Theory and Applications of Graphs

Volume 4 | Issue 2 Article 1

2017

Application of an Extremal Result of Erdős and Gallai to the (n,k,t) Problem

Matt Noble
Middle Georgia State University, matthewhnoble@gmail.com

Peter Johnson
Auburn University Main Campus, johnspd@auburn.edu

Dean Hoffman
Auburn University, hoffmdg@auburn.edu

Jessica McDonald
Auburn University, mcdonald@auburn.edu

Recommended Citation

Noble, Matt; Johnson, Peter; Hoffman, Dean; and McDonald, Jessica (2017) "Application of an Extremal Result of Erdős and Gallai to the (n,k,t) Problem," *Theory and Applications of Graphs*: Vol. 4: Iss. 2, Article 1.
DOI: 10.20429/tag.2017.040201
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol4/iss2/1

Abstract

An extremal result about vertex covers, attributed by Hajnal [4] to Erdős and Gallai [2], is applied to prove the following: If n, k, and t are integers satisfying $n \geq k \geq t \geq 3$ and $k \leq 2t-2$, and G is a graph with the minimum number of edges among graphs on n vertices with the property that every induced subgraph on k vertices contains a complete subgraph on t vertices, then every component of G is complete.

Keywords and phrases: vertex cover, independent set, matching, (n, k, t) problem, Erdős-Stone Theorem, Turán's Theorem, Turán graph

1 Introduction

All graphs here will be finite, non-null, and simple. A *vertex cover* of a graph G is a set $S \subset V(G)$ that contains at least one endpoint of every edge in G. The *vertex cover number* of G is the minimum size of a vertex cover of G, and is denoted by $\beta(G)$. This parameter is monotone – that is, $\beta(H) \leq \beta(G)$ for all subgraphs H of G. Deleting a vertex or an edge of G causes the vertex cover number to go down by at most 1. An edge or vertex of G whose removal causes such a decrease is said to be *β-critical* (or vertex-cover critical) for G. The graph G itself is said to be *β-critical* or *vertex-cover critical* if $\beta(H) < \beta(G)$ for every proper subgraph H of G. It is easy to see that G is β-critical if and only if G has no isolated vertices and every edge of G is β-critical for G. In particular, this means that if $\beta(G) > 0$, then G has a vertex-cover critical subgraph H with $\beta(H) = \beta(G)$.

$S \subset V(G)$ is a vertex cover if and only if $V(G) \setminus S$ is independent: from this it is easy to see that, if $\alpha(G)$ is the vertex independence number of G, the size of a largest independent (mutually non-adjacent) set of vertices, then $\alpha(G) + \beta(G) = |V(G)|$. Therefore, a graph G is β-critical if and only if G has no isolated vertices and, for each $e \in E(G)$, $\alpha(G-e) = \alpha(G)+1$. With this in mind, it is easy to verify that the following are β-critical; (i) K_n for $n \geq 2$; (ii) odd cycles; and (iii) matchings.

In conformity with the notation by which $G + H$ denotes the disjoint union of G and H, a matching with s edges will be denoted $sK_2 = K_2 + \cdots + K_2$. Clearly $\beta(sK_2) = s$.

If G is bipartite, the Kőnig-Egerváry Theorem ([1], [6]) says that $\beta(G)$ is the maximum number of edges in a matching in G. Therefore, a non-empty bipartite graph is vertex-cover critical if and only if it is a matching.

