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A Survey on Monochromatic Connections of Graphs

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A Survey on Monochromatic Connections of Graphs

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Abstract

The concept of monochromatic connection of graphs was introduced by Caro and Yuster in 2011. Recently, a lot of results have been published about it. In this survey, we attempt to bring together all the results that dealt with it. We begin with an introduction, and then classify the results into the following categories: monochromatic connection coloring of edge-version, monochromatic connection coloring of vertex-version, monochromatic index, monochromatic connection coloring of total-version.

Keywords: monochromatic connection coloring, monochromatic connection number, vertex-monochromatic connection number, monochromatic index, total monochromatic connection number, computational complexity.

AMS Subject Classification 2010: 05C05, 05C15, 05C20, 05C35, 05C40, 05C69, 05C76, 05C80, 05C85, 05D40, 68Q25, 68R10

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty in [3]. Let H be a nontrivial connected graph with an edge-coloring $f : E(H) \rightarrow \{1, 2, \dots, \ell\}$ (ℓ is a positive integer), where adjacent edges may be colored the same. A path in an edge-colored graph H is a *monochromatic path* if all the edges of the path are colored with a same color. The graph H is called *monochromatically connected*, if any two vertices of H are connected by a monochromatic path. An edge-coloring of H is a *monochromatic connection coloring* (*MC-coloring*) if it makes H monochromatically connected. How colorful can an MC-coloring be? This question is the natural opposite of the well-studied problem of rainbow connection coloring [7, 4, 20, 21, 22], where in the latter we seek to find an edge-coloring with minimum number of colors so that there is a rainbow path joining any two vertices; see [21, 22] for details. As introduced by Caro and Yuster [9], for a connected graph G , the *monochromatic connection number* of G , denoted by $mc(G)$, is the maximum number of colors that are needed in order to make G monochromatically connected. An *extremal MC-coloring* is an MC-coloring that uses $mc(G)$ colors.

Gonzalez-Moreno, Guevara, and Montellano-Ballesteros[12] generalized the above concept to digraphs. Let H be a nontrivial strongly connected digraph with an arc-coloring $f : A(H) \rightarrow \{1, 2, \dots, \ell\}$ (ℓ is a positive integer), where adjacent arcs may be colored the same. A path in an arc-colored graph H is a *monochromatic path* if all the arcs of the path are colored with a same color. The strongly connected digraph H is called *strongly monochromatically connected* if for every pair $\{u, v\}$ of vertices in H , there exist both a (u, v) -monochromatic path and a (v, u) -monochromatic path. An arc-coloring of H is a *strongly monochromatic connection coloring* (*SMC-coloring*) if it makes H strongly monochromatically connected. For a strongly connected digraph D , the *strongly monochromatic connection number* of D , denoted by $smc(D)$, is the maximum number of colors that are needed in order to make D strongly monochromatically connected. An *extremal SMC-coloring* is an SMC-coloring that uses $smc(D)$ colors.

Now we introduce another generalization of the monochromatic connection number by Li and Wu [23]. A tree T in an edge-colored graph H is called a *monochromatic tree* if all the edges of T have the same color. For an $S \subseteq V(H)$, a *monochromatic S -tree* is a monochromatic tree of H containing the vertices of S . Given an integer k with $2 \leq k \leq |V(H)|$, the graph H is called *k -monochromatically connected* if for any set S of k vertices of H , there exists a monochromatic S -tree in H . An edge-coloring of H is called a *k -monochromatic connection coloring* (MX_k -coloring) if it makes H k -monochromatically connected. For a connected graph G and a given integer k such that $2 \leq k \leq |V(G)|$, the *k -monochromatic index* $mx_k(G)$ of G is the maximum number of colors that are needed in order to make G k -monochromatically connected. An *extremal MX_k -coloring* is an MX_k -coloring that uses $mx_k(G)$ colors. By definition, we have $mx_{|V(G)|}(G) \leq \cdots \leq mx_3(G) \leq mx_2(G) = mc(G)$.

Note that the above graph-parameters are defined on edge-colored graphs. Naturally, Cai, Li and Wu [6] introduced a graph-parameter corresponding to monochromatic connection number which is defined on vertex-colored graphs. Let H be a nontrivial connected graph with a vertex-coloring $f : V(H) \rightarrow \{1, 2, \dots, \ell\}$ (ℓ is a positive integer), where adjacent vertices may be colored the same. A path in a vertex-colored graph H is a *vertex-monochromatic path* if all the internal vertices of the path are colored with a same color. The graph H is called *vertex-monochromatically connected*, if any two vertices of H are connected by a vertex-monochromatic path. A vertex-coloring of H is a *vertex-monochromatic connection coloring* (VMC -coloring) if it makes H vertex-monochromatically connected. For a connected graph G , the *vertex-monochromatic connection number* of G , denoted by $vmc(G)$, is the maximum number of colors that are needed in order to make G vertex-monochromatically connected. An *extremal VMC -coloring* is a VMC -coloring that uses $vmc(G)$ colors.

Li and Wu [23] introduced another graph-parameter corresponding to the k -monochromatic index, which is defined on vertex-colored graphs. A tree T in a vertex-colored graph H is called a *vertex-monochromatic tree* if all the internal vertices of T have the same color. For an $S \subseteq V(H)$, a *vertex-monochromatic S -tree* is a vertex-monochromatic tree of H containing the vertices of S . Given an integer k with $2 \leq k \leq |V(H)|$, the graph H is called *k -vertex-monochromatically connected* if for any set S of k vertices of H , there exists a vertex-monochromatic S -tree in H . A vertex-coloring of H is called a *k -vertex-monochromatic connection coloring* (VMX_k -coloring) if it makes H k -vertex-monochromatically connected. For a connected graph G and a given integer k such that $2 \leq k \leq |V(G)|$, the *k -vertex-monochromatic index* $vmx_k(G)$ of G is the maximum number of colors that are needed in order to make G k -vertex-monochromatically connected. An *extremal VMX_k -coloring* is a VMX_k -coloring that uses $vmx_k(G)$ colors. By definition, we have $vmx_{|V(G)|}(G) \leq \cdots \leq vmx_3(G) \leq vmx_2(G) = vmc(G)$.

