allowing $v_\theta(0) \neq 0$ yields $Z = Y$.

For boundary case II, the tangent directions are given at the end points. That is $\theta(0) = \theta_0,\ \theta(1) = \theta_1$. Assume $\beta_\theta(0) = 0$ which forces $Z = 0$ naturally. Assuming $\beta_\theta(1) = 0 = A(1) + Y + Z$ yields $Y = -A(1)$ since $Z = 0$. We now have the coefficents of β_θ for our boundary cases. Thus we have found $\nabla\Lambda = (\beta_\theta, \beta_L)$. $\square$

We now turn to find the tangential part of ∇E_λ. There is a unique $(\alpha_1\ ,\alpha_2)$ such that $\nabla E_\lambda - \nabla\Lambda$ is tangent which lies in the kernel of $D\Phi$. [8] The equation is as follows

$$\int_I \beta_\theta(s)\dot\gamma(s)\mathrm{d}s + \frac{\beta_L}{L}(b,-a) = \int_I w_\theta(s)\dot\gamma(s)\mathrm{d}s + \frac{w_L}{L}(b,-a).$$

We now establish the following result:

Lemma 6.4. *The pair $\alpha := (\alpha_1, \alpha_2)$ satisfies a linear system $A\alpha = B$, with fixed length,*

$$A = \begin{bmatrix} \int_I cy(s)\cos(\theta(s))\mathrm{d}s & -\int_I cx(s)\cos(\theta(s))\mathrm{d}s \\ \int_I cy(s)\sin(\theta(s))\mathrm{d}s & -\int_I cx(s)\sin(\theta(s))\mathrm{d}s \end{bmatrix} \ \textit{and}\ B = \begin{bmatrix} \int_I cth(s)\cos(\theta(s))\mathrm{d}s \\ \int_I cth(s)\sin(\theta(s))\mathrm{d}s \end{bmatrix}.$$

For boundary conditions I, we have the entries

$$\begin{aligned} cx(s) &= \int_0^s (x(t) - x(0)\mathrm{d}t + b(s+1) \\ cy(s) &= \int_0^s (y(t) - y(0)\mathrm{d}t + a(s+1) \\ cth(s) &= \theta(s) - \theta(0) \end{aligned}$$

For boundary conditions II, we have the entries

$$\begin{aligned} cx(s) &= \int_0^s (x(t) - x(0))\mathrm{d}t - s\int_I (x(t) - x(0))\mathrm{d}t \\ cy(s) &= \int_0^s (y(t) - y(0))\mathrm{d}t - s\int_I (y(t) - y(0))\mathrm{d}t \\ cth(s) &= \theta(s) - (\theta_1 - \theta_0)s - \theta_0 \end{aligned}$$

The pair $\alpha := (\alpha_1, \alpha_2)$ satisfies a linear system $A_L\alpha = B_L$, with variable length,

$$A_L = A + \frac{1}{L^2}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \ \textit{and}\ B_L = B + \begin{bmatrix} \frac{b\lambda}{L} - \frac{b}{2L^3}\int_I \dot\theta^2(s)\mathrm{d}s \\ \frac{a\lambda}{L} - \frac{a}{2L^3}\int_I \dot\theta^2(s)\mathrm{d}s \end{bmatrix}.$$

Proof. The (w_θ, w_L) and (β_θ, β_L) vary from boundary case to boundary case. We use the (w_θ, w_L) from the proof of ∇E_λ found earlier in this chapter. Moreover, using the first boundary condition we find $\nabla\Lambda(\theta, L) = (\beta_\theta, \beta_L)$ where,

$$\beta_L = \frac{1}{L}(\alpha_1 a + \alpha_2 b)$$
$$\beta_\theta = A(s) + (\alpha_1 a - \alpha_2 b)(s+1)$$

where $A(s) = \int_0^s \Big(\alpha_1\big(y(t) - y(0)\big) - \alpha_2\big(x(t) - x(0)\big)\Big)\mathrm{d}t.$

This can be shown from the following equation,

$$\int_I \beta_\theta(s)\gamma'(s)\mathrm{d}s + \frac{\beta_L}{L}(b, -a) = \int_I w_\theta(s)\gamma'(s)\mathrm{d}s + \frac{w_L}{L}(b, -a).$$

We take

$$\begin{aligned}\int_I \beta_\theta(s)\gamma'(s)\mathrm{d}s = &\int_I \Big[\alpha_1 \int_0^s (y(t) - y(0)\mathrm{d}t - \alpha_2 \int_0^2 (x(t) - x(0))\mathrm{d}t \\ &+ \alpha_1 a(s+1) - \alpha_2 b(s+1)\Big]\big(L\cos(\theta(s)), L\sin(\theta(s))\big)\mathrm{d}s \\ &= \alpha_1 L \int_I \Big[\int_0^s (y(t) - y(0))\mathrm{d}t + a(s+1)\Big]\big(\cos(\theta(s)), \sin(\theta(s))\big)\mathrm{d}s \\ &- \alpha_2 L \int_I \Big[\int_0^s (x(t) - x(0))\mathrm{d}t + b(s+1)\Big]\big(\cos(\theta(s), \sin(\theta(s))\big)\mathrm{d}s\end{aligned}$$

$$\begin{aligned}\int_I w_\theta(s)\gamma'(s)\mathrm{d}s &= \int_I \frac{1}{L}\big(\theta(s) - \theta(0)\big)\big(L\cos(\theta(s)), L\sin(\theta(s))\big)\mathrm{d}s \\ &= \int_I \big(\theta(s) - \theta(0)\big)\big(\cos(\theta(s)), \sin(\theta(s))\big)\mathrm{d}s\end{aligned}$$

$$\begin{aligned}\frac{\beta_L}{L}(b, -a) &= \frac{\frac{1}{L}(\alpha_1 a + \alpha_2 b)}{L}(b, -a) \\ &= \alpha_1 \frac{ab - a^2}{L^2} + \alpha_2 \frac{b^2 - ab}{L^2}\end{aligned}$$

$$
\begin{aligned}
\frac{w_L}{L}(b,-a) &= \frac{1}{L}(\lambda - \frac{1}{2L}\int_I \dot{\theta}^2(s)\mathrm{d}s)(b,-a) \\
&= (\frac{b\lambda}{L}, \frac{-a\lambda}{L}) - \frac{1}{2L^3}\int_I \dot{\theta}^2(s)\mathrm{d}s(b,-a)
\end{aligned}
$$