The extremal result of Erdős and Gallai [2] referred to in the title of this paper concerns the function f defined for $s = 1, 2, \ldots$ as

$$f(s) = \max\{|V(G)| : G \text{ is } \beta\text{-critical and } \beta(G) = s\}.$$

The result is that $f(s) = 2s$. We have not been able to obtain a copy of [2]; we found an attribution to [2] of this result in [4], where Hajnal provides a short proof, suggesting that the proof in [2], a 23-page paper, is not very short. Later in [4], Hajnal, apparently without realizing it, provides an even shorter proof that supplies a stronger conclusion: not only is it true that $f(s) = 2s$, but, also, sK_2 is the only β-critical graph on $2s$ vertices with vertex cover number s. We will give this proof here, in a form that the reader will not find in [4]. Hajnal there is driving toward a dual form of the result, based on the fact that S is a vertex cover of G if and only if $V(G) \setminus S$ induces a complete graph in $\overline{G}$, the complement of G. Here is our translation of his second proof. The word "cover" will mean vertex cover. If $S \subset V(G)$, $N_G(S) = \{u \in V(G) : uv \in E(G) \text{ for some } v \in S\}$.

Lemma 1.1 *Suppose that G is β-critical and $I \subset V(G)$ is an independent set of vertices. Then $|I| \leq |N_G(I)|$.*

Proof The proof will be by induction on $|I|$. Since G has no isolated vertices, the conclusion holds when $|I| = 1$.

Suppose $|I| > 1$, and suppose that $|I| \geq |N_G(I)|+1$. We will deduce a contradiction. Let $v \in I$ and $I' = I \setminus \{v\}$. By the induction hypothesis, $|N_G(I)| \leq |I| - 1 = |I'| \leq |N_G(I')| \leq |N_G(I)|$. Therefore, $|I'| = |N_G(I')| = |N_G(I)|$, so $N_G(I') = N_G(I)$. Now let H be the induced subgraph of G with vertex set $V(H) = I' \cup N_G(I)$. Also by induction, for every $J \subset I'$, $|J| \leq |N_G(J)|$. Therefore, by Hall's Theorem, H has a perfect matching M.

Since v is not an isolated vertex, $vw \in E(G)$ for some $w \in N_G(I)$. Since G is β-critical, $G - vw$ has a cover C of size $\beta(G) - 1$. Let $C' = C \setminus (I \cup N_G(I))$ and $C'' = C \cap (I \cup N_G(I))$. Since all edges of $G - vw$ having both ends in $I \cup N_G(I)$ must be covered by C'', C'' must cover M, so $|C''| \geq |M| = |N_G(I)|$. Thus $|C' \cup N_G(I)| = |C'| + |N_G(I)| \leq |C'| + |C''| = |C' \cup C''| = |C| = \beta(G) - 1$.

But C' covers all edges of G with neither end in $I \cup N_G(I)$, and $N_G(I)$ covers each edge of G with at least one end in $I \cup N_G(I)$, because I is independent, so $C' \cup N_G(I)$ is a cover of G. Therefore, $|C' \cup N_G(I)| \leq \beta(G) - 1$ is a contradiction. □

Theorem 1.2 *If G is vertex-cover critical then $|V(G)| \leq 2\beta(G)$, with equality if and only if G is isomorphic to $\beta(G)K_2$.*

Proof Let S be a minimum cover of G; $|S| = \beta(G)$. Let $I = V(G) \setminus S$, an independent set; since S is a cover, $N_G(I) = S$. By Lemma 1.1, $|I| \leq |S|$, so $|V(G) = |I|+|S| \leq 2|S| = 2\beta(G)$. If $|V(G)| = 2\beta(G)$, then $|I| = |S|$. By Lemma 1.1, $|J| \leq |N_G(J)|$ for all $J \subset I$. Therefore, by Hall's Theorem, there is a perfect matching M in G; M is isomorphic to $\beta(G)K_2$. Since $\beta(M) = \beta(G)$ and G is β-critical, it must be that $M = G$. □

2 Application to the (n, k, t) Problem

Suppose $n \geq k \geq t$ are positive integers. An (n, k, t)-graph is a graph on n vertices such that every induced subgraph of order k contains a clique of order t. The (n, k, t) *problem* is to determine, for each triple (n, k, t), all the *minimum* (n, k, t)-graphs – that is, the (n, k, t)-graphs with the fewest edges. When $t = 1$ the only such graph is the graph with n isolated vertices, and when $t = 2$, the problem can be seen as a complementary version of Turán's Theorem [7]; hence the unique minimum $(n, k, 2)$-graphs are $\overline{T}_{n,k-1}$, where $T_{n,r}$ denotes the Turán graph on n vertices with r parts. Other easy cases include $k = t \geq 2$ and $n = k$, where the unique extremal graphs are K_n and $(n - t)K_1 + K_t$, respectively [5].