Jiang, Li and Zhang [15] introduced the monochromatic connection of total-coloring version. Let H be a nontrivial connected graph with a total-coloring $f : V(H) \cup E(H) \rightarrow \{1, 2, \dots, \ell\}$ (ℓ is a positive integer), where any two elements may be colored the same. A path in a total-colored graph H is a *total-monochromatic path* if all the edges and internal vertices of the path are colored with a same color. The graph H is called *total-monochromatically connected* if any two vertices of H are connected by a total-monochromatic path. A total-coloring of H is a *total-monochromatic connection coloring* (TMC -coloring) if it makes H total-monochromatically connected. For a connected graph G , the *total-monochromatic con-*

nection number of G , denoted by $tmc(G)$, is the maximum number of colors that are needed in order to make G total-monochromatically connected. An *extremal TMC-coloring* is a TMC-coloring that uses $tmc(G)$ colors.

For convenience of reading, we summarize the above concepts into the following table.

Edge-coloring Version	
MC-coloring	monochromatic connection coloring
SMC-coloring	strongly monochromatic connection coloring
MX_k -coloring	k -monochromatic connection coloring
$mc(G)$	monochromatic connection number
$smc(G)$	strongly monochromatic connection number
$mx_k(G)$	k -monochromatic index
Vertex-coloring Version	
VMC-coloring	vertex-monochromatic connection coloring
$V MX_k$ -coloring	k -vertex-monochromatic connection coloring
$vmc(G)$	vertex-monochromatic connection number
$vmx_k(G)$	k -vertex-monochromatic index
Total-coloring Version	
TMC-coloring	total-monochromatic connection coloring
$tmc(G)$	total-monochromatic connection number

Next, we recall the definitions of various products of graphs, which will be used in the sequel. The *Cartesian product* of two graphs G and H , denoted by $G \square H$, is defined to have the vertex-set $V(G) \times V(H)$, in which two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. The *lexicographic product* $G \circ H$ of two graphs G and H has the vertex-set $V(G \circ H) = V(G) \times V(H)$, and two vertices $(g, h), (g', h')$ are adjacent if and only if $gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$. The *strong product* $G \boxtimes H$ of two graphs G and H has the vertex-set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $hh' \in E(H)$. The *direct product* $G \times H$ of two graphs G and H has the vertex-set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if the projections on both coordinates are adjacent, i.e., $gg' \in E(G)$ and $hh' \in E(H)$. Finally, the *join* $G + H$ of two graphs G and H has the vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$.

The most frequently occurring probability models of random graphs is the Erdős-Rényi random graph model $G(n, p)$ [10]. The model $G(n, p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p . We say an event $\mathcal{A}$ happens *with high probability* if the probability that it happens approaches 1 as $n \rightarrow \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$. Sometimes, we say *w.h.p.* for short. We will always assume that n is the variable that tends to infinity. Let G and H be two graphs on n vertices. A property P is said to be *monotone* if whenever $G \subseteq H$ and G satisfies P , then H also satisfies P . For a graph property P , a function $p(n)$ is called a threshold function of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ w.h.p. satisfies P ; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ w.h.p. does not satisfy P .

Furthermore, $p(n)$ is called a sharp threshold function of P if there exist two positive constants c and C such that:

- for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ w.h.p. satisfies P ; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ w.h.p. does not satisfy P .

It is well known that all monotone graph properties have a sharp threshold function; see [2] and [11] for details.

2 The edge-coloring version

2.1 Upper and lower bounds for $mc(G)$

In [9], Caro and Yuster observed that a general lower bound for $mc(G)$ is $m(G) - n(G) + 2$. Simply color the edges of a spanning tree of G with one color, and each of the remaining edges with a distinct new color. Then, Caro and Yuster gave some sufficient conditions for graphs attaining this lower bound.

Theorem 2.1. [9] *Let G be a connected graph with $n > 3$ vertices and m edges. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

- $\overline{G}$ is 4-connected.
- G is triangle-free.
- $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G) \leq (n+1)/2$ or $\Delta(G) \leq n - 2m/n$.
- $\text{diam}(G) \geq 3$.
- G has a cut vertex.

Jin, Li and Wang got some conditions on graphs containing triangles.

Theorem 2.2. [18] *Let G be a connected graph of order $n \geq 7$. If G does not have subgraphs isomorphic to K_4^- , then $mc(G) = m - n + 2$, where K_4^- denotes the graph obtained from K_4 by deleting an edge.*

Theorem 2.3. [18] *Let G be a connected graph of order $n \geq 7$. If G does not have two triangles that have exactly one common vertex, then $mc(G) = m - n + 2$.*

Theorem 2.4. [18] *Let G be a connected graph of order $n \geq 7$. If G does not have two vertex-disjoint triangles, then $mc(G) = m - n + 2$.*

Caro and Yuster [9] also showed some nontrivial upper bounds for $mc(G)$ in terms of the chromatic number, the connectivity, and the minimum degree. Recall that a graph is called *s-perfectly-connected* if it can be partitioned into $s + 1$ parts $\{v\}, V_1, \dots, V_s$, such that each V_j induces a connected subgraph, any pair V_j, V_r induces a corresponding complete bipartite graph, and v has precisely one neighbor in each V_j . Notice that such a graph has minimum degree s , and v has degree s .

Theorem 2.5. [9]

- (1) If G is a complete r -partite graph, then $mc(G) = m - n + r$.
- (2) Any connected graph G satisfies $mc(G) \leq m - n + \chi(G)$.
- (3) If G is not k -connected, then $mc(G) \leq m - n + k$. This is sharp for any k .
- (4) If $\delta(G) = s$, then $mc(G) \leq m - n + s$, unless G is s -perfectly-connected, in which case $mc(G) = m - n + s + 1$.

As an application of Theorem 2.5(4), Caro and Yuster got the upper bounds for the following planar graphs.

Corollary 2.6. [9]

- (1) For $n \geq 5$, the wheel $G = W_n$ has $mc(G) = m - n + 3$.
- (2) If G is an outerplanar graph, then $mc(G) = m - n + 2$, except that $mc(K_1 \vee P_{n-1}) = m - n + 3$.
- (3) If G is a planar graph with minimum degree 3, then $mc(G) \leq m - n + 3$, except that $mc(K_2 \vee P_{n-2}) = m - n + 4$.

