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On the Graceful Cartesian Product of Alpha-Trees

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Abstract

A *graceful labeling* of a graph G of size n is an injective assignment of integers from the set $\{0, 1, \ldots, n\}$ to the vertices of G such that when each edge has assigned a *weight*, given by the absolute value of the difference of the labels of its end vertices, all the weights are distinct. A graceful labeling is called an α-labeling when the graph G is bipartite, with stable sets A and B, and the labels assigned to the vertices in A are smaller than the labels assigned to the vertices in B.

In this work we study graceful and α-labelings of graphs. We prove that the Cartesian product of two α-trees results in an α-tree when both trees admit α-labelings and their stable sets are balanced. In addition, we present a tree that has the property that when any number of pendant vertices are attached to the vertices of any subset of its smaller stable set, the resulting graph is an α-tree. We also prove the existence of an α-labeling of three types of graphs obtained by connecting, sequentially, any number of paths of equal size.

1 Introduction

A *difference vertex labeling* of a graph G of size n is an injective mapping f from $V(G)$ into a set N of nonnegative integers, such that every edge uv of G has assigned a *weight* defined by $|f(u) - f(v)|$. All labelings considered in this work are difference vertex labelings. A labeling is called *graceful* when $N = \{0, 1, \ldots, n\}$ and the induced weights are $1, 2, \ldots, n$. If G admits such a labeling, then it is called a *graceful graph*.

Let G be a bipartite graph where $\{A, B\}$ is the natural bipartition of $V(G)$, we refer to A and B as the *stable sets* of $V(G)$. A *bipartite labeling* of G is an injection $f : V(G) \to \{0, 1, \ldots, t\}$ for which there is an integer λ, named the *boundary value* of f, such that $f(u) \le \lambda < f(v)$ for every $(u, v) \in A \times B$, that induces n different weights. This is an extension of the definition given by Rosa and Širáň in [11]; there, they focused on bipartite labelings of trees. From the definition we may conclude that $t \ge |E(G)|$, furthermore, the labels assigned by f on the vertices of A and B are in the integer intervals $[0, \lambda]$ and $[\lambda+1, t]$, respectively. If $t = n$, the bipartite labeling f is an *α-labeling*. By an *α-graph* we mean a graph that admits an α-labeling. If G is an α-graph, λ is the smaller of the two vertex labels of the edge of weight 1. If G is an α-tree, then $\lambda = |A| - 1$.

Let $f : V(G) \to \{0, 1, ..., t\}$ be a labeling of a graph G of size n :

- $\overline{f} : V(G) \to \{0, 1, ..., t\}$, defined for every $v \in V(G)$ as $\overline{f}(v) = t - f(v)$, is the *complementary* labeling of f. If f is graceful, $\overline{f}$ is also graceful. Moreover, if f is an α-labeling with boundary value λ, then $\overline{f}$ is an α-labeling with boundary value $n - \lambda - 1$.

- $g : V(G) \to \{c, c+1, ..., c+t\}$, defined for every $v \in V(G)$ and $c \in \mathbb{Z}$ as $g(v) = c + f(v)$, is the *shifting* of f in c units. Note that this labeling preserves the weights induced by f.

- $h : V(G) \to \{0, \kappa, ..., t\kappa\}$, defined for every $v \in V(G)$ and $\kappa \in \mathbb{Z}^+$ as $h(v) = \kappa f(v)$, is the *amplification* of f in κ units. If w_1, w_2, ..., w_t are the weights induced by f, then the weights induced by h are κw_1, κw_2, ..., κw_t.

Suppose now that $f : V(G) \to \{0, 1, ..., t\}$ is a bipartite labeling with boundary value λ.

- $f_r : V(G) \to \{0, 1, ..., t\}$, defined for every $v \in V(G)$ as, $f_r(v) = \lambda - f(v)$ if $f(v) \leq \lambda$, and $f_r(v) = t + \lambda + 1 - f(v)$ if $f(v) > \lambda$, is the *reverse* labeling of f. Note that if f is an α-labeling, then f_r is also an α-labeling with boundary value λ.
- $f^d : V(G) \to \{0, 1, ..., t+d-1\}$, defined for every $v \in V(G)$ and $d \in \mathbb{Z}$ as, $f^d(v) = f(v)$ if $f(v) \leq \lambda$ and $f^d(v) = f(v)+d-1$ if $f(v) > \lambda$, is the *bipartite d-labeling* of G obtained from f. This labeling uses labels from $\{0, 1, ..., \lambda\} \cup \{\lambda + d, \lambda + d + 1, ..., t + d - 1\}$ and induces the weights $d, d + 1, ..., t + d - 1$. In other terms, this labeling shifts the weights induced by f in $d-1$ units. Thus, if f is an α-labeling of G and d is a positive constant, then f^d is the, well-known, d-graceful labeling of G.

Let f be an α-labeling of a tree G of size n with boundary value λ. Suppose that f is transformed into a d-graceful labeling shifted c units. Then the stable set A receives the labels $c, c+1, ..., c+\lambda$ and the stable set B receives the labels $c+\lambda+d, c+\lambda+d+1, ..., c+n+d-1$.

