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Minimal Graphs with a Specified Code Map Image

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Minimal Graphs with a Specified Code Map Image

Cover Page Footnote

The author thank Dr. C.Kang for suggesting that graphs be studied based on the image of their code maps.

Abstract

Let G be a graph and $e_1, \dots, e_n$ be n distinct vertices. Let ρ be the metric on G . The code map on vertices, corresponding to this list, is $c(x) = (\rho(x, e_1), \dots, \rho(x, e_n))$. This paper introduces a variation: begin with $V \subseteq \mathbb{Z}^n$ for some n , and consider assignments of edges E such that the identity function on V is a code map for $G = (V, E)$. Refer to such a set E as a *code-match*.

An earlier paper classified subsets of V for which at least one code-match exists. We prove

- If there is a code-match E for which (V, E) is bipartite, then (V, E) is bipartite for every code-match E .
- If there is a code-match E for which (V, E) is a tree, then E is unique.
- There exists a code-match E such that (V, E) has a $(2^{n-1} + 1)$ -vertex-coloring.

Keywords: Metric Dimension; Distance in Graph; Coloring; Trees

AMS Classification: Primary 05C12, Secondary 05C62

Introduction

Assume all graphs are undirected and contain no multiple edges nor loops. For all other basic terminology, see [2]. We denote the shortest path distance function on vertices of G by ρ_G or, when the graph is clear from context, ρ . Let G be a graph and let $w_1, \dots, w_n$ an irredundant list of n vertices of G . Define the *code map* of this selection to be the function from V to n -tuples of integers given by

$$c : v \mapsto (\rho(v, w_1), \dots, \rho(v, w_n)) .$$

We refer to the image of v under this as its *code*.

Given $n \in \mathbb{N}$ and $V \subseteq \mathbb{Z}^n$, we say V is *codeable* if there is a set of edges E for which (V, E) becomes a connected graph and V contains a list of n vertices whose associated code map is the identity function. Such a set E is said to *code-match* V .

In [4], codeable set are characterized. A result is that if V is codeable, there are many choices for code-matches. Given a codeable set V , we pose the question: can we select a code-match for which (V, E) has certain properties? This paper proves three results in this direction.

- (1.a) If there is a code-match E for which (V, E) is bipartite, then (V, E) is bipartite for every code-match E .
- (1.b) If there is E for which (V, E) is a tree, then that E is unique.
- (1.c) There is E for which (V, E) is $(2^{n-1} + 1)$ -vertex-colorable.

The bound $2^{n-1} + 1$ should be something that can be improved for large n . For $n = 2$, this means there is a code-match whose chromatic number is 3 or less.

1 Notation and Formal Details

For $n \in \mathbb{N}$, let $Ind(n)$ be the set of natural numbers from 1 to n , inclusive.

Let $G = (V, E)$ be a graph. We model each edge as an unordered pair of two different vertices. For $x \in V$, let $N_G(x)$ be the neighbors of x . When the graph is clear from context, write $N(x)$ for $N_G(x)$.

Let $n \in \mathbb{N}$. We consider members of $\mathbb{Z}^n$ as vertices of graphs. In this context, when we use a variable x to represent a member of $\mathbb{Z}^n$, then, for each $i \in Ind(n)$, we represent the i -th coordinate in x by x_i . An edge is an unordered pair of n -tuples.

The earlier paper restricts the set of unordered pairs that can be edges in the constructions of interest. Let $Ed(n)$ be the set of unordered pairs of members xy from $\mathbb{Z}^n$ such that

(2.a) $x \neq y$, and

(2.b) for $i \in Ind(n)$, $|x_i - y_i| \leq 1$.

Let $V \subseteq \mathbb{Z}^n$ be a finite, non-empty subset. Define $Ed(V)$ to be all $xy \in Ed(n)$ such that $x, y \in V$. We say V is *codeable (of rank n)* if there is $E \subseteq Ed(V)$ for which

(3.a) (V, E) is a connected graph, and

(3.b) there is a list $e_1, \dots, e_n \in V$ with respect to which the code map is the identity function.

Furthermore, any such E is said to *code-match* V . We refer to $G = (V, E)$ as *self-coded graph (of rank n)*.