2.2 Erdős-Gallai-type problems for $mc(G)$

Cai, Li and Wu [5] studied the following two kinds of Erdős-Gallai-type problems for $mc(G)$.

Problem A. Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the minimum integer $f(n, k)$ such that for any graph G of order n , if $|E(G)| \geq f(n, k)$ then $mc(G) \geq k$.

Problem B. Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the maximum integer $g(n, k)$ such that for any graph G of order n , if $|E(G)| \leq g(n, k)$ then $mc(G) \leq k$.

It is worth mentioning that the two parameters $f(n, k)$ and $g(n, k)$ are equivalent to another two parameters. Let $t(n, k) = \min\{|E(G)| : |V(G)| = n, mc(G) \geq k\}$ and $s(n, k) = \max\{|E(G)| : |V(G)| = n, mc(G) \leq k\}$. It is easy to see that $t(n, k) = g(n, k - 1) + 1$ and $s(n, k) = f(n, k + 1) - 1$. In [5] the authors determined the exact values of $f(n, k)$ and $g(n, k)$ for all integers n, k with $1 \leq k \leq \binom{n}{2}$.

Theorem 2.7. [5] Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$,

$$f(n, k) = \begin{cases} n + k - 2 & \text{if } 1 \leq k \leq \binom{n}{2} - 2n + 4 \\ \binom{n}{2} + \left\lceil \frac{k - \binom{n}{2}}{2} \right\rceil & \text{if } \binom{n}{2} - 2n + 5 \leq k \leq \binom{n}{2}. \end{cases}$$

Theorem 2.8. [5] Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$,

$$g(n, k) = \begin{cases} \binom{n}{2} & \text{if } k = \binom{n}{2} \\ k + t - 1 & \text{if } \binom{n-t}{2} + t(n-t-1) + 1 \leq k \leq \binom{n-t}{2} + t(n-t) - 1 \\ k + t - 2 & \text{if } k = \binom{n-t}{2} + t(n-t) \end{cases}$$

for $2 \leq t \leq n - 1$.

2.3 Results for graph classes

Gu, Li, Qin and Zhao[14] characterized all connected graphs G of size m with small and large values of $mc(G)$.

Theorem 2.9. [14] *Let G be a connected graph. Then $mc(G) = 1$ if and only if G is a tree.*

Theorem 2.10. [14] *Let G be a connected graph. Then $mc(G) = 2$ if and only if G is a unicyclic graph except for K_3 .*

Theorem 2.11. [14] *Let G be a connected graph. Then $mc(G) = 3$ if and only if G is either K_3 or a bicyclic graph except for $K_4 - e$.*

Fig 1: Graphs from Theorem 2.12 and Theorem 2.15

Theorem 2.12. [14] *Let G be a connected graph. Then $mc(G) = 4$ if and only if G is either $K_4 - e$ or a tricyclic graph except for G_1, G_2, K_4 , where G_1, G_2 are shown in Fig 1.*

Theorem 2.13. [14] *Let G be a connected graph. Then $mc(G) = m - 1$ if and only if $G = K_n - e$.*

Theorem 2.14. [14] *Let G be a connected graph. Then $mc(G) = m - 2$ if and only if $G \in \{K_n - 2K_2, K_n - P_3, K_n - K_3, K_n - P_4\}$.*

Theorem 2.15. [14] *Let G be a connected graph. Then $mc(G) = m - 3$ if and only if $G \in \{K_n - 3K_2, K_n - C_4, K_n - C_5, K_n - (P_2 \cup P_3), K_n - (P_2 \cup K_3), K_n - P_5, K_n - (P_2 \cup P_4), K_n - S_4, K_n - K_4, K_n - G_i\}$, where i is an integer with $3 \leq i \leq 10$ and G_i is shown in Fig 1.*

From the above theorems, they also verified the following corollary.

Corollary 2.16. [14] *Let G be a connected graph of order n . Then*

- (1) $mc(G) \neq \binom{n}{2} - 1, mc(G) \neq \binom{n}{2} - 3$.
- (2) $mc(G) = \binom{n}{2} - 2$ if and only if $G = K_n - e$.
- (3) $mc(G) = \binom{n}{2} - 4$ if and only if $G \in \{K_n - 2K_2, K_n - P_3\}$.

2.4 Results for graph products

Mao, Wang, Yanling and Ye [24] studied the monochromatic connection numbers of the following graph products.

Theorem 2.17. [24] *Let G and H be connected graphs.*

(1) *If neither G nor H is a tree, then*

$$\begin{aligned} \max\{|E(G)||V(H)|, |E(H)||V(G)|\} + 2 &\leq \text{mc}(G \square H) \\ &\leq |E(G)||V(H)| + (|E(H)| - 1)|V(G)| + 1. \end{aligned}$$

(2) *If G is not a tree and H is a tree, then*

$$|E(H)||V(G)| + 2 \leq \text{mc}(G \square H) \leq |E(G)||V(H)| + 1.$$

(3) *If both G and H are trees, then*

$$|E(G)||E(H)| + 1 \leq \text{mc}(G \square H) \leq |E(G)||E(H)| + 2.$$

Moreover, the lower bounds are sharp.

Corollary 2.18. [24] *Let G and H be a connected graph.*

(1) *If neither G nor H is a tree, then*

$$\text{mc}(G \square H) \geq \max\{\text{mc}(G)|V(H)| + 2, \text{mc}(H)|V(G)| + 2\}.$$

(2) *If G is not a tree and H is a tree, then $\text{mc}(G \square H) \geq \text{mc}(H)|V(G)| + 2$.*

(3) *If both G and H are trees, then $\text{mc}(G \square H) \geq \text{mc}(G)\text{mc}(H) + 1$.*

Theorem 2.19. [24] *Let G and H be connected graphs, and let G be noncomplete.*

(1) *If neither G nor H is a tree, then*

$$\begin{aligned} |E(G)||V(H)|^2 + 2 &\leq \text{mc}(G \circ H) \\ &\leq |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(H)| + 1. \end{aligned}$$

(2) *If G is not a tree and H is a tree, then*

$$\begin{aligned} |E(H)||V(G)|(|V(H)| + 1) + 2 &\leq \text{mc}(G \circ H) \\ &\leq |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(H)| + 1. \end{aligned}$$

(3) *If H is not a tree and G is a tree, then*

$$\begin{aligned} |E(H)||V(G)|^2 + 2 &\leq \text{mc}(G \circ H) \\ &\leq |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(H)| + 1. \end{aligned}$$

(4) *If both G and H are trees, then*

$$\begin{aligned} |E(H)||E(G)|(|V(H)| + 1) + 1 &\leq \text{mc}(G \circ H) \\ &\leq |E(H)||E(G)|(|V(H)| + 1) + |V(H)|. \end{aligned}$$

Moreover, the lower bounds are sharp.