Several authors have studied graceful and α-labelings of graphs that are the Cartesian product of two graphs. Maheo [9] proved that if G is strongly graceful (a special type of graceful labeling with some specific restrictions), then $G \times K_2$ is strongly graceful. Fu and Wu [5] showed, in one way, a stronger result. They proved that if T is a tree with an α-labeling and the sizes of the two stable sets of T differ by at most one, then $T \times P_m$ has an α-labeling. In [13], Wu went even further, proving that if G has size n, order $n + 1$, and admits an α-labeling with boundary value λ, where $|n - 2\lambda - 1| \leq 1$, then $G \times P_m$ is graceful for all m. Jungreis and Read [8] proved, among other results, that $P_m \times P_n$ is graceful. This result is a special case of the main result of Fu and Wu [5]. El-Zanati and Vanden Eynden [3] used a different hypothesis to prove that $G \times P_m$ is graceful. They proved that if G has a strongly graceful labeling, then $G \times P_m$ has an α-labeling. Note that they used the name strong α-labeling instead of strongly graceful labeling.

In Section 2 we consider the family $\mathscr{R}$ of all trees of size n such that when $T \in \mathscr{R}$, the cardinality of the two stable sets of T differ by at most one. We prove that for any two α-trees, T_1 and T_2, in $\mathscr{R}$, the Cartesian product $T_1 \times T_2$ results in an α-graph.

Frucht and Harary [4] defined the *corona* of two graphs as the graph obtained by taking one copy of a graph G of order n, and n copies of a graph H, and then joining, by an edge, the kth vertex of G to every vertex in the kth copy of H. The corona of G and H is denoted by $G \odot H$. As a consequence of the results of Stanton and Zarnke [12], we know that if G is a graceful tree, then $G \odot rK_1$ is graceful for every $r \geq 1$; in other terms, when r pendant vertices are attached to every vertex of a graceful tree G, the resulting tree, $G \odot rK_1$ is also graceful. In Section 3 we study a related problem, presenting a tree of size 12 and stable sets with cardinalities 5 and 8. This tree has the property that we can attach any number of pendant vertices to the vertices of any nonempty subset of the stable set with five elements and the resulting tree can be α-labeled. The path P_n is the only graph, that we know, with the same property; that is, we can attach any number of vertices to any subset of vertices of P_n.

In Section 4, we give α-labelings for three families of trees obtained by connecting, sequentially, any number of copies of P_n with paths of length 2. We discuss here the connection between these trees and other types of combinatorial structures.

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The reader interested in graph labeling is referred to Gallian' survey [6] for more information about the subject. In this paper we follow the notation and terminology used in [2] and [6].

2 Cartesian Product of Regular α-Trees

Jungreis and Read [8] proved that the Cartesian product of the paths P_m and P_n is an α-graph. For any given path P_n, the difference of the cardinalities of its stable sets is at most one and P_n is an α-tree. These facts motivate the study of the Cartesian product of other trees satisfying these conditions. A tree of order n is said to be *regular* when the cardinalities of its stable sets are equal or differ by one. We claim that the Cartesian product of two regular α-trees is an α-graph.

Theorem 2.1. *If S and T are regular α-trees, then $S \times T$ is an α-graph.*

Proof. Suppose that S and T are regular α-trees of order m and n, respectively. Thus, $S \times T$ is a graph of order $\mu = mn$ and size $\xi = (m-1)n + (n-1)m$. Let S_1, S_2, ..., S_n be the n copies of S in $S \times T$. Assume that f is an α-labeling of S with boundary value λ such that the label λ is assigned to a vertex of the largest stable set of $V(S)$. Recall that $\overline{f}$ corresponds to the complementary labeling of f. For all odd values of i, the labeling f_i of S_i is obtained by transforming f into a $(1+(2m-1)(n-i))$-graceful labeling shifted $(2m-1)(i-1)/2$ units. When i is even, the labeling f_i of S_i is obtained by transforming $\overline{f}$ into a $(1+(2m-1)(n-i))$-graceful labeling shifted $m+(2m-1)(i-2)/2$ units. Hence, the weights obtained on the lth copy of S form the set $V_{n+1-l} = \{j + (2m-1)(l-1) : 1 \leq j \leq m-1 \text{ and } 1 \leq l \leq n\}$. The smallest label used is 0 and the largest one is $m-1+(2m-1)(n-1) = m(n-1)+n(m-1) = \xi$. In addition, the shiftings guarantee that every label is used exactly once.

Fu and Wu (see [5], Proposition 2.6) proved that if S is a regular α-tree of order m, then $S \times P_2$ is an α-graph. In their labeling of $S \times P_2$, the first copy of S has an α-labeling f transformed into a $2m$-graceful labeling; the second copy of S is labeled using $\overline{f}$ shifted m units. The edges connecting both copies of S have weights m, $m+1$, ..., $2m-1$. In our case, for every edge uv of T, we have a graph of the form $S \times P_2$.

Applying the result of Fu and Wu to our case, we have that for every edge uv of T, there is a subgraph $S \times P_2$ of $S \times T$, such that the weights of the edges, connecting the two copies of S, are consecutive integers. Hence, the weights of the edges connecting the copies of S_l and S_{l+1} in $S \times T$ form the set $H_{n-l} = \{(2m-1)l + j - m : 1 \leq j \leq m \text{ and } 1 \leq l \leq n-1\}$.