The (n, k, t) *conjecture* is that whenever $n \geq k \geq t$, some minimum (n, k, t)-graph has complete components. The *strong* (n, k, t) conjecture is that *every* minimum (n, k, t)-graph has complete components. If the strong (n, k, t) conjecture holds then the (n, k, t) problem is essentially solved in [5] – the extremal graphs are all $aK_1 + \overline{T}_{n-a,b}$ for particular non-negative integers a,b – although there is room for improvement in the determination of a and b given in [5].

Theorem 2.1 (Erdős and Stone [3]) *Suppose $\mathcal{F}$ is a family of graphs containing no empty graph, and let*

$$g(n) = \max\{|E(G)| : |V(G)| = n \textit{ and no member of } \mathcal{F} \textit{ is a subgraph of } G\}.$$

Let $\chi(\mathcal{F}) = \min\{\chi(H) : H \in \mathcal{F}\}$, and suppose that $\chi(\mathcal{F}) > 2$. Let $r = \chi(\mathcal{F}) - 1$. Then

$$\frac{|E(T_{n,r})|}{g(n)} \to 1 \textit{ as } n \to \infty.$$

Explanation: The name $\mathcal{F}$ was chosen to connote forbidden subgraphs. Clearly no graph with chromatic number $r = \chi(\mathcal{F}) - 1$ can contain a subgraph from $\mathcal{F}$, and clearly the Turán graph $T_{n,r}$ is the graph on n vertices of that chromatic number with the most edges, if $n \geq r$. Therefore, $|E(T_{n,r})| \leq g(n)$, for $n \geq r$. The Erdős-Stone Theorem asserts that if $\mathcal{F}$ contains no bipartite graph, then, asymptotically, $|E(T_{n,r})| \sim g(n)$.

In the original Erdős-Stone Theorem, $\mathcal{F}$ was a singleton; but the more general theorem follows easily from the original, by the following argument. Given $\mathcal{F}$, let $H \subset \mathcal{F}$ be such that $\chi(H) = \chi(\mathcal{F}) > 2$, and set $\mathcal{F}' = \{H\}$. Let g' be defined with reference to $\mathcal{F}'$ as g was defined with reference to $\mathcal{F}$. Clearly, $g'(n) \geq g(n)$ for all n, so, for $n \geq r = \chi(\mathcal{F}) - 1$, $1 \geq \frac{|E(T_{n,r})|}{g(n)} \geq \frac{|E(T_{n,r})|}{g'(n)} \to 1$ as $n \to \infty$.

To apply the Erdős-Stone Theorem to the (n,k,t) problem, we define an $(\overline{n,k,t})$-graph to be the complement of an (n,k,t)-graph. In other words, an $(\overline{n,k,t})$-graph is a simple graph on n vertices such that every subgraph H of order k has vertex independence number $\alpha(H) \geq t$. (Notice the absence of the word "induced" in this description.) Clearly the (n,k,t) problem is equivalent to the problem of describing the $(\overline{n,k,t})$-graphs with the most edges.

Fix $k > t > 2$. For $n \geq k$, an $(\overline{n,k,t})$-graph is a graph on n vertices with no subgraph from $\mathcal{F} = \{H : |V(H)| = k \text{ and } \alpha(H) \leq t-1\}$. Since $\chi(H) \geq \frac{|V(H)|}{\alpha(H)}$ for any graph H, $\chi(\mathcal{F}) \geq \lceil \frac{k}{t-1} \rceil$. On the other hand, there exists a complete multipartite graph H with $\lceil \frac{k}{t-1} \rceil \geq 2$ parts on k vertices with maximum part size $t-1$. Clearly $H \in \mathcal{F}$ and $\chi(H) = \lceil \frac{k}{t-1} \rceil$. Therefore, $\chi(\mathcal{F}) = \lceil \frac{k}{t-1} \rceil$.