Corollary 2.20. [24] *Let G and H be connected graphs.*

- (1) *If neither G nor H is a tree, then $\text{mc}(G \circ H) \geq \text{mc}(G)|V(H)|^2 + 2$.*
- (2) *If G is not a tree and H is a tree, then $\text{mc}(G \circ H) \geq \text{mc}(H)|V(G)|(|V(H)| + 1) + 2$.*
- (3) *If H is not a tree and G is a tree, then $\text{mc}(G \circ H) \geq \text{mc}(H)|V(G)|^2 + 2$.*
- (4) *If both G and H are trees, then $\text{mc}(G \circ H) \geq \text{mc}(G)\text{mc}(H)(|V(H)| + 1) + 1$.*

Theorem 2.21. [24] *Let G and H be a connected graph, and at least one of G and H is not a complete graph.*

- (1) *If neither G nor H is a tree, then*

$$\text{mc}(G \boxtimes H) \geq \max\{|E(G)||V(H)| + 2|E(H)||E(G)| + 2, \\ |E(H)||V(G)| + 2|E(H)||E(G)| + 2\}$$

and

$$\text{mc}(G \boxtimes H) \leq |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| - \\ \min\{|V(H)|, |V(G)|\} + 1.$$

- (2) *If G not a tree and H is a tree, then*

$$|E(H)||V(G)| + 2|E(H)||E(G)| + 2 \leq \text{mc}(G \boxtimes H) \\ \leq |E(G)||V(H)| + 2|E(H)||E(G)| + 1.$$

- (3) *If both G and H are trees, then*

$$3|E(H)||E(G)| + 1 \leq \text{mc}(G \boxtimes H) \leq 3|E(H)||E(G)| + \min\{|V(G)|, |V(H)|\}.$$

Moreover, the lower bounds are sharp.

Corollary 2.22. [24] *Let G and H be a connected graph.*

- (1) *If neither G nor H is a tree, then*

$$\text{mc}(G \boxtimes H) \geq \max\{|\text{mc}(G)||V(H)| + 2|\text{mc}(H)||\text{mc}(G)| + 2, \\ |\text{mc}(H)||V(G)| + 2|\text{mc}(H)||\text{mc}(G)| + 2\}.$$

- (2) *If G not a tree and H is a tree, then*

$$\text{mc}(G \boxtimes H) \geq |\text{mc}(H)||V(G)| + 2|\text{mc}(H)||\text{mc}(G)| + 2.$$

- (3) *If both G and H are trees, then*

$$\text{mc}(G \boxtimes H) \geq 3|\text{mc}(H)||\text{mc}(G)| + 1.$$

Theorem 2.23. [24] *Let G and H be nonbipartite graphs. Then*

$$|E(H)||E(G)| + 2 \leq \text{mc}(G \times H) \leq 2|E(H)||E(G)| + 1.$$

Moreover, the lower bounds are sharp.

Corollary 2.24. [24] *Let one of G and H be a non-bipartite connected graph. Then*

$$\text{mc}(G \times H) \geq |\text{mc}(H)| |\text{mc}(G)| + 2.$$

As an application of the above results, they also studied the following graph classes.

We call $P_n \square P_m$ a *two-dimensional grid graph*, where P_n and P_m are paths on n and m vertices, respectively.

Proposition 2.25. [24]

(1) *For the network $P_n \square P_m$ ($n \geq 3, m \geq 2$),*

$$\text{mc}(P_n \square P_m) = nm - n - m + 2.$$

(2) *For network $P_n \circ P_m$ ($n \geq 4, m \geq 3$),*

$$\text{mc}(P_n \circ P_m) = m^2n - m^2 - n + 2.$$

An *n -dimensional mesh* is the Cartesian product of n linear arrays. Particularly, two-dimensional grid graph is a 2-dimensional mesh. An *n -dimensional hypercube* is an n -dimensional mesh, in which all the n linear arrays are of size 2.

Proposition 2.26. [24]

(1) *For n -dimensional mesh $P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}$ ($n \geq 4$),*

$$\text{mc}(P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}) \geq (2\ell_1\ell_2 - \ell_1 - \ell_2)(\ell_3\ell_4 \cdots \ell_n) + 2.$$

(2) *For network $P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}$,*

$$\text{mc}(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) \geq (\ell_1\ell_2^2 + \ell_1\ell_2 - \ell_1 - \ell_2^2)(\ell_3\ell_4 \cdots \ell_n)^2 + 2.$$

An *n -dimensional torus* is the Cartesian product of n cycles $R_1, R_2, \dots, R_n$ of size at least three.

Proposition 2.27. [24]

(1) *For network $R_1 \square R_2 \square \cdots \square R_n$, $n \geq 4$*

$$\text{mc}(R_1 \square R_2 \square \cdots \square R_n) \geq r_1r_2 \cdots r_n + 2,$$

where r_i is the order of R_i and $3 \leq i \leq n$.

(2) *For network $R_1 \circ R_2 \circ \cdots \circ R_n$, $n \geq 4$*

$$\text{mc}(R_1 \circ R_2 \circ \cdots \circ R_n) \geq r_1(r_2 \cdots r_n)^2 + 2.$$

Let K_m be a clique of m vertices, $m \geq 2$. An *n -dimensional generalized hypercube* is the Cartesian product of n cliques.

Proposition 2.28. [24]

(1) *For network $K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}$ ($m_i \geq 2, n \geq 3, 1 \leq i \leq n$)*

$$\text{mc}(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) \geq \binom{m_1}{2} m_2 \cdots m_n + 2.$$

(2) *For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$,*

$$\text{mc}(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = \binom{m_1 m_2 \cdots m_n}{2}.$$

An n -dimensional hyper Petersen network HP_n is the Cartesian product of Q_{n-3} and the well-known Petersen graph, where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube.