Therefore, for a fixed value of $l \in \{1, 2, \ldots, n-1\}$,

$$\begin{aligned}
&V_{n+1-l} \cup H_{n-l} = \\
&[(2m-1)(l-1)+1, (2m-1)(l-1)+m-1] \cup [(2m-1)(l-1)+1, (2m-1)l-m] = \\
&[(2m-1)(l-1)+1, (2m-1)l-m] \cup [(2m-1)l-m+1, (2m-1)l] = \\
&[(2m-1)(l-1)+1, (2m-1)l].
\end{aligned}$$

In addition,

$$\bigcup_{l=1}^{n-1} (V_{n+1-l} \cup H_{n-l}) = [1, (2m-1)(n-1)],$$

and

$$V_1 \cup [1, (2m-1)(n-1)] = [1, m(n-1) + n(m-1)] = [1, \xi].$$

Since any pair of "adjacent" copies of S are labeled using f and $\overline{f}$, the final labeling of $S \times T$ is an α-labeling with boundary value $m(n-1) - 1$. □

In Figure 1 we show an example of the α-labeling of $S \times T$ where S and T are caterpillars of order $m = 7$ and $m = 8$, respectively; the original α-labelings of the graphs are exhibited on the left and on the top of $S \times T$. In addition we show the labelings f_1 and f_2 of the first two copies of S.

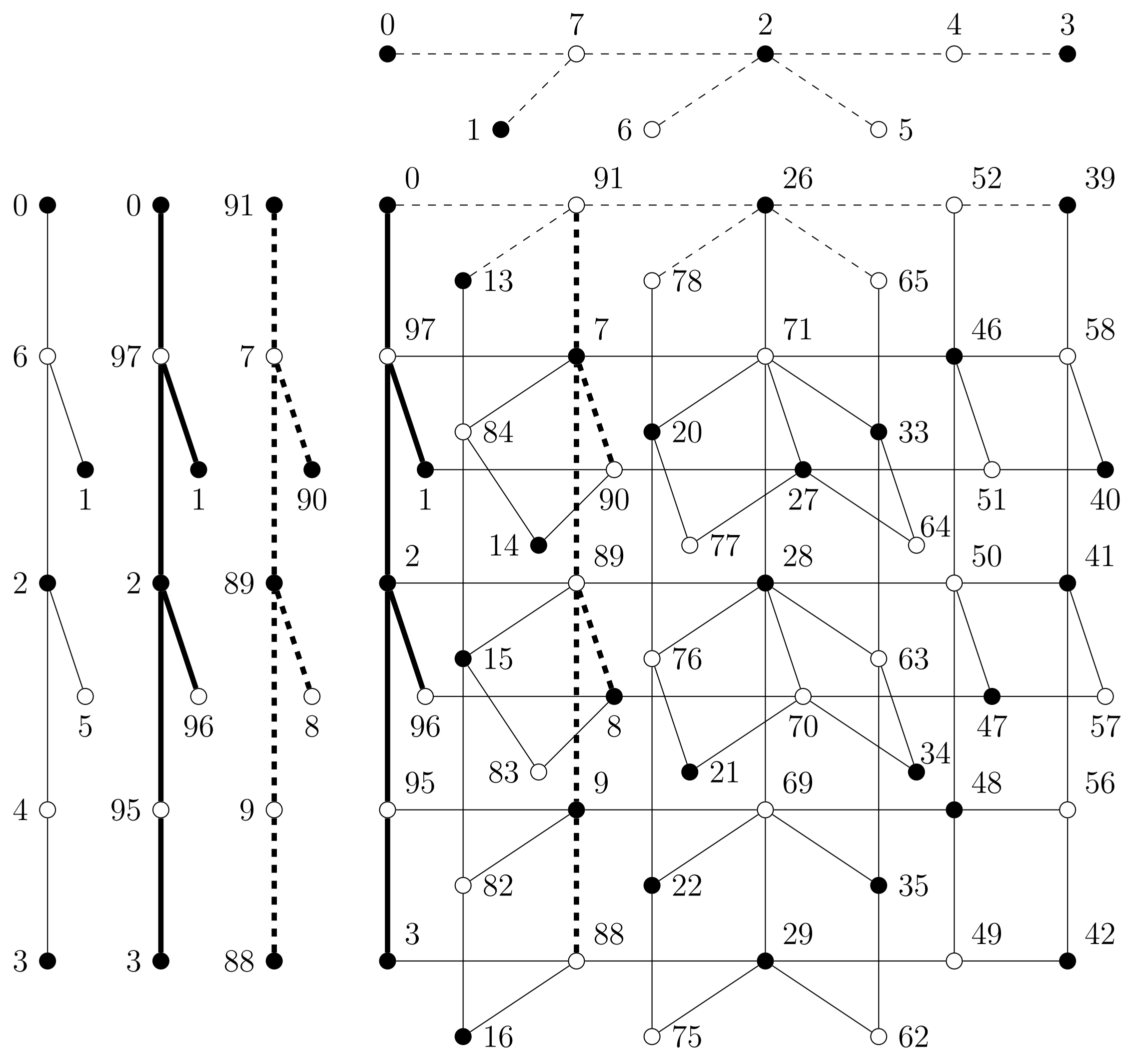

Figure 1: α-labeling of $S \times T$

A graph G of size n is *harmonious* if there exists an injection $f : V(G) \to \mathbb{Z}_n$ such that when each edge uv is assigned the *weight* $f(u) + f(v)$ (mod n), the resulting weights are distinct. When each edge uv is assigned the weight $f(u) + f(v)$, we say that f is *sequential* if the weights are consecutive integers. Thus, it is possible to transform a sequential labeling into a harmonious labeling by reducing the weights $f(u) + f(v)$ modulo n. Grace [7] has shown how to transform an α-labeling f of a graph G of size n with boundary value λ, into a labeling g such that the weights $g(u) + g(v)$ are $\lambda + 1$, $\lambda + 2$, ..., $\lambda + n$. The labeling g is defined for all the vertices in $V(G)$ by:

$$g(v) = \begin{cases} f(v) & \text{if } f(v) > \lambda, \\ \lambda - f(v) & \text{if } f(v) \leq \lambda. \end{cases}$$