Let $V \subseteq \mathbb{Z}^n$ be a codeable subset and let $E \subseteq Ed(V)$. We say E meets the *match criterion (for V)* if and only if

(4) for every $i \in Ind(n)$ and every $x \in V$ for which $x_i > 0$, there is $y \in V$ for which $xy \in E$ and $y_i = x_i - 1$.

We paraphrase Definition 2.1 and Proposition 2.1 from [4] as follows:

Proposition 1.1. *Let $V \subseteq \mathbb{Z}^n$ be a codeable set. Let $W \subseteq V$, and let $E \subseteq Ed(W)$.*

(A) *If E satisfies the match criterion for W , then W is codeable.*

(B) *Suppose W is codeable. Then E code-matches W if and only if it satisfies the match criterion for W .*

Clearly, not every subset $V \subseteq \mathbb{Z}^n$ is codeable. Any member of a codeable set V must have non-negative entries. Furthermore, for each $i \in Ind(n)$, exactly one member (which must be the e_i of (3)) has 0 for its i -th coordinate.

Codeable sets are characterized in [4]. In particular, Corollary 1.1 of that paper states that

(5) The image of a code map on a connected graph and a choice of n vertices is codeable.

The assertion is true even if the code map is not injective.

2 Future and History

The primary purpose of this paper is to argue that minimal code-matches merit more study. In passing, we identified two questions, one concerning homotopy and the other about coloring.

An important motivation for the present author was **Hernando et al.**[6]. Those mathematicians obtained a tight bound on the maximum size of graph of dimension n and diameter d by constructing, in the current language, a codeable set of n -tuples in which every coordinate was d or less. The set of *all* such tuples is not codeable. However, a step in [6] constructs the largest such set (and establishes that its size is maximum). That paper required a “maximal” code-match of edges. The set of edge needed to be large in order to imply that the diameter of the graph is d . (In a graph in which every member is d units from a fixed vertex, the diameter could be as large as $2d$.)

This powerful result, and a problem posed by **Dr. Cong Kang**, spurred the current investigation of codeable sets.

Are there implications about metric dimension, if we switch from maximal choices (of vertices and edge)s to minimal choices? The history of metric dimensions suggests that small changes to a graph can change dimension radically. **Eroh et al.** in [3] give examples in which an added edge can wildly alter dimension. The same paper considers a family of graphs for which an additional edge has a very predictable effect. Examples in Section 2 of [4], and the discussion about Figure 1 above, show that there are many ways to assign edges to a specific codeable set. This flexibility on edges gives a heuristic reason to *doubt* that a general assertion related to metric dimension can be formulated in the present language.

There are two limited problems for which solutions might lead to results related to metric aspects.:

- (6.a) Let E be a minimal code-match for a codeable set V . Let d be the maximum value of any coordinate of any member of V . Under a reasonably broad hypothesis, is there a lower bound for the diameter of (V, E) which is significantly greater than d ? Likewise, is there an upper bound which is significantly $< 2d$?
- (6.b) Let V be a codeable of rank n . For each list $\omega = w_1, \dots, w_{n-1}$ of $n - 1$ members of V and each code-match E , define the following set. Let f be the code map for ω . Let $\delta(\omega, E)$ be all $v \in V$ for which there is $x \neq v$ such that $f(x) = f(v)$. Now let Δ be the minimum of all sizes of all $\delta(\omega, E)$. Under a reasonably broad hypothesis, is there an bound for $\Delta/|V|$?

3 Minimal Submatches

Let $V \subseteq \mathbb{Z}^n$ be a codeable set. For E a code-match, we refer to a subset of E that is also a code-match as a *submatch*. Refer to a code-match E as *minimal* if there is no proper submatch. Then every code-match contains at least one minimal submatch.

A general challenge is to identify graph properties of (V, E) in which E is a minimal code-match.

Obviously, a code-match E for which $G = (V, E)$ is a tree is minimal.

Lemma 3.1. *Let $G = (V, E)$ be a self-coded graph of rank n . Fix $i \in \text{Ind}(n)$. Then G is a tree if and only if*

(7.a) *For $xy \in E$, $x_i \neq y_i$, and*

(7.b) *For $x \in V$ such that $x_i > 0$, there is a unique $y \in V$ such that $y_i = x_i - 1$ and $xy \in E$.*

Proof. Recall that a graph G is *not* a tree if and only if there is a circuit of G that uses some edge exactly once. Furthermore, if G is not a tree, there is a circuit in which no edge is repeated.