The network HL_n is the lexicographical product of Q_{n-3} and the Petersen graph, where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube.

Proposition 2.29. [24]

- (1) For network HP_3 and HL_3 , $mc(HP_3) = mc(HL_3) = 7$;
- (2) For network HL_4 and HP_4 , $mc(HP_4) = 22$ and $112 \leq mc(HL_4) \leq 124$.

For the join of two graphs, Jin, Li and Wang got the following results.

Theorem 2.30. [18] Let G_1 and G_2 be two disjoint connected graphs and $G = G_1 + G_2$. Then $mc(G) = mc(G_1) + mc(G_2) + |V(G_1)||V(G_2)|$.

Theorem 2.31. [18] Let G be the join of a connected graph G_1 and a disconnected graph G_2 , where $V(G_1) \cap V(G_2) = \emptyset$. Then $mc(G) = mc(G_1) + |E(G_2)| + |V(G_1)||V(G_2)| - |V(G_2)| + 1$.

Theorem 2.32. [18] Let G be the join of two disjoint disconnected graphs G_1 and G_2 . Then $mc(G) = |E(G_1)| + |E(G_2)| + |V(G_1)||V(G_2)| - |V(G_1)| - |V(G_2)| + 2$.

2.5 Results for random graphs

The goal of MC -coloring of a graph is to find as many as colors to make the graph monochromatically connected. So it is interesting to consider the threshold function of property $mc(G(n, p)) \geq f(n)$, where $f(n)$ is a function of n . For any graph G with n vertices and any function $f(n)$, having $mc(G) \geq f(n)$ is a monotone graph property (adding edges does not destroy this property), so it has a sharp threshold function.

Gu, Li, Qin and Zhao[14] showed a sharp threshold function for $mc(G)$ as follows.

Theorem 2.33. [14] Let $f(n)$ be a function satisfying $1 \leq f(n) < \frac{1}{2}n(n-1)$. Then

$$p = \begin{cases} \frac{f(n) + n \log \log n}{n^2} & \text{if } \ell n \log n \leq f(n) < \frac{1}{2}n(n-1), \text{ where } \ell \in \mathbb{R}^+, \\ \frac{\log n}{n} & \text{if } f(n) = o(n \log n). \end{cases}$$

is a sharp threshold function for the property $mc(G(n, p)) \geq f(n)$.

Remark 2.33. Note that $mc(G(n, p)) \leq \frac{1}{2}n(n-1)$ for any function $0 \leq p \leq 1$, and $mc(G(n, p)) = \frac{1}{2}n(n-1)$ if and only if $G(n, p)$ is isomorphic to the complete graph K_n . Hence we only concentrate on the case $f(n) < \frac{1}{2}n(n-1)$.

3 The vertex-coloring version

3.1 Upper and lower bounds for $vmc(G)$

For a connected graph G of order 1 or 2, it is easy to check $vmc(G) = 1, 2$, respectively. For a connected graph G of order at least 3, Cai, Li and Wu [6] got that a general lower

bound for $vmc(G)$ is $\ell(T) + 1 \geq 3$, where T is a spanning tree of G , and $\ell(T)$ is the number of leaves in T . Simply take a spanning tree T of G . Then, give all the non-leaves in T one color, and each leaf in T a distinct new color. Clearly, this is a VMC-coloring of G using $\ell(T) + 1$ colors.

By the known results about spanning trees with many leaves in [8, 13, 19], Cai, Li and Wu [6] got the following lower bounds.

Proposition 3.1. [6] *Let G be a connected graph with n vertices and minimum degree δ .*

- (1) *If $\delta \geq 3$, then $vmc(G) \geq \frac{1}{4}n + 3$.*
- (2) *If $\delta \geq 4$, then $vmc(G) \geq \frac{2}{5}n + \frac{13}{5}$.*
- (3) *If $\delta \geq 5$, then $vmc(G) \geq \frac{1}{2}n + 3$.*
- (4) *If $\delta \geq 3$, then $vmc(G) \geq \left(1 - \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))\right)n + 1$.*

They also got an upper bound for $vmc(G)$.

Proposition 3.2. [6] *Let G be a connected graph with n vertices and diameter d .*

- (1) *$vmc(G) = n$ if and only if $d \leq 2$;*
- (2) *If $d \geq 3$, then $vmc(G) \leq n - d + 2$, and the bound is sharp.*

3.2 Erdős-Gallai-type problems for $vmc(G)$

Cai, Li and Wu [6] studied two Erdős-Gallai-type problems for the graph parameter $vmc(G)$.

Problem A: Given two positive integers n, k with $3 \leq k \leq n$, compute the minimum integer $f_v(n, k)$ such that for any graph G of order n , if $|E(G)| \geq f_v(n, k)$ then $vmc(G) \geq k$.

Problem B: Given two positive integers n, k with $3 \leq k \leq n$, compute the maximum integer $g_v(n, k)$ such that for any graph G of order n , if $|E(G)| \leq g_v(n, k)$ then $vmc(G) \leq k$.

Note that $g_v(n, n) = \binom{n}{2}$, and $g_v(n, k)$ does not exist for $3 \leq k \leq n - 1$. This is because for a star S_n on n vertices, we have $vmc(S_n) = n$. For this reason, Cai, Li and Wu [6] just studied **Problem A**. They got the value of $f_v(n, k)$.

Theorem 3.3. [6] *Given two integers n, k with $3 \leq k \leq n$,*

$$f_v(n, k) = \begin{cases} n - 1 & \text{if } k = 3 \\ n + \binom{k-2}{2} & \text{if } 4 \leq k \leq n - 2 \\ n - 1 + \binom{k-2}{2} & \text{if } n - 1 \leq k \leq n \end{cases}$$

3.3 Nordhaus-Gaddum-type theorem for $vmc(G)$

Cai, Li and Wu [6] got the following Nordhaus-Gaddum-type result for $vmc(G)$

Theorem 3.4. [6] *Let G be a connected graph on $n \geq 5$ vertices with connected complement $\overline{G}$. Then $n + 3 \leq vmc(G) + vmc(\overline{G}) \leq 2n$, and $3n \leq vmc(G) \cdot vmc(\overline{G}) \leq n^2$. Moreover, these bounds are sharp.*

4 The arc-coloring version for digraphs

Gonzalez-Moreno, Guevara, and Montellano-Ballesteros [12] got the following result for strongly connected oriented graph.