Jungreis and Read [8] used Grace's result to prove that the Cartesian product of two paths is sequential when one of the paths has even order. In the case $P_{2m} \times P_{2n+1}$, they started with an α-labeling f of $P_{2m} \times P_{2n+1}$, which is transformed into the labeling h which is defined as:

$$h(v) = \begin{cases} f(v) - 2m & \text{if } f(v) > \lambda, \\ \lambda - f(v) & \text{if } f(v) \leq \lambda. \end{cases}$$

The case $P_{2m} \times P_{2n}$ is a little more complicated. Start with an α-labeling f of $P_{2m} \times P_{2n}$ and subtract 1 from every vertex label of the $2n$th copy of P_{2m}. The new labeling of $P_{2m} \times P_{2n}$, denoted by f^*, is also an α-labeling with boundary value λ^*. They followed the same steps, used in the previous case, replacing f by f^*.

Suppose that S is a regular tree of even order; following the same steps of Jungreis and Read, we can convert the α-labeling of $S \times T$ into a sequential labeling, which can be transformed into a harmonious labeling. Thus, as a consequence of Theorem 1, and the results in [7] and [8], we have the following two corollaries.

Corollary 2.2. *If S and T are regular α-trees and S has even order, then $S \times T$ admits a sequential labeling.*

Corollary 2.3. *If S and T are regular α-trees and S has even order, then $S \times T$ admits a harmonious labeling.*

3 α-Labelings of a Special Tetrapod

A *tetrapod* is a rooted tree whose root r has degree four and all the internal vertices have degree two, in other terms, the one-point union of two paths, where the amalgamated vertices are internal vertices. Within this class of trees, we distinguish the tetrapod obtained by amalgamating the central vertices of two copies of the path P_7. Let S denote this special tetrapod; suppose that A and B are its stable sets, where $|A| = 5$ and $|B| = 8$. A picture of S is shown in Figure 2. We claim that any tree obtained from S, by attaching any number of pendant vertices to the elements of A, is an α-tree. (Note that the number of vertices attached may vary from vertex to vertex in A.) This property is interesting to us, because it resembles the fact that caterpillars of diameter d can be obtained in the same form, by attaching pendant vertices to the internal vertices of P_{d+1}. We are not aware of any other tree with the same property. This raises the question about the existence of other trees with the same property.

Theorem 3.1. *Any tree T of size $n > 12$, obtained from S by attaching any number of pendant vertices to the elements of A, is an α-tree.*

Proof. Recall that the star with $m \geq 1$ leaves is the graph $S_m \cong K_{1,m}$. For practical reasons, we allow here $m \geq 0$, that is, an isolated vertex is considered a star with 0 leaves.

The tree T can be constructed by connecting a leaf, from each of four stars, to the central vertex of a new star. For each $0 \leq i \leq 4$, let S^i be the star of size n_i, where $n_i \geq 2$ except for $i = 2$, where $n_2 \geq 0$. Let f_i be the α-labeling of S^i that places the label 0 at the center of the star. The labeling f_i is transformed into a k-graceful labeling shifted i units, where $k = 1$ when $i = 4$ and $k = 5 - i + \sum_{j=i+1}^{4} n_j$, otherwise.

Suppose that $n = 4 + \sum_{j=0}^{4} n_j$, so n is the size of T. Consequently, the labels assigned to the vertices of $\bigcup_{i=0}^{4} S^i$ form the set $\{0, 1, ..., n\}$, and the induced weights form the set $\{1, 2, ..., n\} - \{n_4 + 1,\ n_4 + n_3 + 2,\ n_4 + n_3 + n_2 + 3,\ n_4 + n_3 + n_2 + n_1 + 4\}$. Since the centers of the stars S^i are labeled $0, 1, 2, 3,$ and 4, respectively, and S^4 has a leaf labeled $n_4 + 3$, S^3 has a leaf labeled $n_4 + n_3 + 4$, S^1 has a leaf labeled $n_4 + n_3 + n_2 + 5$, and S^0 has a leaf labeled $n_4 + n_3 + n_2 + n_1 + 6$, we can connect these leaves to the central vertex of S^2, labeled 2, to obtain four edges whose weights correspond to the missing weights mentioned before. Summarizing, the new edges connect four leaves to a central vertex, therefore the resulting labeling is a bipartite labeling of T. Moreover, the labels assigned to the vertices of T are $0, 1, \ldots, n$, the induced weights are $1, 2, \ldots, n$, and 4 is the largest label assigned to a central vertex of the starts S^i. Hence, we have obtained an α-labeling of T with boundary value $\lambda = 4$. □

In Figure 2 we show a possible tree T with the α-labeling described in the proof of Theorem 4. In this example, $n_0 = 5$, $n_1 = 7$, $n_2 = 4$, $n_3 = 4$, and $n_4 = 8$.

