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Application of an Extremal Result of Erdős and Gallai to the \((n,k,t)\) Problem

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Abstract

An extremal result about vertex covers, attributed by Hajnal [4] to Erdős and Gallai [2], is applied to prove the following: If $n$, $k$, and $t$ are integers satisfying $n \geq k \geq t \geq 3$ and $k \leq 2t - 2$, and $G$ is a graph with the minimum number of edges among graphs on $n$ vertices with the property that every induced subgraph on $k$ vertices contains a complete subgraph on $t$ vertices, then every component of $G$ is complete.

**Keywords and phrases:** vertex cover, independent set, matching, $(n,k,t)$ problem, Erdős-Stone Theorem, Turán’s Theorem, Turán graph

1 Introduction

All graphs here will be finite, non-null, and simple. A vertex cover of a graph $G$ is a set $S \subset V(G)$ that contains at least one endpoint of every edge in $G$. The vertex cover number of $G$ is the minimum size of a vertex cover of $G$, and is denoted by $\beta(G)$. This parameter is monotone— that is, $\beta(H) \leq \beta(G)$ for all subgraphs $H$ of $G$. Deleting a vertex or an edge of $G$ causes the vertex cover number to go down by at most 1. An edge or vertex of $G$ whose removal causes such a decrease is said to be $\beta$-critical (or vertex-cover critical) for $G$. The graph $G$ itself is said to be $\beta$-critical or vertex-cover critical if $\beta(H) < \beta(G)$ for every proper subgraph $H$ of $G$. It is easy to see that $G$ is $\beta$-critical if and only if $G$ has no isolated vertices and every edge of $G$ is $\beta$-critical for $G$. In particular, this means that if $\beta(G) > 0$, then $G$ has a vertex-cover critical subgraph $H$ with $\beta(H) = \beta(G)$.

$S \subset V(G)$ is a vertex cover if and only if $V(G) \setminus S$ is independent: from this it is easy to see that, if $\alpha(G)$ is the vertex independence number of $G$, the size of a largest independent (mutually non-adjacent) set of vertices, then $\alpha(G) + \beta(G) = |V(G)|$. Therefore, a graph $G$ is $\beta$-critical if and only if $G$ has no isolated vertices and, for each $e \in E(G)$, $\alpha(G-e) = \alpha(G)+1$. With this in mind, it is easy to verify that the following are $\beta$-critical: (i) $K_n$ for $n \geq 2$; (ii) odd cycles; and (iii) matchings.

In conformity with the notation by which $G + H$ denotes the disjoint union of $G$ and $H$, a matching with $s$ edges will be denoted $sK_2 = K_2 + \cdots + K_2$. Clearly $\beta(sK_2) = s$.

If $G$ is bipartite, the König-Egerváry Theorem ([1], [6]) says that $\beta(G)$ is the maximum number of edges in a matching in $G$. Therefore, a non-empty bipartite graph is vertex-cover critical if and only if it is a matching.

The extremal result of Erdős and Gallai [2] referred to in the title of this paper concerns the function $f$ defined for $s = 1, 2, \ldots$ as

$$f(s) = \max\{|V(G)| : G \text{ is } \beta\text{-critical and } \beta(G) = s\}.$$

The result is that $f(s) = 2s$. We have not been able to obtain a copy of [2]; we found an attribution to [2] of this result in [4], where Hajnal provides a short proof, suggesting that the proof in [2], a 23-page paper, is not very short. Later in [4], Hajnal, apparently without realizing it, provides an even shorter proof that supplies a stronger conclusion: not only is it true that $f(s) = 2s$, but, also, $sK_2$ is the only $\beta$-critical graph on $2s$ vertices with vertex cover number $s$. We will give this proof here, in a form that the reader will not find in [4]. Hajnal there is driving toward a dual form of the result, based on the fact that $S$ is a vertex...
cover of $G$ if and only if $V(G) \setminus S$ induces a complete graph in $G$, the complement of $G$.

Here is our translation of his second proof. The word “cover” will mean vertex cover. If $S \subset V(G)$, $N_G(S) = \{u \in V(G) : uv \in E(G) \text{ for some } v \in S\}$.

**Lemma 1.1** Suppose that $G$ is $\beta$-critical and $I \subset V(G)$ is an independent set of vertices. Then $|I| \leq |N_G(I)|$.

**Proof** The proof will be by induction on $|I|$. Since $G$ has no isolated vertices, the conclusion holds when $|I| = 1$.