We have a series of cases. Let e_i be the i -th generator of the code map. That is, e_i is the unique member of V whose i -th coordinate is 0.

Suppose $xy \in E$ and $x_i = y_i$. Consider the circuit which starts with a geodesic from e_i to x , then xy , and then a geodesic from y to e_i . The circuit traverses the edge xy exactly once. Hence, there is a non-degenerate circuit in G , and G is not a tree.

Suppose y, z are different neighbors of x such that $y_i = x_i - 1 = z_i$. Consider the circuit composed of a geodesic from e_i to y , then yx and xz , and then a geodesic from z to e_i . In this circuit, yx and xz are each traversed exactly once.

Conversely, suppose that C is a circuit in which no edge is repeated. Let $x \in C$ be a vertex for which $\rho(x, e_i) = x_i$ is a maximum among vertices in C . Then x has two different neighbors in C , and $\rho(y, e_i)$ is x_i or $x_i - 1$ for each neighbor y . Hence, at least one of (7.a,b) fails. $\square$

Theorem 3.2. *Let V be a codeable set of rank n . Then there is at most one code-match E for V such that $G = (V, E)$ is a tree.*

Proof. First, suppose the rank n of V is odd. Let $n = 2k + 1$.

We start with notation. Let $P(n)$ be the set of subsets of $\{1, \dots, n\}$. Let $x \in V$. Define

$$f(x) = \sum_{i=1}^n x_i .$$

For $J \in P(n)$, let $x(J)$ be the n -tuple such that, for each index i ,

(8.a) $x(J)_i = x_i - 1$ if $i \in J$, and

(8.b) $x(J)_i = x_i + 1$ if $i \notin J$.

Observe that $f(x(J)) > f(x)$ if and only if $|J| \leq k$, and that $f(x(J)) < f(x)$ otherwise.

Suppose that E code-matches V , and $G = (V, E)$ is a tree. By Lemma 3.1, for $x \in V$, each neighbor of x in G is $x(J)$ for exactly one $J \in P(n)$.

Let $S(x)$ be the set of $i \in \text{Ind}(n)$ for which $x_i > 0$. The matching criterion and Lemma 3.1 imply that there is an indexed partition $J_1, \dots, J_r$ of $S(x)$ such that the neighborhood of x is

$$N(x) = \{x(J_1), \dots, x(J_r)\} .$$

Note that we allow $J_u = \emptyset$ for at most one index u .

We proceed by contradiction. Suppose that E_1 and E_2 are different two code-matches for V . For $j = 1, 2$, let $G_j = (V, E_j)$. We let $N_j(x)$ be the neighborhood of $x \in V$ with respect to G_j . Assume each G_j is a tree. Let X be the set of $x \in V$ such that $N_1(x) \neq N_2(x)$.

Assume $X \neq \emptyset$. Let $x \in X$ so that $f(x)$ is maximal for f restricted to X .

Then there are partitions $J_1, \dots, J_r$ and $K_1, \dots, K_s$ of $S(x)$ such that

$$N_1(x) = \{x(J_1), \dots, x(J_r)\} \quad \text{and} \quad N_2(x) = \{x(K_1), \dots, x(K_s)\}.$$

Without loss of generality, assume J_1 is not equal to any K_v .

If, for any index u , J_u has size $\leq k$, then $x(J_u)$ is a neighbor of x and $f(x(J_u)) > f(x)$. The choice of x implies that $x(J_u)$ is a neighbor in both G_1 and G_2 . Consequently, $J_u = K_v$ for some index v . Likewise, each K_v with k or fewer members must be J_u for some index u . It follows that

$$(9.a) \quad |J_1| \geq k + 1,$$

$$(9.b) \quad \text{For } 2 \leq u \leq r, |J_u| < k \text{ and } J_u = K_v \text{ for some index } v.$$

$$(9.c) \quad \text{We can renumber the } K_v \text{ list so that for some index } t \geq 2, \text{ the sublist } K_1, \dots, K_t \text{ lists all members not in the } J\text{-partition and this sublist is a partition of } J_1 \text{ which does not contain } J_1.$$

But then there is some K_v which is a non-empty subset of J_1 with $\leq k$ members. This K_v would also appear in the J partition, which contradicts the partition property.

It remains to address the assertion when n is even. We resolve this case by embedding it in the $n + 1$ case.