Consequently, if $\frac{k}{t-1} > 2$, $r = \lceil \frac{k}{t-1} \rceil - 1$, and $g(n)$ is defined as in Theorem 2.1 with reference to $\mathcal{F}$, then $\frac{|E(T_{n,r})|}{g(n)} \to 1$ as $n \to \infty$. Therefore, the minimum number of edges in an (n,k,t)-graph, for k and t satisfying $k > t > 2$ and $k > 2t-2$, is asymptotically equivalent, as $n \to \infty$, to $|E(\overline{T}_{n,r})|$, where $r = \lceil \frac{k}{t-1} \rceil - 1$. This conclusion by no means proves that $\overline{T}_{n,r}$ is a minimum (n,k,t)-graph for all n sufficiently large, which is a good thing, because that conclusion would be false. For example, if $t = 3$, $k = 6$, so $\lceil \frac{k}{t-1} \rceil = 3$, by applying the main result of [5] it can be seen that for all $n \geq 8$ the unique $(n,6,3)$-graph with the fewest edges among those with all components complete is $K_1 + \overline{T}_{n-1,2}$. In this case, and in many others, $\overline{T}_{n,r}$ is an (n,k,t)-graph with number of edges (asymptotically as $n \to \infty$) close to smallest, but not smallest, among (n,k,t)-graphs.

However, the application of the Erdős-Stone Theorem to the (n, k, t) problem is intriguing. For those sharing our prejudices, the asymptotic result reinforces a belief in the truth of the (n, k, t) conjecture. It also points out the following, a nice result that we neglected to include in [5].

Theorem 2.2 *Suppose that $k > t > 2$ are integers, $\frac{k}{t-1} > 2$, $r = \lceil \frac{k}{t-1} \rceil - 1$, and $a = k-1-r(t-1)$. For all sufficiently large n, the unique (n, k, t)-graph with the fewest number of edges among those with every component complete is $aK_1 + \overline{T}_{n-a,r}$.*

Proof By Corollary 1 of [5], for $n \geq k + r - 1$ an (n, k, t)-graph having only complete components and with as few edges as possible will be one of $(k-1-b(t-1))K_1 + \overline{T}_{n-(k-1-b(t-1)),b}$ for $1 \leq b \leq r$. In [5], $r = \lfloor \frac{k-1}{t-1} \rfloor$; but this is equal to $\lceil \frac{k}{t-1} \rceil - 1$. Since, for each fixed pair (s, b) with $s \geq 0$ and $b \geq 0$, $|E(\overline{T}_{n-s,b})| \sim \frac{n^2}{2b}$, for n sufficiently large the choice of b must be $b = r$. $\square$

The application of Theorem 1.2 to the (n, k, t) problem concerns values of k and t such that $\frac{k}{t-1} \leq 2$, the values about which the Erdős-Stone Theorem has nothing to say.

The *join* of two graphs G and H, denoted $G \vee H$, is the graph obtained from the disjoint union of G and H by adding a complete bipartite graph between $V(G)$ and $V(H)$.

Lemma 2.3 *Suppose that $n > s \geq 1$ are integers. The unique graph of order n with vertex cover number s with the most edges is $K_s \vee \overline{K}_{n-s}$.*

Proof Suppose $|V(G)| = n$ and $\beta(G) = s$, and let $S \subset V(G)$ be a minimum vertex cover. Then $V(G) \setminus S$ is an independent set of vertices; clearly G can have no more edges than the copy of $K_s \vee \overline{K}_{n-s}$ obtained by putting in all S-S edges and all S-$(V(G) \setminus S)$ edges.