Theorem 4.1. [12] *Let D be a strongly connected oriented graph of size m , and let $\Omega(D)$ be the minimum size of a strongly connected spanning subdigraph of D . Then*

$$smc(D) = m - \Omega(D) + 1.$$

As an application of Theorem 4.1, they found a sufficient and necessary condition to determine whether a strongly connected oriented graph is Hamiltonian.

Corollary 4.2. [12] *Let D be a strongly connected oriented graph of size m and order n . Then D is Hamiltonian if and only if $smc(D) = m - n + 1$.*

From Corollary 4.2, one can see that computing $\Omega(D)$ is NP-hard.

5 Monochromatic indices

5.1 Edge version

Li and Wu [23] completely determined the k -monochromatic index for $k \geq 3$.

Theorem 5.1. [23] *Let G be a connected graph with n vertices and m edges. Then $mx_k(G) = m - n + 2$ for each k with $3 \leq k \leq n$.*

5.2 Vertex version

Li and Wu [23] studied the hardness for computing $vmx_k(G)$. They showed that given a connected graph $G = (V, E)$, and a positive integer L with $L \leq |V|$, to decide whether $vmx_k(G) \geq L$ is NP-complete for each k with $2 \leq k \leq |V|$. In particular, computing $vmx_k(G)$ is NP-hard.

5.3 Nordhaus-Gaddum-type results

Recall that Cai, Li and Wu [6] got the Nordhaus-Gaddum-type result for $vmc(G)$. Li and Wu [23] got the following Nordhaus-Gaddum-type lower bounds of vmx_k for k with $3 \leq k \leq n$.

Theorem 5.2. [23] *Suppose that both G and $\overline{G}$ are connected graphs on n vertices. For $n = 5$, $vmx_k(G) + vmx_k(\overline{G}) \geq 6$ for k with $3 \leq k \leq 5$. For $n = 6$, $vmx_k(G) + vmx_k(\overline{G}) \geq 8$ for k with $3 \leq k \leq 6$. For $n \geq 7$, if n is odd, then $vmx_k(G) + vmx_k(\overline{G}) \geq n + 3$ for k with $3 \leq k \leq \frac{n-1}{2}$, and $vmx_k(G) + vmx_k(\overline{G}) \geq n + 2$ for k with $\frac{n+1}{2} \leq k \leq n$; if $n = 4t$, then $vmx_k(G) + vmx_k(\overline{G}) \geq n + 3$ for k with $3 \leq k \leq \frac{n}{2} - 1$, and $vmx_k(G) + vmx_k(\overline{G}) \geq n + 2$ for k with $\frac{n}{2} \leq k \leq n$; if $n = 4t + 2$, then $vmx_k(G) + vmx_k(\overline{G}) \geq n + 3$ for k with $3 \leq k \leq \frac{n}{2}$, and $vmx_k(G) + vmx_k(\overline{G}) \geq n + 2$ for k with $\frac{n}{2} + 1 \leq k \leq n$. Moreover, all the above bounds are sharp.*

They also got the following Nordhaus-Gaddum-type upper bound of vmx_k for k with $\lceil \frac{n}{2} \rceil \leq k \leq n$.

Theorem 5.3. [23] Suppose that both G and $\overline{G}$ are connected graphs on $n \geq 5$ vertices. Then, for any k with $\lceil \frac{n}{2} \rceil \leq k \leq n$, we have that $vmx_k(G) + vmx_k(\overline{G}) \leq 2n - 2$, and this bound is sharp.

6 The total-coloring version

Jiang, Li and Zhang [16] studied the hardness for computing $tmc(G)$. They showed that given a connected graph $G = (V, E)$, and a positive integer L with $L \leq |V| + |E|$, to decide whether $tmc(G) \geq L$ is NP-complete. In particular, computing $tmc(G)$ is NP-hard.

6.1 Upper and low bounds for $tmc(G)$

Let $l(T)$ denote the number of leaves in a tree T . For a connected graph G , let $l(G) = \max\{l(T) \mid T \text{ is a spanning tree of } G\}$. Jiang, Li and Zhang [15] got the following lower bound of $tmc(G)$.

Theorem 6.1. [15] For a connected graph G of order n and size m , we have $tmc(G) \geq m - n + 2 + l(G)$.

They also gave some sufficient conditions for graphs attaining this lower bound.

Theorem 6.2. [15] Let G be a connected graph of order $n > 3$ and size m . If G has any of the following properties, then $tmc(G) = m - n + 2 + l(G)$.

- (a) The complement $\overline{G}$ of G is 4-connected.
- (b) G is K_3 -free.
- (c) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$.
- (d) $\text{diam}(G) \geq 3$.
- (e) G has a cut vertex.

The upper bound of $\Delta(G)$ in Theorem 6.2(c) is best possible. For example, let $G = K_{n-2,1,1}$. Then $tmc(G) = m - n + 3 + l(G)$ and $\Delta(G) = n - 1 = n - \frac{2m-3(n-1)}{n-3}$.

Jiang, Li and Zhang [15] computed the total monochromatic connection numbers of wheel graphs and complete multipartite graphs.

Proposition 6.3. [15] Let G be a wheel W_{n-1} of order $n \geq 5$ and size m . Then $tmc(G) = m - n + 2 + l(G)$.

Proposition 6.4. [15] Let $G = K_{n_1, \dots, n_r}$ be a complete multipartite graph with $n_1 \geq \dots \geq n_t \geq 2$ and $n_{t+1} = \dots = n_r = 1$. Then $tmc(G) = m + r - t$.

6.2 Comparing $tmc(G)$ with $vmc(G)$ and $mc(G)$

Jiang, Li and Zhang [15] compared $tmc(G)$ with $vmc(G)$ from different aspects.