Let $G = (V, E)$ be self-coded graph of rank n , where n is even. For $i \in \text{Ind}(n)$, let e_i be the member of V with 0 for its i -th coordinate. Define $H = (W, F)$ to be the graph created by adding (to G) a new vertex, labeled e_{n+1} , and a single edge $e_n e_{n+1}$. The tuple represented by e_n is

$$e_n = (a_1, \dots, a_{n-1}, 0).$$

Let c be the code map for H based on the list $e_1, \dots, e_n, e_{n+1}$. Then

$$(10.a) \quad \text{The image of } x = (x_1, \dots, x_n) \in V \text{ is } c(x) = (x_1, \dots, x_n, x_n + 1).$$

$$(10.b) \quad c(e_{n+1}) = (a_1 + 1, \dots, a_{n-1} + 1, 1, 0).$$

$$(10.c) \quad \text{Let } V^+ = c(W) \text{ and } E^+ = \{c(x)c(y) : xy \in E\}. \text{ Note that } V^+ \text{ depends only on } V.$$

$$(10.d) \quad \text{Then } G^+ = (V^+, E^+) \text{ is a self-coded graph and } c \text{ is a graph isomorphism } H \longrightarrow G^+.$$

$$(10.e) \quad G \text{ is a tree if and only if } G^+ \text{ is a tree.}$$

Now suppose V is codeable of rank n and E_1, E_2 are code-matching sets that generate two trees $G_j = (V, E_j)$. Then (V^+, E_1^+) and (V^+, E_2^+) are trees based on the same codeable set. By the previous case, $E_1^+ = E_2^+$. Therefore $E_1 = E_2$. $\square$

Figure 1: An Initial Graph

3.1 Minimal and Not a Tree

An example illustrates the variability among minimal matches. Consider the following code-able set of rank 3. We assign a variable name to each vertex:

$$\begin{aligned} &A(0, 2, 2), \quad B(2, 0, 2), \quad C(2, 2, 0), \\ &D(1, 3, 3), \quad E(3, 1, 3), \quad F(3, 3, 1), \\ &G(1, 1, 3), \quad H(1, 3, 1), \quad I(3, 1, 1). \end{aligned}$$

The list of vertices for the canonical code map is A, B, C . Each of the other triples has at least one coordinate equal to 1, which indicates it has an edge to the corresponding member of A, B, C . Figure 1 illustrates the graph created by adjoining these edges. As it happens, this initial choice of edges is a code-match.

Now consider adding a new vertex $P(2, 2, 2)$. Here are three ways to expand the edges:

(11.a) Add PD , PE and PI , or

(11.b) Add PD and PF , or

(11.c) Add PF and PH .

These options result in three non-isomorphic graphs. Each is minimal. Note that the number of edges is not the same for the three.

Adding six edges

(12) from P to each of D, E, F, G, H and I respectively,

also produces a code-match. This observation shows that the minimality condition identifies comparatively simple graphs among all that arise from code-matches.

There is variation among minimal matches. But how varied?

Question 1. For each $n \in \mathbb{N}$, is there a codeable set V which has two different minimal code matches, of which one is a tree?

In our previous example, we generated distinct minimal submatches by adjusting edges attached to a single vertex. Lemma 3.1 suggests that creating a second minimal match on vertices from a tree is much more complicated. The lemma reduces the criterion for being a tree to a feature attached to a *fixed* index i . But, to fail to be a tree, there must be a “bad” edge for *every* i . The failure for all indices cannot be shifted to a lone vertex.

4 Coloring

By a coloring of a graph $G = (V, E)$, we mean a proper vertex coloring; that is, each vertex is assigned a color so that no edge links vertices of the same color.

We open this section with an observation related to Question 1. Suppose there is a codeable set V which has a code-match that determines a tree. A comment in [4] implies that every code-match is bipartite! We briefly expand (15) from that paper into a formal theorem.

Proposition 4.1. *Let V be a codeable set of rank n . The following assertions are equivalent*

(13.a) *There is a code-match E such that (V, E) is bipartite.*

(13.b) *For each $x = (x_1, \dots, x_r) \in V$, either $x_i - x_j$ is even for all indices $i \neq j$ or $x_i - x_j$ is odd for all $i \neq j$.*

(13.c) *For every code-match E , (V, E) is bipartite.*

Proof. In this argument, define the *parity* of a natural number to be its congruence mod(2).