Suppose $|I| > 1$, and suppose that $|I| \geq |N_G(I)| + 1$. We will deduce a contradiction. Let $v \in I$ and $I' = I \setminus \{v\}$. By the induction hypothesis, $|N_G(I)| \leq |I| - 1 = |I'| \leq |N_G(I')| \leq |N_G(I)|$. Therefore, $|I'| = |N_G(I')| = |N_G(I)|$, so $N_G(I') = N_G(I)$. Now let $H$ be the induced subgraph of $G$ with vertex set $V(H) = I' \cup N_G(I)$. Also by induction, for every $J \subset I'$, $|J| \leq |N_G(J)|$. Therefore, by Hall’s Theorem, $H$ has a perfect matching $M$.

Since $v$ is not an isolated vertex, $vw \in E(G)$ for some $w \in N_G(I)$. Since $G$ is $\beta$-critical, $G - vw$ has a cover $C$ of size $\beta(G) - 1$. Let $C' = C \setminus (I \cup N_G(I))$ and $C'' = C \cap (I \cup N_G(I))$. Since all edges of $G - vw$ having both ends in $I \cup N_G(I)$ must be covered by $C''$, $C''$ must cover $M$, so $|C''| \geq |M| = |N_G(I)|$. Thus $|C' \cup N_G(I)| = |C'| + |N_G(I)| \leq |C'| + |C''| = |C' \cup C''| = |C| = \beta(G) - 1$.

But $C'$ covers all edges of $G$ with neither end in $I \cup N_G(I)$, and $N_G(I)$ covers each edge of $G$ with at least one end in $I \cup N_G(I)$, because $I$ is independent, so $C' \cup N_G(I)$ is a cover of $G$. Therefore, $|C' \cup N_G(I)| \leq (\beta(G) - 1)$ is a contradiction. \hfill $\square$

**Theorem 1.2** If $G$ is vertex-cover critical then $|V(G)| \leq 2\beta(G)$, with equality if and only if $G$ is isomorphic to $\beta(G)K_2$.

**Proof** Let $S$ be a minimum cover of $G$; $|S| = \beta(G)$. Let $I = V(G) \setminus S$, an independent set; since $S$ is a cover, $N_G(I) = S$. By Lemma 1.1, $|I| \leq |S|$, so $|V(G)| = |I| + |S| \leq 2|S| = 2\beta(G)$.

If $|V(G)| = 2\beta(G)$, then $|I| = |S|$. By Lemma 1.1, $|J| \leq |N_G(J)|$ for all $J \subset I$. Therefore, by Hall’s Theorem, there is a perfect matching $M$ in $G$; $M$ is isomorphic to $\beta(G)K_2$. Since $\beta(M) = \beta(G)$ and $G$ is $\beta$-critical, it must be that $M = G$. \hfill $\square$

## 2 Application to the $(n, k, t)$ Problem

Suppose $n \geq k \geq t$ are positive integers. An $(n, k, t)$-graph is a graph on $n$ vertices such that every induced subgraph of order $k$ contains a clique of order $t$. The $(n, k, t)$ problem is to determine, for each triple $(n, k, t)$, all the minimum $(n, k, t)$-graphs – that is, the $(n, k, t)$-graphs with the fewest edges. When $t = 1$ the only such graph is the graph with $n$ isolated vertices, and when $t = 2$, the problem can be seen as a complementary version of Turán’s Theorem [7]; hence the unique minimum $(n, k, 2)$-graphs are $\overline{T_{n,k-1}}$, where $T_{n,r}$ denotes the Turán graph on $n$ vertices with $r$ parts. Other easy cases include $k = t \geq 2$ and $n = k$, where the unique extremal graphs are $K_n$ and $(n-t)K_1 + K_t$, respectively [5].

The $(n, k, t)$ conjecture is that whenever $n \geq k \geq t$, some minimum $(n, k, t)$-graph has complete components. The strong $(n, k, t)$ conjecture is that every minimum $(n, k, t)$-graph has complete components. If the strong $(n, k, t)$ conjecture holds then the $(n, k, t)$ problem is
essentially solved in [5] – the extremal graphs are all $aK_1 + T_{n-a,b}$ for particular non-negative integers $a, b$ – although there is room for improvement in the determination of $a$ and $b$ given in [5].

**Theorem 2.1** (Erdős and Stone [3]) *Suppose $\mathcal{F}$ is a family of graphs containing no empty graph, and let*

$$g(n) = \max\{|E(G)| : |V(G)| = n \text{ and no member of } \mathcal{F} \text{ is a subgraph of } G\}.$$

*Let $\chi(\mathcal{F}) = \min\{\chi(H) : H \in \mathcal{F}\}$, and suppose that $\chi(\mathcal{F}) > 2$. Let $r = \chi(\mathcal{F}) - 1$. Then*

$$\frac{|E(T_{n,r})|}{g(n)} \to 1 \text{ as } n \to \infty.$$

*Explanation:* The name $\mathcal{F}$ was chosen to denote forbidden subgraphs. Clearly no graph with chromatic number $r = \chi(\mathcal{F}) - 1$ can contain a subgraph from $\mathcal{F}$, and clearly the Turán graph $T_{n,r}$ is the graph on $n$ vertices of that chromatic number with the most edges, if $n \geq r$. Therefore, $|E(T_{n,r})| \leq g(n)$, for $n \geq r$. The Erdős-Stone Theorem asserts that if $\mathcal{F}$ contains no bipartite graph, then, asymptotically, $|E(T_{n,r})| \sim g(n)$.

