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Application of an Extremal Result of Erdős and Gallai to the (n,k,t) Problem

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Abstract

An extremal result about vertex covers, attributed by Hajnal [4] to Erdős and Gallai [2], is applied to prove the following: If \( n, k, \) and \( t \) are integers satisfying \( n \geq k \geq t \geq 3 \) and \( k \leq 2t - 2 \), and \( G \) is a graph with the minimum number of edges among graphs on \( n \) vertices with the property that every induced subgraph on \( k \) vertices contains a complete subgraph on \( t \) vertices, then every component of \( G \) is complete.

Keywords and phrases: vertex cover, independent set, matching, \((n,k,t)\) problem, Erdős-Stone Theorem, Turán’s Theorem, Turán graph

1 Introduction

All graphs here will be finite, non-null, and simple. A vertex cover of a graph \( G \) is a set \( S \subset V(G) \) that contains at least one endpoint of every edge in \( G \). The vertex cover number of \( G \) is the minimum size of a vertex cover of \( G \), and is denoted by \( \beta(G) \). This parameter is monotone – that is, \( \beta(H) \leq \beta(G) \) for all subgraphs \( H \) of \( G \). Deleting a vertex or an edge of \( G \) causes the vertex cover number to go down by at most 1. An edge or vertex of \( G \) whose removal causes such a decrease is said to be \( \beta \)-critical (or vertex-cover critical) for \( G \). The graph \( G \) itself is said to be \( \beta \)-critical or vertex-cover critical if \( \beta(H) < \beta(G) \) for every proper subgraph \( H \) of \( G \). It is easy to see that \( G \) is \( \beta \)-critical if and only if \( G \) has no isolated vertices and every edge of \( G \) is \( \beta \)-critical for \( G \). In particular, this means that if \( \beta(G) > 0 \), then \( G \) has a vertex-cover critical subgraph \( H \) with \( \beta(H) = \beta(G) \).

If \( S \subset V(G) \) is a vertex cover if and only if \( V(G) \setminus S \) is independent: from this it is easy to see that, if \( \alpha(G) \) is the vertex independence number of \( G \), the size of a largest independent (mutually non-adjacent) set of vertices, then \( \alpha(G) + \beta(G) = |V(G)| \). Therefore, a graph \( G \) is \( \beta \)-critical if and only if \( G \) has no isolated vertices and, for each \( e \in E(G) \), \( \alpha(G-e) = \alpha(G)+1 \). With this in mind, it is easy to verify that the following are \( \beta \)-critical: (i) \( K_n \) for \( n \geq 2 \); (ii) odd cycles; and (iii) matchings.

In conformity with the notation by which \( G + H \) denotes the disjoint union of \( G \) and \( H \), a matching with \( s \) edges will be denoted \( sK_2 = K_2 + \cdots + K_2 \). Clearly \( \beta(sK_2) = s \).

If \( G \) is bipartite, the König-Egerváry Theorem ([1], [6]) says that \( \beta(G) \) is the maximum number of edges in a matching in \( G \). Therefore, a non-empty bipartite graph is vertex-cover critical if and only if it is a matching.

The extremal result of Erdős and Gallai [2] referred to in the title of this paper concerns the function \( f \) defined for \( s = 1, 2, \ldots \) as

\[
f(s) = \max\{|V(G)| : G \text{ is } \beta \text{-critical and } \beta(G) = s\}.
\]

The result is that \( f(s) = 2s \). We have not been able to obtain a copy of [2]; we found an attribution to [2] of this result in [4], where Hajnal provides a short proof, suggesting that the proof in [2], a 23-page paper, is not very short. Later in [4], Hajnal, apparently without realizing it, provides an even shorter proof that supplies a stronger conclusion: not only is it true that \( f(s) = 2s \), but, also, \( sK_2 \) is the only \( \beta \)-critical graph on \( 2s \) vertices with vertex cover number \( s \). We will give this proof here, in a form that the reader will not find in [4]. Hajnal there is driving toward a dual form of the result, based on the fact that \( S \) is a vertex...
cover of $G$ if and only if $V(G) \setminus S$ induces a complete graph in $G$, the complement of $G$.

Here is our translation of his second proof. The word “cover” will mean vertex cover. If $S \subset V(G)$, $N_G(S) = \{u \in V(G) : uv \in E(G) \text{ for some } v \in S\}$.

**Lemma 1.1** Suppose that $G$ is $\beta$-critical and $I \subset V(G)$ is an independent set of vertices. Then $|I| \leq |N_G(I)|$.

**Proof**  The proof will be by induction on $|I|$. Since $G$ has no isolated vertices, the conclusion holds when $|I| = 1$.