First, let G be connected graph and $x \in V(G)$. Let $Ev_G(x)$ and $Odd_G(x)$ be the subsets of vertices whose distance from x is even and odd, respectively. If G is bipartite, then the unordered pair $\{Ev_G(x), Odd_G(x)\}$ is the unique partition of $V(G)$ into two sets such that every edge has an end in each subset.

Assume (13.a). Let $G = (V, E)$ for the specified E . Let i, j be two different indices. Let e_i and e_j be the corresponding members in the list for the canonical code map for V . For each $x \in V$, x_i and x_j are distances from e_i and e_j , respectively. By our first observation, x_i and x_j must always have the same parity or must always have opposite parity. This translates to condition (13.b)

Assume (13.b). Suppose E is any code-match. Let $xy \in E$. There is some index i for which $|x_i - y_i| = 1$. In other words, y_i has the parity opposite to x_i . An easy consequence of (13.b) is that for any other index j , x_j and y_j must have opposite parity. In particular, the parity of y_1 is opposite to that of x_1 . It follows that $\{Ev_G(e_1), Odd_G(e_1)\}$ is a bipartite decomposition for (V, E) .

Condition (13.c) implies (13.a) tautologically. □

A general, if trivial, fact is

Lemma 4.1. *Let V be a codeable set of rank n . Let E be a code-match for V . Then $G = (V, E)$ can be colored with 2^n colors.*

Proof. Assign a different color $c(S)$ to each subset $S \subseteq \text{Ind}(n)$. Color each $x \in V$ using $C(S)$ where

$$S = \{ i \in \text{Ind}(n) : x_i \text{ is even} \} .$$

□

The proof of Proposition 4.1 in [4] offers an example in which 2^n colors are needed. It provides a set V , a code-match E and a number b , so that $(x_1, \dots, x_n) \in V$ for every tuple in which each $x_i \in \{b-1, b, b+1\}$. Just the tuples with coordinates b or greater yield a copy of K_{2^n} in (V, E) .

The previous example uses an E with every possible edge. That leaves open the possibility that a smaller choice of edges might allow for a smaller covering. We prove a result of this kind.

Proposition 4.2. *Let V be a codeable set of rank n . Then there exists a code-match E such that $G = (V, E)$ is $(2^{n-1} + 1)$ -vertex-colorable.*

When $n = 2$, it is simple to find an example which requires $2^1 + 1 = 3$ colors. However, the author expects that the bound can be improved for larger n .

Proof. The assertion is trivial if $n = 1$. Assume $n \geq 2$.

Let

$$V_1 = \{(x_1, \dots, x_{n-1}) : \exists y \in \mathbb{Z}, (x_1, \dots, x_{n-1}, y) \in V\} .$$

There is an alternate description. For $i \in \text{Ind}(n)$, let e_i be the only member of V with 0 as its i -component. Then V_1 is the image of the code map based on $e_1, \dots, e_{n-1}$. This map is not injective, but (5) implies that V_1 is codeable.

In this argument, represent a member $(x_1, \dots, x_n) \in V$ as $(\alpha; x_n)$ where

$$\alpha = (x_1, \dots, x_{n-1}) \in V_1 .$$

We select a code-match in two stages. First, let A be the set of edges $(\alpha; y)(\beta; z) \in \text{Ed}(V)$ in which $\alpha \neq \beta$. Now let W be those $x \in V$ such that $x_n > 0$ and there is no $xy \in A$ for which $y_n = x_n - 1$. Since V is codeable, the match criterion implies that if $(\alpha; t) \in W$, then $(\alpha; t-1) \in V$. Let B be the set of edges $(\alpha; t)(\alpha; t-1)$ for which $(\alpha; t) \in W$.

Let $E = A \cup B$. Then E satisfies the matching criterion with respect to V . Put $G = (V, E)$. The remainder of the arguments refers to this choice of graph.