In the original Erdős-Stone Theorem, $\mathcal{F}$ was a singleton; but the more general theorem follows easily from the original, by the following argument. Given $\mathcal{F}$, let $H \subset \mathcal{F}$ be such that $\chi(H) = \chi(\mathcal{F}) > 2$, and set $\mathcal{F}' = \{H\}$. Let $g'$ be defined with reference to $\mathcal{F}'$ as $g$ was defined with reference to $\mathcal{F}$. Clearly, $g'(n) \geq g(n)$ for all $n$, so, for $n \geq r = \chi(\mathcal{F}) - 1$, $1 > \frac{|E(T_{n,r})|}{g(n)} \geq \frac{|E(T_{n,r})|}{g'(n)} \to 1 \text{ as } n \to \infty$.

To apply the Erdős-Stone Theorem to the $(n, k, t)$ problem, we define an $(n, k, t)$-graph to be the complement of an $(n, k, t)$-graph. In other words, an $(n, k, t)$-graph is a simple graph on $n$ vertices such that every subgraph $H$ of order $k$ has vertex independence number $\alpha(H) \geq t$. (Notice the absence of the word “induced” in this description.) Clearly the $(n, k, t)$ problem is equivalent to the problem of describing the $(n, k, t)$-graphs with the most edges.

Fix $k > t > 2$. For $n \geq k$, an $(n, k, t)$-graph is a graph on $n$ vertices with no subgraph from $\mathcal{F} = \{H : |V(H)| = k \text{ and } \alpha(H) \leq t - 1\}$. Since $\chi(H) \geq \frac{|V(H)|}{\alpha(H)}$ for any graph $H$, $\chi(\mathcal{F}) \geq \lceil \frac{k}{t-1} \rceil$. On the other hand, there exists a complete multipartite graph $H$ with $\lceil \frac{k}{t-1} \rceil \geq 2$ parts on $k$ vertices with maximum part size $t-1$. Clearly $H \in \mathcal{F}$ and $\chi(H) = \lceil \frac{k}{t-1} \rceil$. Therefore, $\chi(\mathcal{F}) = \lceil \frac{k}{t-1} \rceil$.

Consequently, if $\frac{k}{t-1} > 2$, $r = \lceil \frac{k}{t-1} \rceil - 1$, and $g(n)$ is defined as in Theorem 2.1 with reference to $\mathcal{F}$, then $\frac{|E(T_{n,r})|}{g(n)} \to 1$ as $n \to \infty$. Therefore, the minimum number of edges in an $(n, k, t)$-graph, for $k$ and $t$ satisfying $k > t > 2$ and $k > 2t - 2$, is asymptotically equivalent, as $n \to \infty$, to $|E(T_{n,r})|$, where $r = \lceil \frac{k}{t-1} \rceil - 1$. This conclusion by no means proves that $T_{n,r}$ is a minimum $(n, k, t)$-graph for all $n$ sufficiently large, which is a good thing, because that conclusion would be false. For example, if $t = 3$, $k = 6$, so $\lceil \frac{k}{t-1} \rceil = 3$, by applying the main result of [5] it can be seen that for all $n \geq 8$ the unique $(n, 6, 3)$-graph with the fewest edges among those with all components complete is $K_1 + T_{n-1,2}$. In this case, and in many others, $T_{n,r}$ is an $(n, k, t)$-graph with number of edges (asymptotically as $n \to \infty$) close to smallest, but not smallest, among $(n, k, t)$-graphs.
However, the application of the Erdős-Stone Theorem to the \((n, k, t)\) problem is intriguing. For those sharing our prejudices, the asymptotic result reinforces a belief in the truth of the \((n, k, t)\) conjecture. It also points out the following, a nice result that we neglected to include in [5].

**Theorem 2.2** Suppose that \(k > t > 2\) are integers, \(\frac{k}{t-1} > 2\), \(r = \left\lceil \frac{k}{t-1} \right\rceil - 1\), and \(a = k - 1 - r(t - 1)\). For all sufficiently large \(n\), the unique \((n, k, t)\)-graph with the fewest number of edges among those with every component complete is \(aK_1 + \overline{T}_{n-a,r}\).