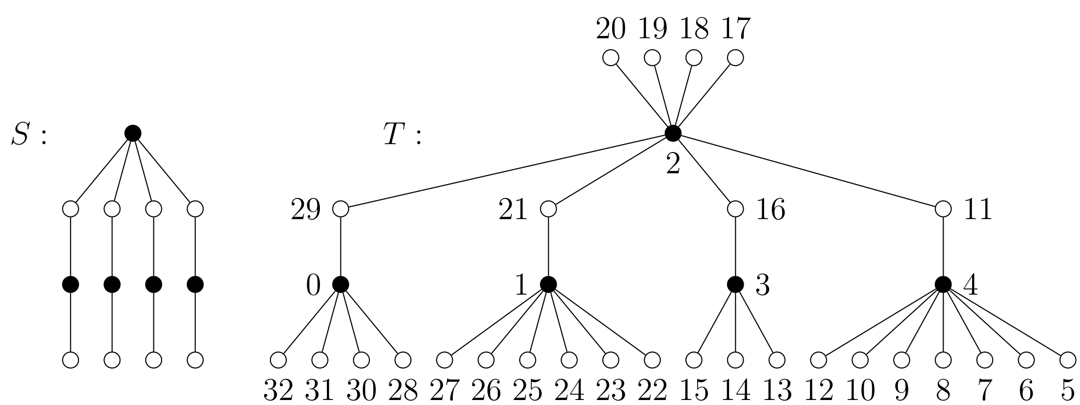

Figure 2: α-labeling of a tree T

4 α-Labelings of Linked Paths

4.1 α-Labelings of Some Fences

Suppose that for every $1 \leq i \leq t$, P_{n_i} is a path of order $n_i \geq 3$. For each $1 \leq i \leq t-1$, a *fence* F is a tree obtained by connecting, with a path of length l_i, an internal vertex of P_{n_i} with an internal vertex of $P_{n_{i+1}}$. The *length of the fence* is given by $l = \sum_{i=1}^{t-1} l_i$.

In [10], Rosa proved that for any n and any vertex v of P_{n+1}, there exists an α-labeling f of P_{n+1} such that $f(v) = 0$, if and only if, v is not the central vertex of P_5. If λ is the boundary value of f, then $f_r(v) = \lambda$. In [1], Barrientos proved that given two α-labeled graphs, G_1 and G_2, the vertex amalgamation of the vertices labeled 0 in G_1 and G_2, produces a new α-graph. Thus, this argument can be used to prove that all trees with up to four leaves are graceful; some special cases need *ad hoc* arguments. Extending the result in [1], it is possible to prove that any fence, constructed connecting, sequentially, the vertices labeled 0 of the P_{n_i}, is an α-tree. This fact motivates our study of graceful and α-labelings of other types of fences, that is, where the vertices labeled 0 are not the ones connected by a path of length l_i.

In this subsection, we study a type of fence where all the P_{n_i} are isomorphic to the path P_{2n+1}. We also show an α-labeling for a fence constructed using the path P_{2n}. Let P^1, P^2, ..., P^t be copies of the path P_{2n+1} and $V(P^i) = \{v_0^i, v_1^i, ..., v_{2n}^i\}$. We denote by $\mathscr{R}_{n,t}$ the family of all fences of length $2(t-1)$ obtained by connecting, with a path of length 2, the vertex $v_{q_i}^i$ of P^i to the vertex $v_{q_i}^{i+1}$ of P^{i+1}, where $1 \leq i \leq t-1$ and $q_i \in \{2, 4, ..., 2n-2\}$. Note that the family $\mathscr{R}_{n,t}$ is quite robust; for instance, when $n = 4$ and $t = 5$, there are 25 nonisomorphic fences in $\mathscr{R}_{4,5}$. There is a bijection between $\mathscr{R}_{4,t}$ and the set of equivalence classes of $(t-1)$-tuples of elements of $\mathbb{Z}_3^{t-1}$, where two $(t-1)$-tuples are equivalent if one can be obtained from the other by a sequence of operations R and C, where R denotes reversal and C denotes the 2's complements (i.e., $C(x) = 2 - x$). More information about the number of $(t-1)$-tuples with elements in $\mathbb{Z}_3$ can be found in the Online Encyclopedia of Integer Sequences, sequence $A001998$ [14]. It is mentioned there that $a(19) = 290,585,050$; that is, the number of elements in $\mathscr{R}_{4,20}$ is $290,585,050$.

Theorem 4.1. *If $F \in \mathscr{R}_{n,t}$, then F is an α-tree for all positive values of n and t.*

Proof. Let $C^1 = P^1$ and for every $2 \leq i \leq t$, C^i be the caterpillar obtained from P^i, by attaching a pendant vertex to the vertex $v_{q_i}^i$, where q_i is any number in $\{2, 4, ..., 2n-2\}$. We denote the new vertex by $u_{q_i}^i$.

Suppose that f is an α-labeling of C^i such that $f(v_0^i) = 0$ and f_r is its reverse α-labeling, so $f_r(v_0^i) = n$. For each $1 \leq i \leq t$, we assume that the initial labeling f_i of C^i is the labeling f when i is odd, and the labeling f_r when i is even. Thus, f_i is transformed into a k_i-graceful labeling g_i shifted d_i units, where $k_i = 2(n+1)(t-i)+1$ and $d_i = (n+1)(i-1)$.