On the other hand, $G = K_s \vee \overline{K}_{n-s}$ has order n and vertex cover number $n - \alpha(G) = n - (n - s) = s$. $\square$

Lemma 2.4 *Let $n > k > t > 2$ be integers, and let G be a graph on n vertices. G is an (n, k, t)-graph if and only if $\overline{G}$ contains no β-critical subgraph X such that $|V(X)| \leq k$ and $\beta(X) = k - t + 1$.*

Proof If G is an (n, k, t)-graph then $\overline{G}$ is an $\overline{(n, k, t)}$-graph; so for every subgraph Y of $\overline{G}$ of order k, $\alpha(Y) \geq t$, so $\beta(Y) = k - \alpha(Y) \leq k - t$. Therefore, every subgraph of $\overline{G}$ on k or fewer vertices has vertex cover number less than $k - t + 1$.

However, if G is not an (n, k, t)-graph then G has an induced subgraph H on k vertices with clique number $\omega(H) \leq t - 1$. Then $\overline{H}$ is a subgraph of $\overline{G}$ of order k with $\alpha(\overline{H}) = \omega(H) \leq t - 1$; we have that $\beta(\overline{H}) = k - \alpha(\overline{H}) \geq k - t + 1$. Hence we can find a β-critical subgraph X of $\overline{H}$ with $\beta(X) = k - t + 1$. $\square$

Theorem 2.5 *Suppose that $k > t > 2$. If $k \leq 2t - 2$, then for every $n > k$ the unique (n, k, t)-graph with the fewest edges is $(k - t)K_1 + K_{n-k+t}$.*

Proof Suppose that $k \leq 2t-2$, $n > k$, and G is an (n,k,t)-graph with the minimum number of edges possible. Then $\overline{G}$ is an $\overline{(n,k,t)}$-graph with the maximum number of edges possible. By Lemma 2.4, $\overline{G}$ has no β-critical subgraph X on k or fewer vertices such that $\beta(X) = k-t+1$. As Theorem 1.2 gives $f(k-t+1) = 2(k-t+1) \leq k$, it follows that $\overline{G}$ has no β-critical subgraph X with $\beta(X) = k-t+1$, because such an X could have no more than $f(k-t+1) \leq k$ vertices.

Therefore, $\beta(\overline{G}) \leq k-t$. By Lemma 2.3, $\overline{G}$ can have no more edges than does $K_{k-t} \vee \overline{K}_{n-k+t}$, and, if $\overline{G}$ has as many edges as that graph, then $\overline{G} = K_{k-t} \vee \overline{K}_{n-k+t}$. Since $K_{k-t} \vee \overline{K}_{n-k+t}$ is an $\overline{(n,k,t)}$-graph, it follows that $\overline{G} = K_{k-t} \vee \overline{K}_{n-k+t}$, so $G = \overline{K}_{k-t} + K_{n-k+t} = (k-t)K_1 + K_{n-k+t}$. □

References

[1] E. Egerváry, On combinatorial properties of matrices (in Hungarian with German summary), *Math. Lapok* 38 (1931), 16-28.

[2] P. Erdős and T. Gallai, On the minimal number of vertices representing the edges of a graph, *Publ. Math. Inst. Hung. Acad. Sci.* 6 (1961), 181-203.

[3] P. Erdős and A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946), 1087-1091.

[4] A. Hajnal, A theorem on k-saturated graphs, *Canad. J. Math.* 17 (1965), 720-724.

[5] D. G. Hoffman, P. Johnson, and J. McDonald, Minimum (n,k,t) clique graphs, *Congr. Numer.* 223 (2015), 199-204.

[6] D. Kőnig, Graphen und Matrizen, *Math. Lapok* 38 (1931), 116-119.

[7] P. Turán, On an extremal problem in graph theory (in Hungarian), *Matematikai ès Fizikai Lapok* 48 (1941), 436-452.