Theorem 6.5. [15] *Let G be a connected graph of order n , size m and diameter d . If $m \geq 2n - d - 2$, then $tmc(G) > vmc(G)$.*

Theorem 6.6. [15] *Let G be a connected graph of order n , diameter 2 and maximum degree Δ . If $\Delta \geq \frac{n+1}{2}$, then $tmc(G) > vmc(G)$.*

Note that $tmc(C_5) = 4 < vmc(C_5) = 5$, where $m < 2n - d - 2$ and $\Delta < \frac{n+1}{2}$. This implies that the conditions of Theorems 6.5 and 6.6 cannot be improved. If G is a star, then $tmc(G) = vmc(G) = n$. However, they could not show whether there exist other graphs with $tmc(G) \leq vmc(G)$. Then they proposed the following problem.

Problem 6.7. [15] *Dose there exists a graph of order $n \geq 6$ except for the star graph such that $tmc(G) \leq vmc(G)$?*

In addition, they proposed the following conjecture.

Conjecture 6.8. [15] *For a connected graph G , it always holds that $tmc(G) > mc(G)$.*

Finally, they compared $tmc(G)$ with $mc(G) + vmc(G)$.

Theorem 6.9. [15] *Let G be a connected graph. Then $tmc(G) \leq mc(G) + vmc(G)$, and the equality holds if and only if G is a complete graph.*

6.3 Results for graph classes

Jiang, Li and Zhang [16] characterized all connected graphs G of order n and size m with $tmc(G) \in \{3, 4, 5, 6, m + n - 2, m + n - 3, m + n - 4\}$, respectively. Let $\mathcal{T}_i$ denote the set of the trees with $l(G) = i$, where $2 \leq i \leq n - 1$. Note that if G is a connected graph with $l(G) = 2$, then G is either a path or a cycle.

Theorem 6.10. [16] *Let G be a connected graph. Then $tmc(G) = 3$ if and only if G is a path.*

Theorem 6.11. [16] *Let G be a connected graph. Then $tmc(G) = 4$ if and only if $G \in \mathcal{T}_3$ or G is a cycle except for K_3 .*

Theorem 6.12. [16] *Let G be a connected graph. Then $tmc(G) = 5$ if and only if $G \in \mathcal{T}_4$ or $G \in \mathcal{G}_i$, where $1 \leq i \leq 4$; see Fig 2.*

Theorem 6.13. [16] *Let G be a connected graph. Then $tmc(G) = 6$ if and only if $G = K_3$, $G \in \mathcal{T}_5$ or $G \in \mathcal{H}_i$, where $1 \leq i \leq 18$; see Fig 3.*

Theorem 6.14. [16] *Let G be a connected graph. Then $tmc(G) = m + n - 2$ if and only if $G = K_n - K_2$.*

Theorem 6.15. [16] *Let G be a connected graph. Then $tmc(G) = m + n - 3$ if and only if G is either $K_n - K_3$ or $K_n - P_3$.*

Theorem 6.16. [16] *Let G be a connected graph. Then $tmc(G) = m + n - 4$ if and only if $G \in \{K_n - P_4, K_n - 2K_2, K_n - K_4, K_n - (K_4 - K_2), K_n - (K_4 - P_3), K_n - C_4, K_n - K_{1,3}\}$.*

Fig 2: Unicyclic graphs with $l(G) = 3$.

Fig 3: Graphs from Theorem 6.13.

6.4 Results for random graphs

For a property P of graphs and a positive integer n , define $\text{Prob}(P, n)$ to be the ratio of the number of graphs with n labeled vertices having property P over the total number of graphs with these vertices. If $\text{Prob}(P, n)$ approaches 1 as n tends to infinity, then we say that *almost all* graphs have property P . More details can be found in [1]. Jiang, Li and Zhang [15] got the following result for $\text{tmc}(G)$.

Theorem 6.17. [15] *For almost all graphs G of order n and size m , we have $\text{tmc}(G) = m - n + 2 + l(G)$.*

Jiang, Li and Zhang [16] showed a sharp threshold function for $\text{tmc}(G)$ as follows.

Theorem 6.18. [16] *Let $f(n)$ be a function satisfying $1 \leq f(n) < \frac{1}{2}n(n-1) + n$. Then*

$$p = \begin{cases} \frac{f(n) + n \log \log n}{n^2} & \text{if } \ln \log n \leq f(n) < \frac{1}{2}n(n-1) + n, \\ & \text{where } l \in \mathbb{R}^+, \\ \frac{\log n}{n} & \text{if } f(n) = o(n \log n). \end{cases}$$

is a sharp threshold function for the property $\text{tmc}(G(n, p)) \geq f(n)$.

Remark 6.19. Note that if $f(n) = \frac{1}{2}n(n-1) + n$, then $G(n, p)$ is a complete graph K_n and $p = 1$. Hence we only concentrate on the case $f(n) < \frac{1}{2}n(n-1) + n$.

6.5 Erdős-Gallai-type problems for $\text{tmc}(G)$

Jiang, Li and Zhang [17] studied the following two kinds of Erdős-Gallai-type problems for $\text{tmc}(G)$.

Problem A. Given two positive integers n and k with $3 \leq k \leq \binom{n}{2} + n$, compute the minimum integer $f_T(n, k)$ such that for any graph G of order n , if $|E(G)| \geq f_T(n, k)$ then $\text{tmc}(G) \geq k$.

Problem B. Given two positive integers n and k with $3 \leq k \leq \binom{n}{2} + n$, compute the maximum integer $g_T(n, k)$ such that for any graph G of order n , if $|E(G)| \leq g_T(n, k)$ then $\text{tmc}(G) \leq k$.

They completely determined the values of $f_T(n, k)$ and $g_T(n, k)$.