Representing the above as a matrix equation yields the result. The second boundary case is found using the $(\beta_\theta, \beta_L), (w_\theta, w_L)$ appropirate for case II. The details are omitted, however are quite similar to case I, with the difference shown in the presentation of the lemma. $\square$

CHAPTER 7

MINIMAX THEORY AND THE PALAIS-SMALE CONDITION

In this section we show how minimax theory can be used to identify critical points (non-min/max) of functionals, and how to establish a gradient flow for these critical points. For this, the main tool needed is the Palais Smale condition. When this condition holds, we can show that saddle points exists. To do so, we will apply the mountain pass theorem of Rabinowitz-Ambrosetti. The **Palais-Smale Condition** is a compactness condition. We formulate the condition in terms relating to the variational problem. We follow Linnér in [7].

Definition 7.1. *A sequence* $\{x_n\}$ *is a* **Palais-Smale sequence** *if* $\forall\ f(x)$, $f(x_n)$ *are uniformly bounded below and* $||\nabla f(x_n)|| \to 0$.

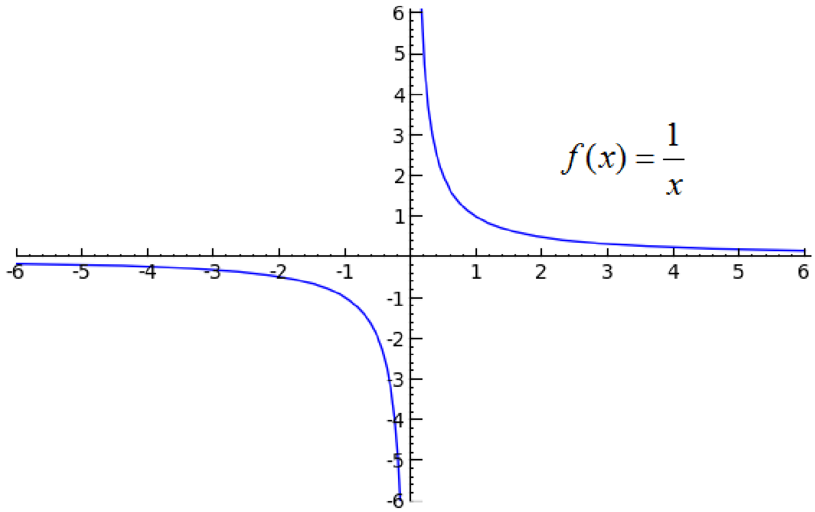
We say f satisfies the Palais-Smale condition if any Palais-Smale sequence has a convergent subsequence.

Definition 7.2. *Let* $\Lambda\ : X \to [0, \infty)$ *be a smooth nonlinear functional on a complete Hilbert manifold* X*. (Any Hilbert space* H *is a Hilbert manifold.) Let* $\{x_n\}$ *be a Palais-Smale sequence. Then, the* **Palais-Smale Condition** *is satisfied if it is always true the sequence* $\{x_n\}$ *always has a convergent subsequence.*

The condition is known to be satisfied, in the context of curve-straightening, if the domain X consists of curves of fixed length, or by adding a term to include variable length. In both cases, the gradients converge to critical points.[7] Examples of these are the functionals introduced in Chapter 2. Global minimizers exists in both of these cases.

We demonstrate two examples in order to better under stand the condition.

Example: Show that $f(x) = \frac{1}{x}$ does NOT satisfy the Palais-Smale condition, and that there is no minimizer.

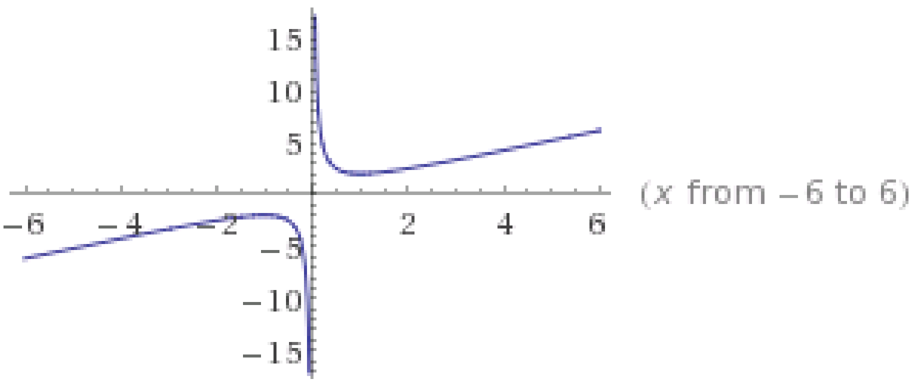


Suppose $\{x_n\}$ is a Palais-Smale sequence, that is, $f(x_n)$ is uniformly bounded below and $||\nabla f(x_n)|| \to 0$. Then,

$$\nabla f(x_n) = \nabla \frac{1}{x_n} = -\frac{1}{x_n^2} \to 0.$$

For the above to hold, it must be true $\{x_n\} \to \infty$. Thus $\{x_n\}$ has no convergent subsequence. Therefore $f(x) = \frac{1}{x}$ fails to satisfy the Palais-Smale condition. Therefore, no minimizer exists.

Now, we show an example that satisfies the Palais-Smale condition. Let $g(x) = x + \frac{1}{x}$. The graph of this function is shown below.