Suppose $|I| > 1$, and suppose that $|I| \geq |N_G(I)| + 1$. We will deduce a contradiction. Let $v \in I$ and $I' = I \setminus \{v\}$. By the induction hypothesis, $|N_G(I)| \leq |I| - 1 = |I'| \leq |N_G(I')| \leq |N_G(I)|$. Therefore, $|I'| = |N_G(I')| = |N_G(I)|$, so $N_G(I') = N_G(I)$. Now let $H$ be the induced subgraph of $G$ with vertex set $V(H) = I' \cup N_G(I)$. Also by induction, for every $J \subset I'$, $|J| \leq |N_G(J)|$. Therefore, by Hall’s Theorem, $H$ has a perfect matching $M$.

Since $v$ is not an isolated vertex, $vw \in E(G)$ for some $w \in N_G(I)$. Since $G$ is $\beta$-critical, $G - vw$ has a cover $C$ of size $\beta(G) - 1$. Let $C' = C \setminus (I \cup N_G(I))$ and $C'' = C \cap (I \cup N_G(I))$. Since all edges of $G - vw$ having both ends in $I \cup N_G(I)$ must be covered by $C''$, $C''$ must cover $M$, so $|C''| \geq |M| = |N_G(I)|$. Thus $|C' \cup N_G(I)| = |C'| + |N_G(I)| \leq |C'| + |C''| = |C' \cup C''| = |C| = \beta(G) - 1$.

But $C'$ covers all edges of $G$ with neither end in $I \cup N_G(I)$, and $N_G(I)$ covers each edge of $G$ with at least one end in $I \cup N_G(I)$, because $I$ is independent, so $C'' \cup N_G(I)$ is a cover of $G$. Therefore, $|C' \cup N_G(I)| \leq \beta(G) - 1$ is a contradiction. \hfill $\square$

**Theorem 1.2** If $G$ is vertex-cover critical then $|V(G)| \leq 2\beta(G)$, with equality if and only if $G$ is isomorphic to $\beta(G)K_2$.

**Proof**  Let $S$ be a minimum cover of $G$; $|S| = \beta(G)$. Let $I = V(G) \setminus S$, an independent set; since $S$ is a cover, $N_G(I) = S$. By Lemma 1.1, $|I| \leq |S|$, so $|V(G)| = |I| + |S| \leq 2|S| = 2\beta(G)$. If $|V(G)| = 2\beta(G)$, then $|I| = |S|$. By Lemma 1.1, $|J| \leq |N_G(J)|$ for all $J \subset I$. Therefore, by Hall’s Theorem, there is a perfect matching $M$ in $G$; $M$ is isomorphic to $\beta(G)K_2$. Since $\beta(M) = \beta(G)$ and $G$ is $\beta$-critical, it must be that $M = G$. \hfill $\square$

# 2 Application to the $(n, k, t)$ Problem

Suppose $n \geq k \geq t$ are positive integers. An $(n, k, t)$-graph is a graph on $n$ vertices such that every induced subgraph of order $k$ contains a clique of order $t$. The $(n, k, t)$ problem is to determine, for each triple $(n, k, t)$, all the minimum $(n, k, t)$-graphs – that is, the $(n, k, t)$-graphs with the fewest edges. When $t = 1$ the only such graph is the graph with $n$ isolated vertices, and when $t = 2$, the problem can be seen as a complementary version of Turán’s Theorem [7]; hence the unique minimum $(n, k, 2)$-graphs are $T_{n,k-1}$, where $T_{n,r}$ denotes the Turán graph on $n$ vertices with $r$ parts. Other easy cases include $k = t \geq 2$ and $n = k$, where the unique extremal graphs are $K_n$ and $(n-t)K_1 + K_t$, respectively [5].

The $(n, k, t)$ conjecture is that whenever $n \geq k \geq t$, some minimum $(n, k, t)$-graph has complete components. The strong $(n, k, t)$ conjecture is that every minimum $(n, k, t)$-graph has complete components. If the strong $(n, k, t)$ conjecture holds then the $(n, k, t)$ problem is
essentially solved in [5] – the extremal graphs are all $aK_1 + T_{n-a,b}$ for particular non-negative integers $a,b$ – although there is room for improvement in the determination of $a$ and $b$ given in [5].

**Theorem 2.1 (Erdős and Stone [3])** Suppose $F$ is a family of graphs containing no empty graph, and let

$$g(n) = \max\{|E(G)| : |V(G)| = n \text{ and no member of } F \text{ is a subgraph of } G\}.$$

Let $\chi(F) = \min\{\chi(H) : H \in F\}$, and suppose that $\chi(F) > 2$. Let $r = \chi(F) - 1$. Then

$$\frac{|E(T_{n,r})|}{g(n)} \to 1 \text{ as } n \to \infty.$$

**Explanation:** The name $F$ was chosen to connote forbidden subgraphs. Clearly no graph with chromatic number $r = \chi(F) - 1$ can contain a subgraph from $F$, and clearly the Turán graph $T_{n,r}$ is the graph on $n$ vertices of that chromatic number with the most edges, if $n \geq r$. Therefore, $|E(T_{n,r})| \leq g(n)$, for $n \geq r$. The Erdős-Stone Theorem asserts that if $F$ contains no bipartite graph, then, asymptotically, $|E(T_{n,r})| \sim g(n)$.