Theorem 6.19. [17] *Given two positive integers n and k with $3 \leq k \leq \binom{n}{2} + n$,*

$$f_T(n, k) = \begin{cases} n-1 & \text{if } k=3, \\ n+k-t-2 & \text{if } k = \binom{t}{2} + t + 2 - s, \\ & \text{where } 0 \leq s \leq t-1 \text{ and } 2 \leq t \leq n-2, \\ k & \text{if } \binom{n}{2} - n + 4 \leq k \leq \binom{n}{2} + n - 3\lfloor \frac{n}{2} \rfloor, \\ & \text{except for } n \text{ is odd and } k = \binom{n}{2} + n - 3\lfloor \frac{n}{2} \rfloor, \\ \binom{n}{2} - r & \text{if } \binom{n}{2} + n - 3(r+1) < k \leq \binom{n}{2} + n - 3r, \text{ where } 0 \leq r \\ & \leq \lfloor \frac{n}{2} \rfloor - 1 \text{ or } n \text{ is odd, } r = \lfloor \frac{n}{2} \rfloor \text{ and } k = \binom{n}{2} + n - 3\lfloor \frac{n}{2} \rfloor. \end{cases}$$

Theorem 6.20. [17] Given two positive integers n and k with $n \leq k \leq \binom{n}{2} + n$,

$$g_T(n, k) = \begin{cases} k - n + t & \text{if } \binom{n-t}{2} + t(n-t-1) + n \leq k \leq \binom{n-t}{2} + t(n-t) + n - 2, \\ k - n + t - 1 & \text{if } k = \binom{n-t}{2} + t(n-t) + n - 1, \\ \binom{n}{2} - 1 & \text{if } k = \binom{n}{2} + n - 1, \\ \binom{n}{2} & \text{if } k = \binom{n}{2} + n, \end{cases}$$

for $2 \leq t \leq n - 1$.

7 Concluding remarks

This survey tries to summarize all the results on monochromatic connection of graphs in the existing literature. The simple purpose is to promote the research along this subject. As one can see, there are some basic problems remaining unsolved. For example, what is the computational complexity of determining the monochromatic connection number $mc(G)$ for a given connected graph G ? From Theorem 2.1 (d) one can see that this problem is reduced to only considering those graphs with diameter 2. It is easily seen also from Theorem 2.1 (a) that for almost all connected graphs G it holds that $mc(G) = m(G) - n(G) + 2$.

Another problem is to consider more monochromatic paths connecting a pair of vertices. The definitions can be easily given as follows. An edge-colored graph is called monochromatically k -connected if each pair of vertices of the graph is connected by k monochromatic paths in the graph. For a k -connected graph G , the *monochromatic k -connection number*, denoted by $mc_k(G)$, is defined as the maximum number of colors that are needed in order to make G monochromatically k -connected. As far as we knew, there is no paper published on this parameter. We think that to get some bounds for the case $k = 2$ is already quite interesting and not so easy.

It is seen that results for the monochromatic indices are very few, and more efforts are needed for deepening the research. It is also seen that research on $smc(D)$ for digraphs has just started, and one can develop it with many possibilities.

Finally, we point out that we changed some terminology and notation. For examples, we use $vmc(G)$ to replace $mvc(G)$ and $vmx_k(G)$ to replace $mvx_k(G)$, etc. This is because we think that the term “vertex-monochromatic connection” is better than “monochromatic vertex-connection”. This is just a matter of taste, depending on authors and readers.

References

- [1] A. Blass, F. Harary, *Properties of almost all graphs and complexes*, J. Graph Theory 3(3)(1979), 225-240.
- [2] B. Bollobás, A. Thomason, *Threshold functions*, Combinatorica 7(1986), 35–38.
- [3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press, London and Basingstoker, 1976.

- [4] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(2008), 85-98.
- [5] Q. Cai, X. Li, D. Wu, *Erdős-Gallai-type results for colorful monochromatic connectivity of a graph*, J. Comb. Optim. 33(1)(2017), 123-131.
- [6] Q. Cai, X. Li, D. Wu, *Some extremal results on the colorful monochromatic vertex-connectivity of a graph*, arXiv:1503.08941.
- [7] Y. Caro, A. Lev, Y. Roditty, Zs. Tuza, R. Yuster, *On rainbow connection*, Electron. J. Combin. 15(1)(2008), R57.
- [8] Y. Caro, D.B. West, R. Yuster, *Connected domination and spanning trees with many leaves*, SIAM J. Discrete Math. 13(2)(2000), 202-211.
- [9] Y. Caro, R. Yuster, *Colorful monochromatic connectivity*, Discrete Math. 311(2011), 1786-1792.
- [10] P. Erdős, A. Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hungar. Acad. Sci. 5(1960), 17-61.
- [11] E. Friedgut, G. Kalai, *Every monotone graph property has a sharp threshold*, Proc. Amer. Math. Soc. 124(1996), 2993-3002.
- [12] D. Gonzalez-Moreno, M. Guevara, J.J. Montellano-Ballesteros, *Monochromatic connecting colorings in strongly connected oriented graphs*, Discrete Math. 340(4)(2017), 578-584.
- [13] J.R. Griggs, M. Wu, *Spanning trees in graphs of minimum degree 4 or 5*, Discrete Math. 104(2)(1992), 167-183.
- [14] R. Gu, X. Li, Z. Qin, Y. Zhao, *More on the colorful monochromatic connectivity*, Bull. Malays. Math. Sci. Soc. 40(4)(2017), 1769-1779.
- [15] H. Jiang, X. Li, Y. Zhang, *Total monochromatic connection of graphs*, Discrete Math. 340(2017), 175-180.
- [16] H. Jiang, X. Li, Y. Zhang, *More on total monochromatic connection of graphs*, Ars Combin. 136(2018).
- [17] H. Jiang, X. Li, Y. Zhang, *Erdős-Gallai-type results for total monochromatic connection of graphs*, Discuss. Math. Graph Theory, in press.
- [18] Z. Jin, X. Li, K. Wang, *The monochromatic connectivity of some graphs*, Manuscript, 2016.
- [19] D.J. Kleitman, D.B. West, *Spanning trees with many leaves*, SIAM J. Discrete Math. 4(1)(1991), 99-106.

- [20] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory. 63(3)(2010), 185-191.
- [21] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs: A survey*, Graphs & Combin. 29(2013), 1-38.
- [22] X. Li, Y. Sun, *Rainbow Connections of Graphs*, SpringerBriefs in Math. Springer, New York, 2012.
- [23] X. Li, Di Wu, *The (vertex-)monochromatic index of a graph*, J. Comb. Optim. 33(2017), 1443-1453.
- [24] Y. Mao, Z. Wang, F. Yanling, C. Ye, *Monochromatic connectivity and graph products*, Discrete Math, Algorithm. Appl. 8(01)2016, 1650011.