Similarly to the previous example, suppose $g(x_n)$ is bounded below uniformly and let $||\nabla g(x_n)|| \to 0$. Then we have

$$\lim_{n\to\infty} \nabla g(x) = 0$$
$$\lim_{n\to\infty} \nabla\big(x + \frac{1}{x}\big) = 0$$
$$\lim_{n\to\infty} \big(1 - \frac{1}{x_n^2}\big) = 0$$
$$\Rightarrow \lim_{n\to\infty} \frac{1}{x_n^2} = 1$$
$$\lim_{n\to\infty} x_n^2 = 1$$
$$\lim_{n\to\infty} x_n = 1$$

Thus, by any reordering of the $\{x_n\}$ we have a convergent subsequence. Thus, for any Palais-Smale sequence, $g(x)$ satisfies the Palais-Smale Condition.

The Palais-Smale condition is an important compactness condition. If we have such a Palais-Smale sequence, one that attains a maximum or a minimum, it converges. This allows us to travel in tragectories along the gradient. In our case we use steepest decent, so we travel in the negative gradient direction. Secondly, that the minimax actually yields a critical value of f.

For the functional $E_\lambda(\theta, L)$, Linnér has established the following result:

Proposition 7.3. *([7], Proposition 3.2) Let S be one of our constraint manifolds. Suppose that (θ_n, L_n) is a sequence in S such that both $E_\lambda(\theta_n, L_n)$ and L_n are bounded. Assume that $\nabla E_\lambda(\theta_n, L_n)$ converges in $H \times \mathbb{R}$, but not necessarily to zero. Then there is a subsequence of (θ_n, L_n) which converges in S. In particular, it follows that if the lenth is bounded then the Palais-Smale condition is satisfied.*

Now that we have an understanding of the Palais-Smale Condition, we can introduce the mountain pass theorem. A powerful theorem regarding crictical points.

Definition 7.4. *For some $\Lambda \in H$, the* **Mountain Pass Theorem** *assumes the following,*

1. $\Lambda : H \to \mathbb{R}$, *$H$ a Hilbert space*
2. $\Lambda \in C^1(H, \mathbb{R})$ *and Λ' is lipschitz continuous on bounded subsets of H*
3. *Λ satisfies the Palais-Smale condition*
4. $\exists\ r, a \in \mathbb{R}^+$ *such that* $\Lambda[x] \geq a$ *if* $||x|| = r$
5. $\exists v \in H$ *with* $||v|| > r$ *such that* $\Lambda[v] \leq 0$

If the above holds, then we define a set $\Gamma = \{g \in C[0,1] \ : \ g(0) = 0,\ g(1) = v\}$ and let

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} \Lambda[g(t)],$$

then the theorem yields that c is a critical value of Λ.

Using a minimax theorem (not the mountain pass theorem), it was shown in [5] that there exist (unstable) critical points of saddle point type for E_λ. An example of this is provided in the final chapter of this thesis.

We now provide a conjecture regarding the evolution of elastic curves and the rotation number of the intital curve.

Definition 7.5. *The* **rotation index***, k, of a closed curve in the plane is an integer representing the total number of times that curve travels in complete counterclockwise rotations.*

The total curvature fuctional is defined to be $TK(\gamma) = \int_a^b \kappa(\ell)\mathrm{d}\ell$.

Lemma 7.6. *Where k is the rotation index of a curve γ, we have the following,*

$$TK(\gamma) = \int_a^b \kappa(s)\mathrm{d}s = 2\pi k.$$

Proof. We have $\ell : 0 \to L$ where L is the length of the curve and $s : 0 \to 1$. Thus we have $\ell = Ls$ and $\mathrm{d}\ell = L\mathrm{d}s$. Thus, together with the fact $\kappa(s) = \frac{\dot{\theta}(s)}{L}$, we have

$$\begin{aligned} TK(\gamma) &= \int_0^L \kappa(\ell)\mathrm{d}\ell = \int_0^1 \kappa(s)L\mathrm{d}s \\ &= \int_0^1 \frac{\dot{\theta}(s)}{L} L\mathrm{d}s \\ &= \int_0^1 \dot{\theta}(s)\mathrm{d}s \\ &= \theta(1) - \theta(0) = 2\pi k \end{aligned}$$

□

Lemma 7.7 (Conjecture)**.** *Let γ be a smooth, regular, closed curve over the functional E or E_λ with rotation index k. Then, the curve will evolve to,* *(a) A single cover of figure* 8 *with* $k = 0$ *(when stable)*

(b) k *coverings of a circle when* $k \neq 0$.

Note the rotation number of the original curve has its orientation preserved over the evolution. Examples of this can be found in chapter 9. These examples allow the reader to clearly see the conjecture demonstrated.

CHAPTER 8

COMPUTATION OF ELASTIC CURVES

To compute the minimizers (or more generally, the critical points) we evolve the curve in the direction of steepest descent. This is the so-called negative gradient flow. It is therefore important to know that if one evolves in the direction of the negative gradient, then the adherent points exist. This is established by the Palais-Smale condition for $E_\lambda(\theta, L)$, as discussed in the previous chapter, which was proved in [7].

The variables are the tangent indicatrix $\theta(s)$ and the length L. In Section 6 we computed that gradients $\nabla E_\lambda(\theta, L) = (w_\theta, w_L)$ and $\nabla \Lambda(\theta, L) = (\beta_\theta, \beta_L)$ of the energy functionals. This gives the steepest descent step

$$\theta(s) = \theta(s) + h(\beta_\theta(s) - w_\theta(s))$$
$$L = L + h(\beta_L - w_L),$$

were h is a step length along which to evolve our curve. In practice, the step length is set to $h = 1$, and reduced as needed for each curve we are evolving. This steepest descent is equivalent to taking an Euler step for the differential equation along a trajectory. I.e., if θ_u and L_u depend on the parameter u along the trajectory, then

$$\begin{bmatrix} \frac{d\theta_u}{du}(s) \\ \frac{dL_u}{du}(s) \end{bmatrix} = h\Big(\nabla \Lambda_u(\theta_u, L_u)(s) - \nabla E_\lambda(\theta_u, L_u)(s)\Big)$$

In the future, we may consider adaptive step lengths h, and apply Runge-Kutta's method for solving initial value problems.