In this form, we have assigned on the vertices of $\bigcup_{i=1}^{t} C^i$ the labels $0, 1, ..., 2nt + 2(t-1)$; the induced weights form the set $\{1, 2, ..., 2nt + 2(t-1)\} - \{2i(n+1) : 1 \leq i \leq t-1\}$.

Suppose that i is odd. Note that $f_i(v_{q_i}^i) = \frac{q_i}{2}$ and $g_i(v_{q_i}^i) = \frac{q_i}{2} + (n+1)(i-1)$. So, $f_{i+1}(v_{q_i}^{i+1}) = n - \frac{q_i}{2}$, and $g_{i+1}(v_{q_i}^{i+1}) = n - \frac{q_i}{2} + (n+1)i$. Thus, $f_{i+1}(u_{q_i}^{i+1}) = n + 1 + \frac{q_i}{2}$ and

$g_{i+1}(u^{i+1}_{q_i}) = (n+1)(2t-i-1) + \frac{q_i}{2}$. Hence, the edge $v^i_{q_i}u^{i+1}_{q_i}$ has weight $2(n+1)(t-i)$.

Suppose now that i is even. In this case, $f_i(v^i_{q_i}) = n - \frac{q_i}{2}$ and $g_i(v^i_{q_i}) = n - \frac{q_i}{2} + (n+1)(i-1)$. So, $f_{i+1}(v^{i+1}_{q_i}) = \frac{q_i}{2}$, and $g_{i+1}(v^{i+1}_{q_i}) = \frac{q_i}{2} + (n+1)i$. Thus, $f_{i+1}(u^{i+1}_{q_i}) = 2n+1-\frac{q_i}{2}$ and $g_{i+1}(u^{i+1}_{q_i}) = 2n+1+(n+1)(2t-i-2) - \frac{q_i}{2}$. Hence, the edge $v^i_{q_i}u^{i+1}_{q_i}$ has weight $2(n+1)(t-i)$.

Therefore, we have obtained all the weights of the form $2i(n+1)$ with $1 \leq i \leq t-1$. The use of f and f_r in alternated copies of C^i implies that the vertices connected belong to different stable sets of F; therefore, the final labeling of F is a bipartite labeling that uses the labels $0, 1, \ldots, 2(nt+t-1)$ to induce the weights $1, 2, \ldots, 2(nt+t-1)$; the boundary value of this labeling is $\lambda = t(n+1)-1$. □

In Figure 3 we show the α-labeling of a fence in $\mathscr{R}_{5,4}$, constructed using Theorem 4.1, with $q_1 = 4, q_2 = 6$, and $q_3 = 2$.

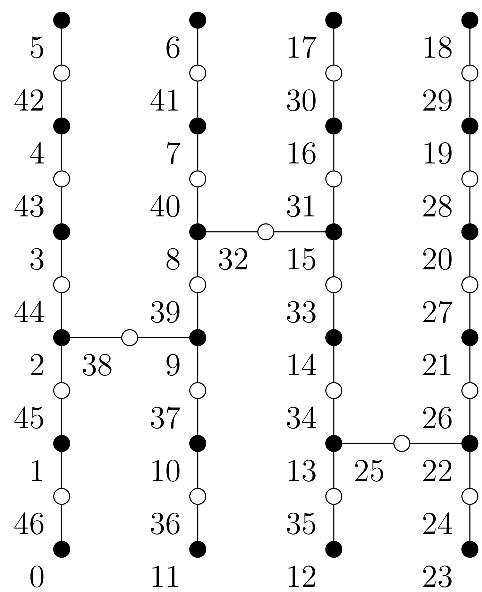

Figure 3: α-labeling of a fence $F \in \mathscr{R}_{5,4}$

Suppose now that $P^1, P^2, \ldots, P^t$ are copies of the path P_{2n}. Let $\mathscr{S}_{n,t}$ be the family of all fences of length $2(t-1)$ obtained by connecting the vertex v^i_{n-1} of P^i to the vertex v^{i+1}_{n-1} of P^{i+1}, for all $1 \leq i \leq t-1$. We claim that if $F \in \mathscr{S}_{n,t}$, then F is an α-tree.

Proposition 4.1. *If $F \in \mathscr{S}_{n,t}$, then F is an α-tree.*

Proof. Let $C^1 = P^1$ and for every $2 \leq i \leq t$, C^i be the caterpillar obtained by attaching a pendant vertex, denoted by u^i_{n-1}, to the vertex v^i_{n-1} of P^i.

Suppose that f_i is the α-labeling of C^i that assigns the label 0 to v^i_0. Thus, f_i is transformed into a k_i-graceful labeling shifted c_i units, where $k_i = (2n+1)(t-i)+1$ and $c_i = n(i-1)$. In this way, the labels used on C^1 form the set $\{0, 1, \ldots, n-1\} \cup \{t(2n+1) - (n+1), t(2n+1)-(n+1)+1, \ldots, t(2n+1)-2\}$; the weights induced on the edges of C^1 are $2n(t-i)+n-i+1, 2n(t-i)+n-i+2, \ldots, 2n(t-i)+t-2$. When $2 \leq i \leq t$, the labels used on the vertices of C^i form the set $\{n(i-1), n(i-1)+1, \ldots, ni-1\} \cup \{t(2n+$

$1) - (n+1)i, t(2n+1) - (n+1)i + 1, \dots, t(2n+1) - (n+1)(i-1) - 1\}$; the weights induced on the edges of C^i are $2n(t-i) + n - i + 1, 2n(t-i) + n - i + 2, \dots, (2n+1)(t-i) + 2n$.