**Proof** By Corollary 1 of [5], for \(n \geq k + r - 1\) an \((n, k, t)\)-graph having only complete components and with as few edges as possible will be one of \((k - 1 - b(t - 1))K_1 + \overline{T}_{n-(k-1-b(t-1)),b}\) for \(1 \leq b \leq r\). In [5], \(r = \left\lfloor \frac{k}{t-1} \right\rfloor\); but this is equal to \(\left\lceil \frac{k}{t-1} \right\rceil - 1\). Since, for each fixed pair \((s, b)\) with \(s \geq 0\) and \(b \geq 0\), \(|E(\overline{T}_{n-s,b})| \sim \frac{n^2}{2b}\), for \(n\) sufficiently large the choice of \(b\) must be \(b = r\).

The application of Theorem 1.2 to the \((n, k, t)\) problem concerns values of \(k\) and \(t\) such that \(\frac{k}{t-1} \leq 2\), the values about which the Erdős-Stone Theorem has nothing to say.

The join of two graphs \(G\) and \(H\), denoted \(G \vee H\), is the graph obtained from the disjoint union of \(G\) and \(H\) by adding a complete bipartite graph between \(V(G)\) and \(V(H)\).

**Lemma 2.3** Suppose that \(n > s \geq 1\) are integers. The unique graph of order \(n\) with vertex cover number \(s\) with the most edges is \(K_s \vee \overline{K}_{n-s}\).

**Proof** Suppose \(|V(G)| = n\) and \(\beta(G) = s\), and let \(S \subset V(G)\) be a minimum vertex cover. Then \(V(G) \setminus S\) is an independent set of vertices; clearly \(G\) can have no more edges than the copy of \(K_s \vee \overline{K}_{n-s}\) obtained by putting in all \(S-S\) edges and all \(S-(V(G) \setminus S)\) edges.

On the other hand, \(G = K_s \vee \overline{K}_{n-s}\) has order \(n\) and vertex cover number \(n - \alpha(G) = n - (n - s) = s\).

**Lemma 2.4** Let \(n > k > t > 2\) be integers, and let \(G\) be a graph on \(n\) vertices. \(G\) is an \((n, k, t)\)-graph if and only if \(\overline{G}\) contains no \(\beta\)-critical subgraph \(X\) such that \(|V(X)| \leq k\) and \(\beta(X) = k - t + 1\).

**Proof** If \(G\) is an \((n, k, t)\)-graph then \(\overline{G}\) is an \((\overline{n}, \overline{k}, \overline{t})\)-graph; so for every subgraph \(Y\) of \(\overline{G}\) of order \(k\), \(\alpha(Y) \geq t\), so \(\beta(Y) = k - \alpha(Y) \leq k - t\). Therefore, every subgraph of \(\overline{G}\) on \(k\) or fewer vertices has vertex cover number less than \(k - t + 1\).

However, if \(G\) is not an \((n, k, t)\)-graph then \(G\) has an induced subgraph \(H\) on \(k\) vertices with clique number \(\omega(H) \leq t - 1\). Then \(\overline{H}\) is a subgraph of \(\overline{G}\) of order \(k\) with \(\alpha(\overline{H}) = \omega(H) \leq t - 1\); we have that \(\beta(\overline{H}) = k - \alpha(\overline{H}) \geq k - t + 1\). Hence we can find a \(\beta\)-critical subgraph \(X\) of \(\overline{H}\) with \(\beta(X) = k - t + 1\).

**Theorem 2.5** Suppose that \(k > t > 2\). If \(k \leq 2t - 2\), then for every \(n > k\) the unique \((n, k, t)\)-graph with the fewest edges is \((k-t)K_1 + K_{n-k+t}\).
Proof Suppose that \( k \leq 2t - 2 \), \( n > k \), and \( G \) is an \((n,k,t)\)-graph with the minimum number of edges possible. Then \( \overline{G} \) is an \((\overline{n},k,t)\)-graph with the maximum number of edges possible. By Lemma 2.4, \( \overline{G} \) has no \( \beta \)-critical subgraph \( X \) on \( k \) or fewer vertices such that \( \beta(X) = k - t + 1 \). As Theorem 1.2 gives \( f(k-t+1) = 2(k-t+1) \leq k \), it follows that \( \overline{G} \) has no \( \beta \)-critical subgraph \( X \) with \( \beta(X) = k - t + 1 \), because such an \( X \) could have no more than \( f(k-t+1) \leq k \) vertices.

Therefore, \( \beta(\overline{G}) \leq k - t \). By Lemma 2.3, \( \overline{G} \) can have no more edges than does \( K_{k-t} \lor K_{n-k+t} \), and, if \( \overline{G} \) has as many edges as that graph, then \( \overline{G} = K_{k-t} \lor K_{n-k+t} \). Since \( K_{k-t} \lor K_{n-k+t} \) is an \((\overline{n},k,t)\)-graph, it follows that \( \overline{G} = K_{k-t} \lor K_{n-k+t} \), so \( G = K_{k-t} + K_{n-k+t} \). \( \square \)

References