Due to the Palais-Smale condition, the gradient descent method will produce critical points (the limits exist) in our infinite dimensional Hilbert Spaces. Unfortunately, curves in an infinite dimensional space are not really computatable, except in some simple cases when characterizations happen to reduce to something simple. Our strategy is therefore to discretize the problem. To do so, we evaluate the initial curve $\gamma(s)$ and it's indicatrix $\theta(s)$ at a large number of points (refinement). This gives us $x(s_i)$, $y(s_i)$ and $\theta(s_i)$ at these points s_i. We then approximate the derivatives by finite differences (we use a non-uniform central difference). and the integrals by numerical integration (currently, we use Matlab's built-in functions *trapz()* and *cumtrapz()*).

The *Matlab* code for solving this includes the following functions m-files[1].

```
%Sample script to evolve a curve (the figure 8 with loop):
clf, clear all, format compact, hold off
ax = [-.5 3.5 -1.5 1.5];
bx1=[0 0 1 2 2]; bx2=[2 2 1.5 1.5 2 2]; bx3 = [2 2 1 0 0];
by1=[0 -1 0 1 0]; by2=[0 -.5 -.5 .5 .5 0]; by3 = [0 -1 0 1 0];
t = linspace(0,1,2000); x1 = bval(bx1,t,0); y1 = bval(by1,t,0);
t = linspace(0,1,2000); x2 = bval(bx2,t,0); y2 = bval(by2,t,0);
t = linspace(0,1,2001); x3 = bval(bx3,t,0); y3 = bval(by3,t,0);
x = [x1, x2(2:end-1), x3(1:end)];
y = [y1, y2(2:end-1), y3(1:end)];
t = linspace(0,1,length(x));
len=1; %variable length
type=3; %closed curve
lambda = 1;
p = [x(1), y(1)];
hh = geth(t);
[t,th,L,x,y] = reparametrize (t,hh,x,y);
hh = geth(t);
a = x(end)-x(1);
b = y(end)-y(1);
subplot(121); plot(x,y,'r'); axis square, axis equal, axis off
inc = .25; %Euler Step
for i=2:500
 [th,L,x,y] = evolve(t,hh,th,L,lambda,p,a,b,inc,len,type);
end
subplot(122); plot(x,y,'r'); axis square, axis equal, axis off
```

```
function [x,y]  = evalcurve(t,th,L,p)
 x = p(1) + L*cumtrapz(t,cos(th));
 y = p(2) + L*cumtrapz(t,sin(th));
```

[1]The *Matlab code is the intellectual property of Scott Kersey*

```
function lmda  = getlambda(t,hh,th,dth,x,y,a,b,L,lambda,len,type)
% len==0 fixed length, len ~= 0, variable length
% type==1 (fixed end points)
% type==2 (fixed end points and tangents)
% type==3 (closed with fixed end points and tangents)
A = []; B=[];
switch type
case 1
 ctx = cumtrapz(t,x-x(1)) + a*(t+1);
 cty = cumtrapz(t,y-y(1)) + b*(t+1);
 tth = th-th(1);
case 2
case 3
 ctx = cumtrapz(t,x-x(1))-t*trapz(t,x-x(1));
 cty = cumtrapz(t,y-y(1))-t*trapz(t,y-y(1));
 tth = th-t*(th(end)-th(1))-th(1);
end
 A(1,1) = L*trapz(t,cty.*cos(th));
 A(1,2) = -L*trapz(t,ctx.*cos(th));
 A(2,1) = L*trapz(t,cty.*sin(th));
 A(2,2) = -L*trapz(t,ctx.*sin(th));
 B(1) = trapz(t,tth.*cos(th));
 B(2) = trapz(t,tth.*sin(th));
if len ~= 0
 A(1,1) = A(1,1) +  a*b/L^2; A(1,2) =  A(1,2) + b^2/L^2;
 A(2,1) =  A(2,1) - a^2/L^2; A(2,2) =  A(2,2) - a*b/L^2;
 B(1) = B(1) + lambda*b/L - b*trapz(t,dth.^2)/(2*L^3);
 B(2) = B(2) - lambda*a/L + a*trapz(t,dth.^2)/(2*L^3);
end
 lmda = A \ B';
```

```
function [t,th,L,x,y] = reparametrize (t,hh,x,y)
dx = diff(x); dy = diff(y);
L = sum(sqrt(dx.*dx + dy.*dy)); %get length
t = cumtrapz(t,[0,sqrt(dx.*dx + dy.*dy)]); t = t/t(end);
ddx = []; ddy = [];
ddx(1) = dx(1); ddy(1) = dy(1);
h = diff(t);
for i=2:length(t)-1
 ddx(i) = (h(i)*dx(i-1) + h(i-1)*dx(i)) / (h(i)+h(i-1));
 ddy(i) = (h(i)*dy(i-1) + h(i-1)*dy(i)) / (h(i)+h(i-1));
end
ddx(1) = (h(1)*dx(end)+h(end)*dx(1))/(h(1)+h(end));
ddy(1) = (h(1)*dy(end)+h(end)*dy(1))/(h(1)+h(end));
ddx(end+1) = dx(1); ddy(end+1) = dy(1);
dx = ddx; dy = ddy;
th = atan2(dy,dx);
dth = th(2:end)-th(1:end-1);
idth = find(abs(dth) > pi/2);
for i=1:length(idth)
 k = idth(i);
 if th(k)>0  dth(k) = th(k) + th(k+1);
 else dth(k) = -th(k) - th(k+1); end
end
th = [th(1), th(1) + cumsum(dth)];
[x,y] = evalcurve(t,hh,th,L,[x(1), y(1)]);
```

```
function [th,L,x,y] = evolve(t,hh,th,L,lambda,p,a,b,inc,len,type)
%len == 0 for fixed length, otherwise variable length
 [x,y] = evalcurve(t,hh,th,L,p);
 dth = diff(th)./diff(t);
 dth = [dth(1), (dth(1:end-1) + dth(2:end))/2, dth(end)];
 lmda = getlambda(t,hh,th,dth,x,y,a,b,L,lambda,len,type);
 A = lmda(1)*cumtrapz(t,y-y(1)) ...
   - lmda(2)*cumtrapz(t,x-x(1));
switch type
case 1
 alphaTh = (th - th(1))/L;
 betaTh = A + (lmda(1)*b-lmda(2)*a)*(t+1);
case 3
 alphaTh = (th -(th(end)-th(1))*t- th(1))/L;
 betaTh = A - A(end)*t;
end
Dth = betaTh - alphaTh;
th = th + Dth*inc;
if len ~= 0
 alphaL = lambda - trapz(t,dth.^2) / (2*L^2);
 betaL = (lmda(1)*a + lmda(2)*b)/L;
 DL = betaL-alphaL;
 L = L + DL*inc;
end
[x,y] = evalcurve(t,hh,th,L,p);
```