Let $c(S)$ be a function which assigns a color to each subset $S \subseteq \text{Ind}(n-1)$. For purposes of discussion, assume none of the $c(S)$ is *black*. Assign a color to each $x \in V$ the color

$$c(\{i \in \text{Ind}(n-1) : x_i \text{ is even}\}) .$$

We refer to this as the *preliminary coloring*. It is clear that if $x \in V \setminus W$, then no neighbor of x has the same color as x . Put

$$C = \{(\alpha, t) \in V : (\alpha, t+1) \in W \text{ and } (\alpha, t+2) \notin W\} .$$

In particular, $(\alpha, t) \in V$ is in C if $(\alpha, t+1) \in W$ and $(\alpha, t+2)$ is a tuple that does not belong to V . We prove that changing the color of these vertices to *black* creates a proper coloring.

Claim A. Suppose $(\alpha; t), (\beta; u) \in W$ such that $\alpha \neq \beta$. Then $(\alpha; t-1)$ and $(\beta; u-1)$ are not neighbors in (V, E) .

Proof of Claim. We argue by contradiction. Assume $(\alpha; t), (\beta; u) \in W$ such that $\alpha \neq \beta$ and

$$(\alpha; t-1)(\beta; u-1) \in E.$$

Without loss of generality, we may assume that $t \geq u$. Then $\alpha\beta \in Ed(V_1)$.

(14.a) Suppose that $u = t$. Then $(\beta; u-1) = (\beta; t-1)$ and $(\alpha; t)(\beta; t-1)$ is an edge in A . But, by choice of W , $(\alpha; t) \notin W$ which contradicts assumption.

(14.b) Suppose that $u = t-1$. Then $(\alpha; t)(\beta; u) \in A$. Once again, this violates $(\alpha; t) \notin W$.

The claim is now established.

Claim B. Assume $(\alpha; t) \in W$ such that $(\alpha; t-1) \in W$. Then $(\alpha; t-2) \notin W$.

Proof of Claim B. Again, we proceed by contradiction. Assume $(\alpha; t), (\alpha; t-1), (\alpha; t-2) \in W$ for some parameters α and t . Expand $(\alpha; t-1) = (x_1, \dots, x_{n-1}, t-1)$. Note that $x_1 \neq 0$, because only one tuple in V has first coordinate 0. By the matching criterion, there is

$$(\beta; u) = (y_1, \dots, y_{n-1}, u) \in V$$

for which

$$(15.a) \ y_1 = x_1 - 1, \text{ and}$$

$$(15.b) \ (\alpha; t-2)(\beta; u) \in A.$$

Consider possible values for u . Note that $|u - (t-2)| \leq 1$.

(16.a) If $u = t-1$, then $(\alpha; t)(\beta; t-1) \in A$ would imply that $(\alpha; t) \notin W$.

(16.b) If $u = t-2$, then $(\alpha; t-1)(\beta; t-2) \in A$ would imply that $(\alpha; t-1) \notin W$.

(16.c) If $u = t-3$, then $(\alpha; t-2)(\beta; t-3) \in A$ would imply that $(\alpha; t-2) \notin W$.

We have a contradiction. This establishes the second claim.

We return to the set C and the modified coloring. Now suppose xy is an edge in G for which x and y have the same colors with respect to the modified assignment.

(17.a) Suppose that x and y are both black. Express $x = (\alpha, t)$ and $y = (\beta, u)$. Then $(\alpha, t+1)$ and $(\beta, u+1)$ belong to W . If $\alpha \neq \beta$, the latter pair share an edge. This situation violates Claim A. Hence, $\alpha = \beta$ and $u \neq t$. Without loss of generality, assume $u = t+1$. Then $(\beta, u+1) = (\alpha, t+2) \in W$ violates the assumption that $x \in C$.

(17.b) Suppose neither x nor y is black. In other words, each has the color assigned in the initial assignment. We may assume $x = (\alpha, t) \in W \setminus C$ and $y = (\alpha, t - 1)$. Since $y \notin C$, $(\alpha, t + 1) \in W$. But then Claim B implies that x has color *black*.

□

The approach of the previous proof is easy to describe: color vertices by the first $n - 1$ coordinates, look for all situations in which neighbors have the same color, and find a corrective. It seems likely that a similar approach could give an estimate that began with a color based on the first 2^{n-2} coordinates; however, there would be a greater variety of neighborhoods with a conflict in the coloring. The author hopes that there is a bound depending less on case-by-case study.

Question 2. For $n \in \mathbb{N}$, find the smallest number $\ell(n)$ such that for any codeable set V of rank n , there is a code-match E and a $\ell(n)$ -coloring of (V, E)

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