Up to this point, we have that the vertices of F are labeled with the integers $0, 1, \dots, t(2n+1) + 2(t-1)$, and the weights induced on the edges of $\bigcup_{i=1}^{t} C^i$ form the set $\{1, 2, \dots, 2t(n+1) + t - 2\} - \{(2n+1)(t-j) : 1 \leq j \leq t-1\}$.

Note that for each $1 \leq i \leq t-1$, the vertex v^i_{n-1} has label $\frac{n-1}{2} + n(i-1)$ and the vertex u^{i+1}_{n-1} has label $t(2n+1) - (n+1)i - \frac{n+1}{2}$. So, the edge $v^i_{n-1}u^{i+1}_{n-1}$ has weight $(2n+1)(t-i)$.

Since the labelings of the C^i are α-labelings shifted conveniently and the edges $v^i_{n-1}u^{i+1}_{n-1}$ connect vertices in different stable sets, we have that the final labeling of F is an α-labeling with boundary value $\lambda = nt - 1$. $\square$

4.2 α-Labelings of 2-Link Fences

Another type of α-graph that can be constructed using α-labeled paths are the 2-link fences. Let $P^1, P^2, \dots, P^r$ be disjoint copies of P_n. For each $1 \leq i \leq r$, let $V(P^i) = \{v^i_1, v^i_2, \dots, v^i_n\}$. A *2-link fence* is a graph F of order rn and size $r(n+1)-2$ obtained by connecting the vertices $v^i_{j_i}$ and $v^i_{k_i}$ of P^i to the vertices $v^{i+1}_{j_i}$ and $v^{i+1}_{k_i}$ of P^{i+1}, respectively, for all $1 \leq i \leq r-1$, where $j_i, k_i \in \{1, 2, \dots, n\}$ and $|j_i - k_i|$ is odd. Let $\mathscr{F}_{r,n}$ be the family of all the 2-link fences constructed using r copies of P_n. Note that $\mathscr{F}_{r,n}$ is a robust family, for instance, when $r = 3$ and $n = 7$, we have counted 42 non-isomorphic elements in $\mathscr{F}_{3,7}$.

The main result of this subsection is associated to a property of an α-labeling of the path P_n. Suppose that $v_1, v_2, \dots, v_n$ are the consecutive vertices of P_n. Let $f : V(P_n) \to \{0, 1, \dots, n-1\}$ be the α-labeling of P_n, given by Rosa [10]; thus,

$$f(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ n - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

Then, the reverse of the complementary labeling is defined as:

$$\overline{f}_r(v_i) = \begin{cases} \frac{n-2+i}{2} & \text{if } i \text{ is odd,} \\ \frac{n-1-i}{2} & \text{if } i \text{ is even,} \end{cases}$$

when n is odd, and

$$\overline{f}_r(v_i) = \begin{cases} \frac{n-1+i}{2} & \text{if } i \text{ is odd,} \\ \frac{n-i}{2} & \text{if } i \text{ is even,} \end{cases}$$

when n is even.

We claim that all the elements of $\mathscr{F}_{r,n}$ are α-graphs. Before proving this result, we prove that all the members of $\mathscr{F}_{2,n}$ are α-graphs.

Lemma 4.2. *Let $n \geq 4$, if $G \in \mathscr{F}_{2,n}$, then G is an α-graph.*

Proof. Suppose that $G \in \mathscr{F}_{2,n}$, then there exist $j, k \in \{1, 2, \dots, n\}$ such that $|j - k|$ is odd and $j_1 = j, k_1 = k$. The vertices of P^1 are labeled using the α-labeling f, described above, the vertices of P^2 are labeled using $\overline{f}_r$. Once these labelings are in place, f is transformed into a $(n+2)$-graceful labeling and $\overline{f}_r$ is shifted $\lceil \frac{n}{2} \rceil$ units. In this way, when i is odd, $f(v^1_i) = \frac{i-1}{2}$ and $\overline{f}_r(v^2_i) = n + \frac{i-1}{2}$. Hence, the edge $v^1_i v^2_i$ has weight n. When i is even,

$f(v_i^1) = 2n + 1 - \frac{i}{2}$ and $\overline{f}_r(v_i^2) = n - \frac{i}{2}$, and the edge $v_i^1 v_i^2$ has weight $n + 1$. Since j and k have different parity, the edges $v_j^1 v_j^2$ and $v_k^1 v_k^2$ have weights n and $n + 1$.

Recall that the induced weights on P^1 are $n+2, n+3, \ldots, 2n$, and on P^2 are $1, 2, \ldots, n-1$. So, the weights of the edges of G are $1, 2, \ldots, 2n$. The use of f and $\overline{f}_r$ guarantees that the final labeling of G is an α-labeling with boundary value $\lambda = n - 1$. □

Theorem 4.3. *Let $n \geq 4$; if $G \in \mathscr{F}_{r,n}$, then G is an α-graph.*

Proof. Let $G \in \mathscr{F}_{r,n}$ and $P^1, P^2, \ldots, P^r$ be the disjoint copies of P_n used to construct G. We label the vertices of P^i using the labeling f when i is odd, and $\overline{f}_r$ when i is even. The labeling of P^i is transformed into a $((n+1)(r-i)+1)$-graceful labeling shifted $(n+1)(i-2)/2$ units except when both, n and i are even, where the labeling is shifted $n/2+(n+1)(i-2)/2$ units.