CHAPTER 9

EXAMPLES

The circle is a stable critical point for $E_\lambda(\theta, L)$ with free length. As λ is increased, more penalty is placed on the length component of the functional, and the circle is correspondingly smaller. In this example, a distorted ellipse is shown to evolve to a circle. The examples also show the effect of increasing λ.

It is known (see [5]) that the multiple coverings of the 'figure eight' (leminiscate) are unstable critical points for $E(\theta, L)$ with fixed length, while one covering is stable. In Figure 9.4, the original curve is a slighty deformed double cover of the leminiscate, which evolves to a single cover of the leminiscate, i.e., a stable configuration.

In Figure 9.1 we have a Polynomial curve in B-form, fixed length. The curve attains what is similar to a section of a circle. In Figure 9.2 a deformed ellipse of fixed length is evolved. Note the conjecture is satisfied in this example, almost trivially. A variable length case is shown in Figure 9.3. Figure 9.5 is to confirm the conjecture for $k = 0$. Figures 9.6 and 9.7 are shown to validate the conjecture. The image in Figure 9.8 is a figure eight shape with an extra loop inside the curve. In Figure 9.9 we have a double loop. These curves evolve into the stable curves predicted by the conjecture.

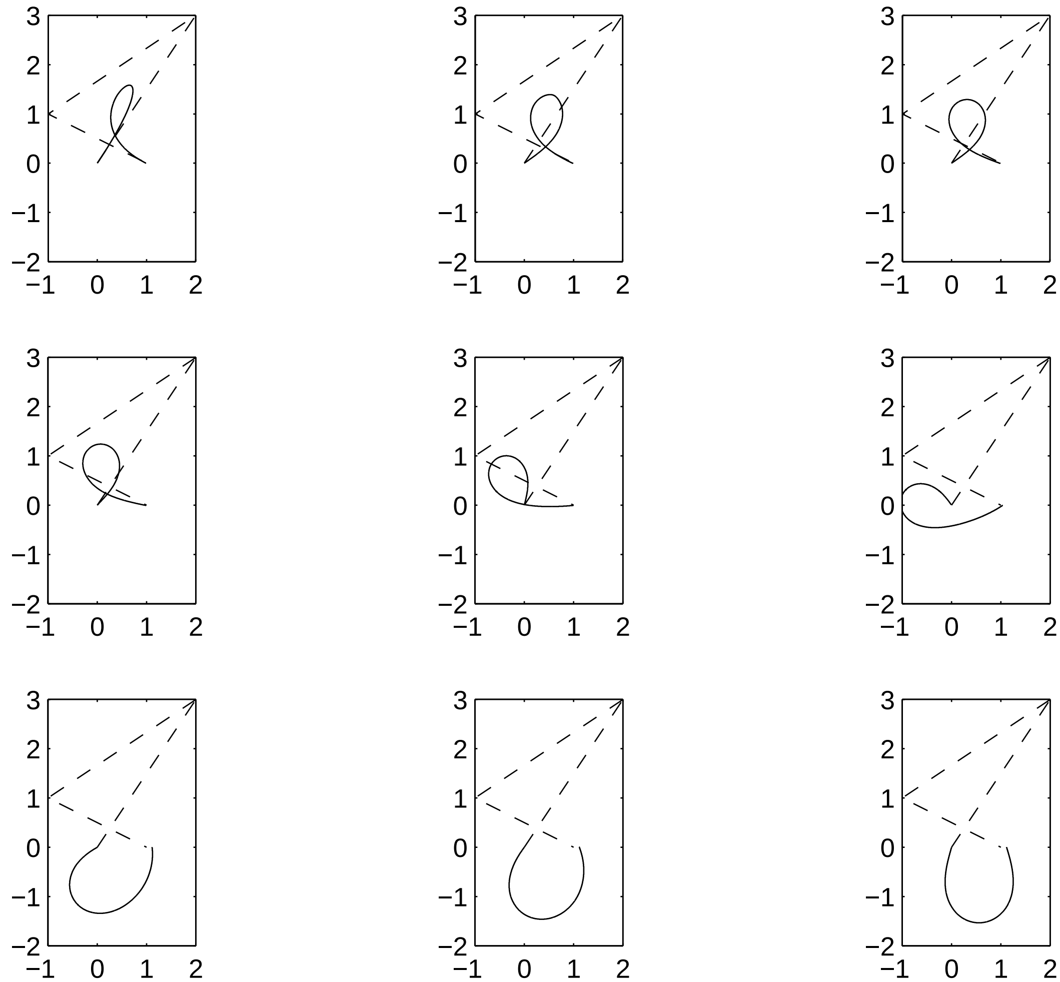


Figure 9.1: Polynomial Curve in B-form, Fixed Length

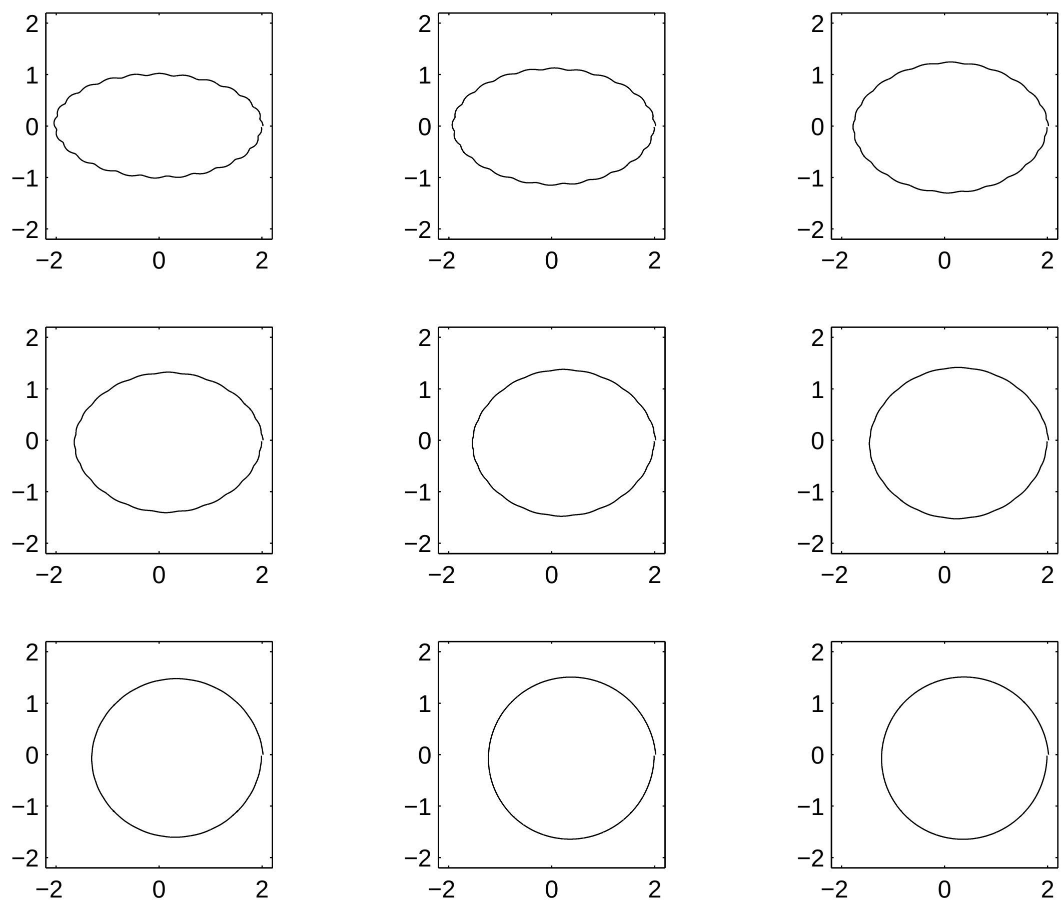


Figure 9.2: Deformed Ellipse, Fixed Length

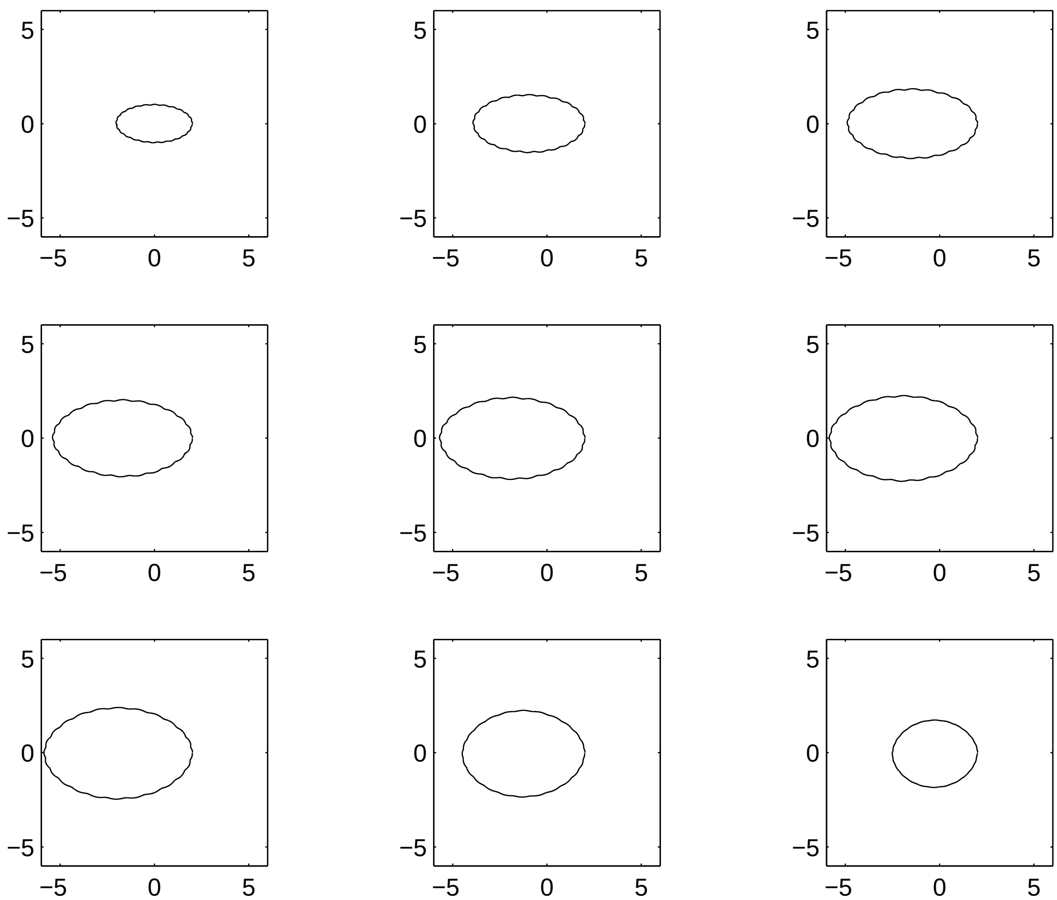


Figure 9.3: Deformed Ellipse, Variable Length

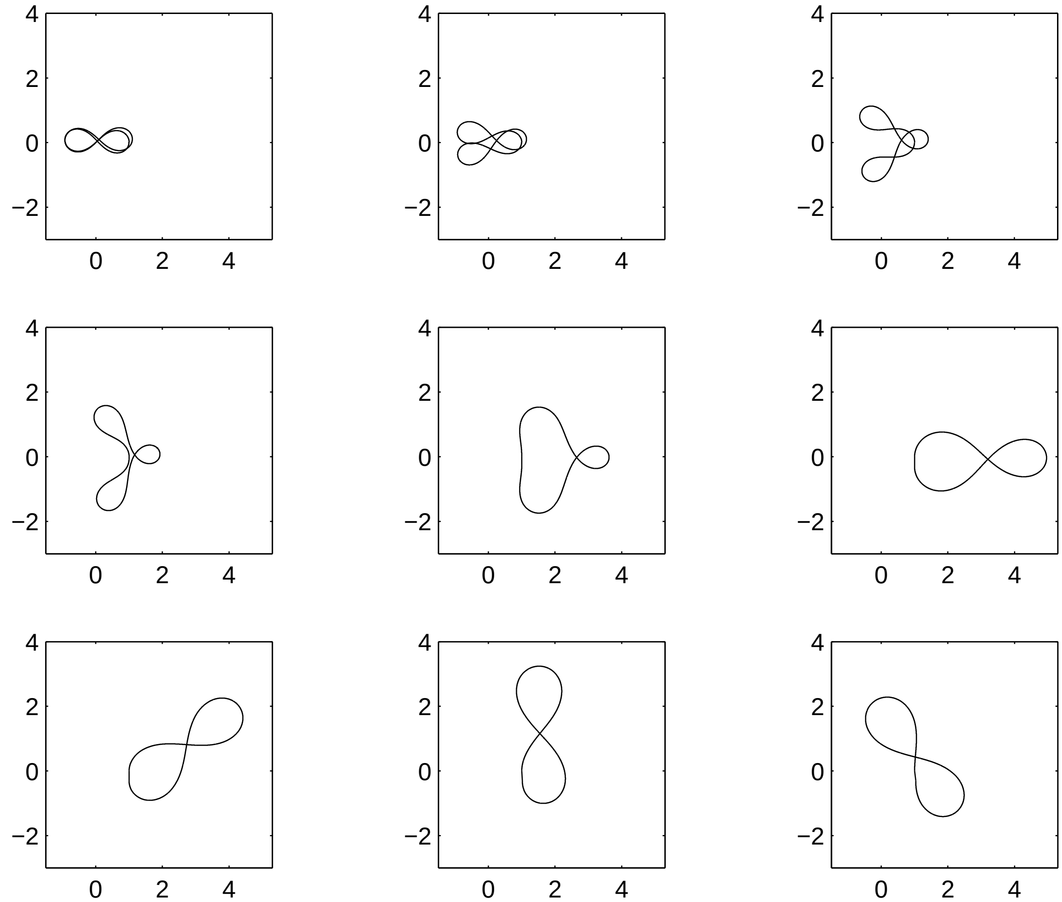


Figure 9.4: Double Figure Eight, Fixed Length