In the original Erdős-Stone Theorem, $F$ was a singleton; but the more general theorem follows easily from the original, by the following argument. Given $F$, let $H \subset F$ be such that $\chi(H) = \chi(F) > 2$, and set $F' = \{H\}$. Let $g'$ be defined with reference to $F'$ as $g$ was defined with reference to $F$. Clearly, $g'(n) \geq g(n)$ for all $n$, so, for $n \geq r = \chi(F) - 1$, 1 $\geq \frac{|E(T_{n,r})|}{g(n)} \geq \frac{|E(T_{n,r})|}{g'(n)} \to 1$ as $n \to \infty$.

To apply the Erdős-Stone Theorem to the $(n,k,t)$ problem, we define an $(\bar{n},\bar{k},\bar{t})$-graph to be the complement of an $(n,k,t)$-graph. In other words, an $(\bar{n},\bar{k},\bar{t})$-graph is a simple graph on $n$ vertices such that every subgraph $H$ of order $k$ has vertex independence number $\alpha(H) \geq t$. (Notice the absence of the word “induced” in this description.) Clearly the $(n,k,t)$ problem is equivalent to the problem of describing the $(\bar{n},\bar{k},\bar{t})$-graphs with the most edges.

Fix $k > t > 2$. For $n \geq k$, an $(\bar{n},\bar{k},\bar{t})$-graph is a graph on $n$ vertices with no subgraph from $F = \{H : |V(H)| = k \text{ and } \alpha(H) \leq t - 1\}$. Since $\chi(H) \geq \frac{|V(H)|}{\alpha(H)}$ for any graph $H$, $\chi(F) \geq \lceil \frac{k}{t-1} \rceil$. On the other hand, there exists a complete multipartite graph $H$ with $\lceil \frac{k}{t-1} \rceil \geq 2$ parts on $k$ vertices with maximum part size $t-1$. Clearly $H \in F$ and $\chi(H) = \lceil \frac{k}{t-1} \rceil$. Therefore, $\chi(F) = \lceil \frac{k}{t-1} \rceil$.

Consequently, if $\frac{k}{t-1} > 2$, $r = \lceil \frac{k}{t-1} \rceil - 1$, and $g(n)$ is defined as in Theorem 2.1 with reference to $F$, then $\frac{|E(T_{n,r})|}{g(n)} \to 1$ as $n \to \infty$. Therefore, the minimum number of edges in an $(n,k,t)$-graph, for $k$ and $t$ satisfying $k > t > 2$ and $k > 2t - 2$, is asymptotically equivalent, as $n \to \infty$, to $|E(T_{n,r})|$, where $r = \lceil \frac{k}{t-1} \rceil - 1$. This conclusion by no means proves that $T_{n,r}$ is a minimum $(n,k,t)$-graph for all $n$ sufficiently large, which is a good thing, because that conclusion would be false. For example, if $t = 3$, $k = 6$, so $\lceil \frac{k}{t-1} \rceil = 3$, by applying the main result of [5] it can be seen that for all $n \geq 8$ the unique $(n,6,3)$-graph with the fewest edges among those with all components complete is $K_1 + T_{n-1,2}$. In this case, and in many others, $T_{n,r}$ is an $(n,k,t)$-graph with number of edges (asymptotically as $n \to \infty$) close to smallest, but not smallest, among $(n,k,t)$-graphs.
However, the application of the Erdős-Stone Theorem to the \((n,k,t)\) problem is intriguing. For those sharing our prejudices, the asymptotic result reinforces a belief in the truth of the \((n,k,t)\) conjecture. It also points out the following, a nice result that we neglected to include in [5].

**Theorem 2.2** Suppose that \(k > t > 2\) are integers, \(\frac{k}{t-1} > 2\), \(r = \left\lceil \frac{k}{t-1} \right\rceil - 1\), and \(a = k - 1 - r(t - 1)\). For all sufficiently large \(n\), the unique \((n,k,t)\)-graph with the fewest number of edges among those with every component complete is \(aK_1 + \overline{T}_{n-a,r}\).