These shiftings of labels and weights guarantees that every label has been used exactly once and that the induced weights form the set $\{1, 2, \ldots, r(n+1) - 2\} - \{(n+1)(r-i), (n+1)(r-i) - 1 : 1 \leq i \leq r-1\}$.

Using the lemma, it is straight forward to prove that the edges connecting P^i and P^{i+1}, $1 \leq i \leq r-1$, have weights $(n+1)(r-i)$ and $(n+1)(r-i) - 1$. Since consecutive copies of P_n use the labelings f and $\overline{f}_r$, the resulting labeling of G is an α-labeling. □

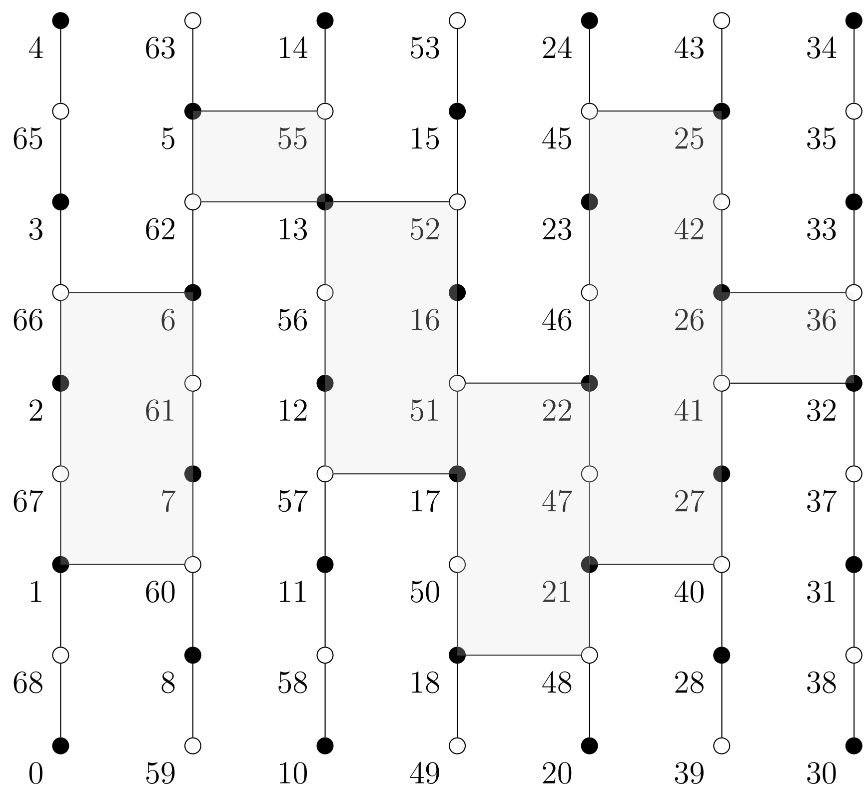

Figure 4: α-labeling of a 2-link fence in $\mathscr{F}_{7,9}$

Consider the 2-link fence F formed by using t copies of P_{2n}, where the vertices v_1^i and v_{2n}^i of P^i are connected to the vertices v_1^{i+1} and v_{2n}^{i+1}, respectively when $1 \leq i \leq t-1$. Since

v_1^i and v_{2n}^i are in different stable sets of $V(P^i)$, F is an α-graph. This 2-link fence can also be seen as a cycle $C_{2(t+2n-2)}$ with $t-2$ parallel P_{2n}-chords.

In fact, let $u_1, u_2, \ldots, u_{2(t+2n-2)}$ be the consecutive vertices of the cycle $C_{2(t+2n-2)}$, where $u_i = v_1^i$ for every $1 \leq i \leq t$, $u_{t+i} = v_{i+1}^t$ for every $1 \leq i \leq 2n-2$, $u_{t+2n-2+i} = v_{2n}^{t+1-i}$ for every $1 \leq i \leq t$, and $u_{2t+2n-2+i} = v_{2n-i}^1$ for every $1 \leq i \leq 2n-2$. The $t-2$ chords connect the vertices u_i and $u_{2t+2n-1-i}$, for every $2 \leq i \leq t-1$.

We define the *cycle with* $t-2$ *parallel* P_{2n}*-chords* as the graph obtained from the cycle $C_{2(t+2n-2)}$, $t \geq 2$ and $n \geq 1$, with consecutive vertices $u_1, u_2, \ldots, u_{2(t+2n-2)}$ by adding disjoint paths P_{2n}, between the vertices u_i and $u_{2t+2n-1-i}$, for all $2 \leq i \leq t-1$. Then, as a consequence of Theorem 4.3, we have the following corollary.

Corollary 4.4. *For all* $t \geq 2$ *and* $n \geq 1$*, the cycle with* $t-2$ *parallel* P_{2n}*-chords is an* α*-graph.*

Remark 1. *Note that the path* P_n*, used in Theorem 4.3, can be replaced by any* α*-tree of size* n *and the result still holds.*

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