Figure 9.5: Figure Eight

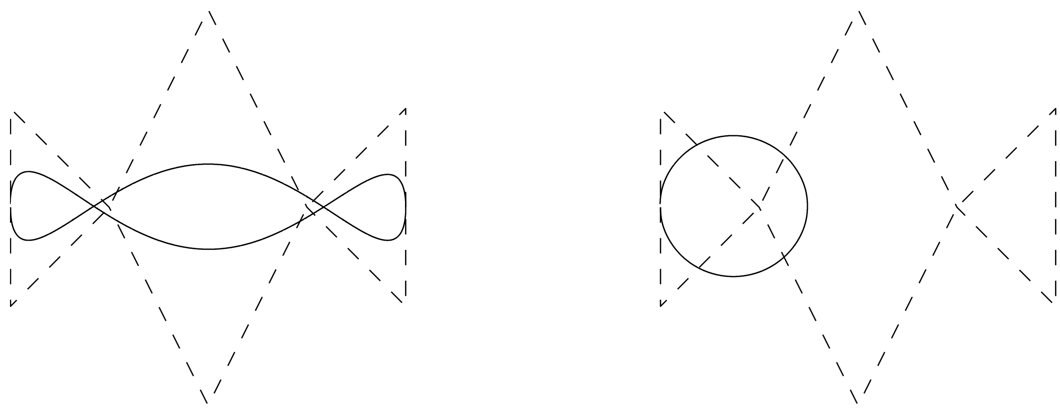


Figure 9.6: Triple Loop Figure Eight

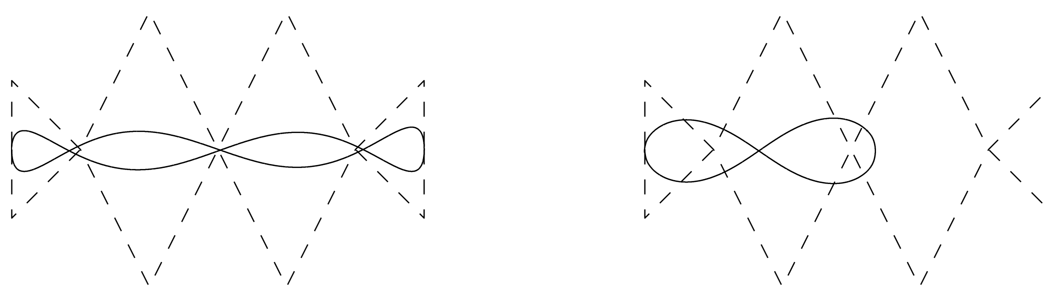


Figure 9.7: Quad Loop Figure Eight

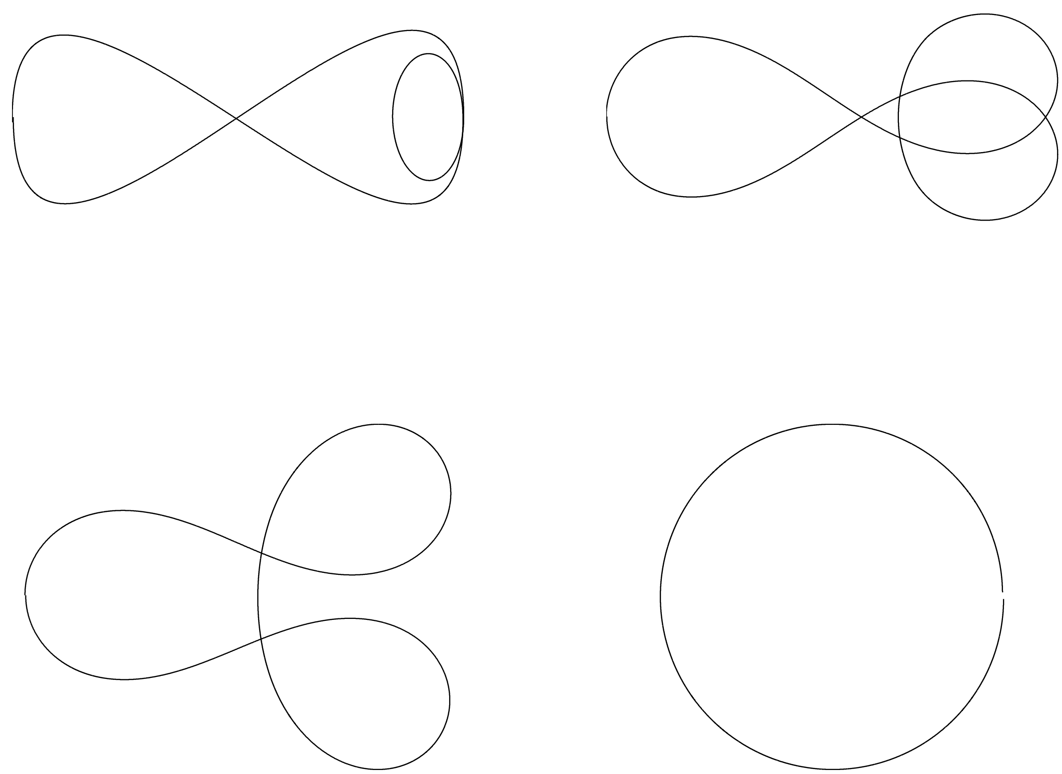


Figure 9.8: Figure Eight With Loop

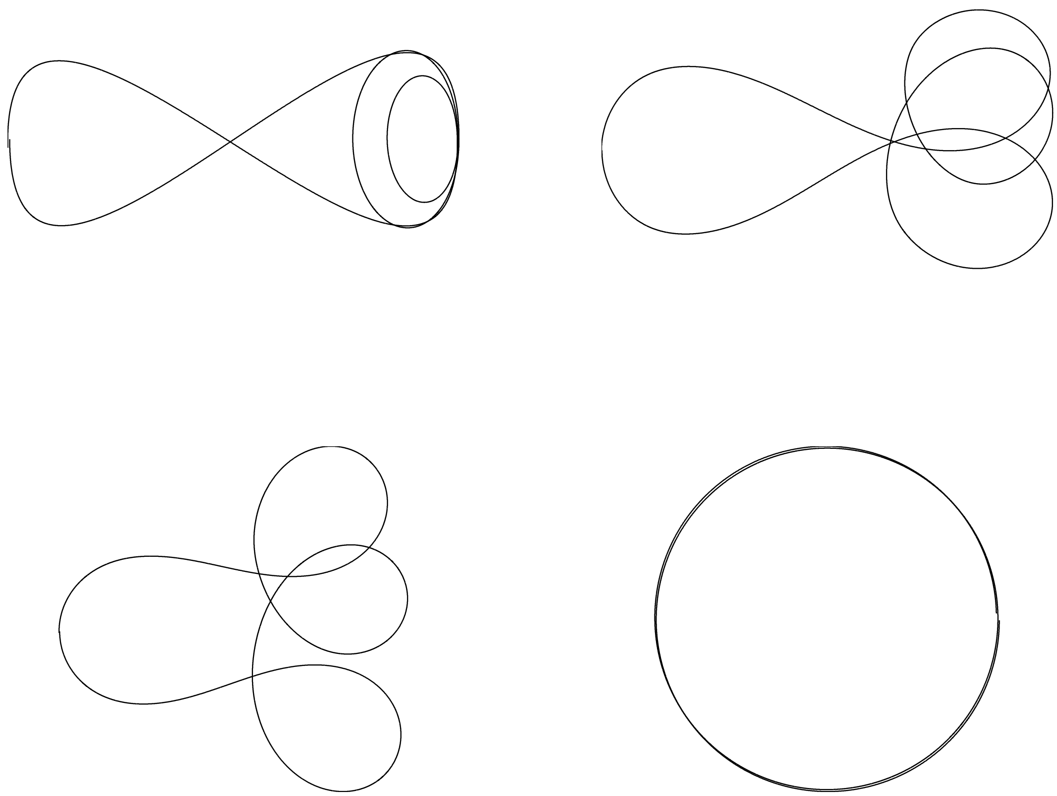


Figure 9.9: Figure Eight With Double Loops

CHAPTER 10
CONCLUSION

In this thesis, we sought out to clarify many details from the literature. By clarifying these details, we aim to make the material more accessable to future problem solvers. These clarifications come from both notational changes and inclusion of many proofs considered to be "obvious." We have left little to the reader, only referring the reader to proofs inappropriate for the thesis, or are not novel.

We provide MATLAB code which is not previously found in the literature. This code allows for the evolution of curves into their minimal bending energy solutions. These MATLAB scripts allowed for the development of the graphics seen in this thesis. Several of the examples are never-before-seen in this context.

The next step would be to examine different boundary conditions. A proper treatment of the problem in three dimensions is needed. Defining a tangent indicatrix becomes bothersome in this case, and requires attention. This problem has been attempted in the framework of a spline. Further work can surely be done on the subject from the viewpoint of a spline. Expanding the applications of critical point theory is highly desired, and would be attempted in this thesis given more time. Also, proof of the conjecture is desired.

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