**Proof** By Corollary 1 of [5], for \(n \geq k + r - 1\) an \((n,k,t)\)-graph having only complete components and with as few edges as possible will be one of \((k - 1 - b(t - 1))K_1 + \overline{T}_{n-(k-1-b(t-1)),b}\) for \(1 \leq b \leq r\). In [5], \(r = \left\lfloor \frac{k-1}{t-1} \right\rfloor\); but this is equal to \(\left\lceil \frac{k}{t-1} \right\rceil - 1\). Since, for each fixed pair \((s,b)\) with \(s \geq 0\) and \(b \geq 0\), \(|E(\overline{T}_{n-s,b})| \sim \frac{n^2}{2b}\), for \(n\) sufficiently large the choice of \(b\) must be \(b = r\).

The application of Theorem 1.2 to the \((n,k,t)\) problem concerns values of \(k\) and \(t\) such that \(\frac{k}{t-1} \leq 2\), the values about which the Erdős-Stone Theorem has nothing to say.

The **join** of two graphs \(G\) and \(H\), denoted \(G \vee H\), is the graph obtained from the disjoint union of \(G\) and \(H\) by adding a complete bipartite graph between \(V(G)\) and \(V(H)\).

**Lemma 2.3** Suppose that \(n > s \geq 1\) are integers. The unique graph of order \(n\) with vertex cover number \(s\) with the most edges is \(K_s \vee \overline{K}_{n-s}\).

**Proof** Suppose \(|V(G)| = n\) and \(\beta(G) = s\), and let \(S \subset V(G)\) be a minimum vertex cover. Then \(V(G) \setminus S\) is an independent set of vertices; clearly \(G\) can have no more edges than the copy of \(K_s \vee \overline{K}_{n-s}\) obtained by putting in all \(S-S\) edges and all \(S-(V(G) \setminus S)\) edges.

On the other hand, \(G = K_s \vee \overline{K}_{n-s}\) has order \(n\) and vertex cover number \(n - \alpha(G) = n - (n - s) = s\).

**Lemma 2.4** Let \(n > k > t > 2\) be integers, and let \(G\) be a graph on \(n\) vertices. \(G\) is an \((n,k,t)\)-graph if and only if \(\overline{G}\) contains no \(\beta\)-critical subgraph \(X\) such that \(|V(X)| \leq k\) and \(\beta(X) = k - t + 1\).

**Proof** If \(G\) is an \((n,k,t)\)-graph then \(\overline{G}\) is an \((\overline{n},k,t)\)-graph; so for every subgraph \(Y\) of \(\overline{G}\) of order \(k\), \(\alpha(Y) \geq t\), so \(\beta(Y) = k - \alpha(Y) \leq k - t\). Therefore, every subgraph of \(\overline{G}\) on \(k\) or fewer vertices has vertex cover number less than \(k - t + 1\).

However, if \(G\) is not an \((n,k,t)\)-graph then \(G\) has an induced subgraph \(H\) on \(k\) vertices with clique number \(\omega(H) \leq t - 1\). Then \(\overline{H}\) is a subgraph of \(\overline{G}\) of order \(k\) with \(\alpha(\overline{H}) = \omega(H) \leq t - 1\); we have that \(\beta(\overline{H}) = k - \alpha(\overline{H}) \geq k - t + 1\). Hence we can find a \(\beta\)-critical subgraph \(X\) of \(\overline{H}\) with \(\beta(X) = k - t + 1\).

**Theorem 2.5** Suppose that \(k > t > 2\). If \(k \leq 2t - 2\), then for every \(n > k\) the unique \((n,k,t)\)-graph with the fewest edges is \((k - t)K_1 + K_{n-k+t}\).
Proof Suppose that \( k \leq 2t - 2 \), \( n > k \), and \( G \) is an \((n,k,t)\)-graph with the minimum number of edges possible. Then \( \overline{G} \) is an \((n,k,t)\)-graph with the maximum number of edges possible. By Lemma 2.4, \( \overline{G} \) has no \( \beta \)-critical subgraph \( X \) on \( k \) or fewer vertices such that \( \beta(X) = k - t + 1 \). As Theorem 1.2 gives \( f(k - t + 1) = 2(k - t + 1) \leq k \), it follows that \( \overline{G} \) has no \( \beta \)-critical subgraph \( X \) with \( \beta(X) = k - t + 1 \), because such an \( X \) could have no more than \( f(k - t + 1) \leq k \) vertices.

Therefore, \( \beta(\overline{G}) \leq k - t \). By Lemma 2.3, \( \overline{G} \) can have no more edges than does \( K_{k-t} \lor K_{n-k+t} \), and, if \( \overline{G} \) has as many edges as that graph, then \( \overline{G} = K_{k-t} \lor K_{n-k+t} \). Since \( K_{k-t} \lor K_{n-k+t} \) is an \((n,k,t)\)-graph, it follows that \( \overline{G} = K_{k-t} \lor K_{n-k+t} \), so \( G = K_{k-t} \lor K_{n-k+t} \).

\[ \square \]

References