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A LAGRANGIAN RELAXATION FOR CAPACITATED SINGLE ALLOCATION *P*-HUB MEDIAN PROBLEM WITH MULTIPLE CAPACITY LEVELS

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Abstract

In this paper we consider a capacitated single allocation *p*-hub median problem with multiple capacity levels (CSApHMPMC) in which the decisions are to determine the location of *p* hubs and their capacity levels, the single allocation of non-hub nodes to hubs in the logistics network. This problem is formulated as an integer programming model with the objective of minimizing the sum of total transportation cost and fixed cost of the selected *p* hubs with established capacity levels. A Lagrangian relaxation (LR) approach is proposed to solve the CSApHMPMC. The Lagrangian function that we formulated decomposed the original problem into smaller subproblems that can be solved easier. We only solve the CSApHMPMC using Gurobi optimizer for the small sized problems. The experimental results show that the proposed LR heuristic can be an effective solution method for the capacitated *p* hub median location problem with multiple capacity levels.

1 Introduction

Hub-and-spoke networks are widely used in a variety of industries such as airline, postal delivery, and telecommunication to efficiently route flows between many origins and destinations. The key feature lies in the use of collection of flows from the origin to the hub, transfer flows between hubs, and distribution of flows from the hub to destinations. Hub location problem (HLP) consists of locating hub facilities and of designing hub networks such that the sum of transportation cost and fixed facility cost is optimized.

The fundamental HLP has been extended in many features, such as node allocation type, hub capacity limitation, and the number of hubs is known or unknown priori. For node allocation, each non-hub node can be allocated either to one hub (single allocation) or to multiple hubs (multiple allocation). The hub capacity could be uncapacitated or capacitated. The latter one is close to the realistic condition. If the number of hubs is pre-determined to *p*, it is called *p*-hub location problem (*p*HLP). Klincewicz [12], Campbell et al. [4], Alumur and Kara [1], Campbell and O'Kelly [5] and Zanjirani Farahani et al. [15] provided a good survey on different hub location problems.

The majority of research has addressed uncapacitated HLPs. There are just a few articles in the literature dealing with capacitated version of HLPs. Campbell [3] produced the first mixed integer linear programming (LP) formulation for the capacitated single allocation HLP (CSAHLP). Aykin [2] studied the capacitated hub location problem with direct links, as well as multiple allocation, and included capacity limitations and fixed costs for hubs. Ernst and Krishnamoorthy [11] extended the Skorin-Kapov et al. [14] formulation to the capacitated case and also proposed a mixed integer programming formulation.

All of the above mentioned capacitated models considered that hub capacities are exogenous, i.e. capacity levels for candidate hub nodes are determined a priori. However, hubs are structural facilities that require several strategic decisions to be made in addition to the location decisions. Especially, the capacity that each hub should have can have a determining impact on locational and routing decisions. For instance, in the distribution center in distribution applications, the crucial capacity is number of docks to handle the inbound traffic. If the number of doors is not large enough, incoming traffic might be blocked and thus the distribution operations will be delayed. In recent years, some researchers have started studying more realistic capacitated models in which the amount of installed capacity is part of the decision process.

Correia et al. [7] studied an extension of capacitated HLPs with single assignment in which the hub capacity (CSAHLPM) is a decision variable. Three different mixed integer linear programming formulations were provided and theoretically compared. Several preprocessing tests of reducing the size of the models were presented for each particular instance. Elhedhli and Wu [9] introduced congestion cost in the objective of a capacitated model in which hub capacity is also a decision variable. Correia et al. [8] extended CSAHLPM and imposed the allocation balance on selected hub locations. Two mixed-integer linear programming formulations were provided and compared with the instances in Correia et al. [7].

Contreras et al. [6] presented models with multiple assignments in which the amount of capacity installed at the hubs is part of the decision process, for both splittable and non-splittable commodity cases. Zarei et al. [16] considered capacitated multiple allocation p-hub median problem with multi-level capacity. Rastani et al. [13] presented a mixed integer linear programming model for a hub network with capacity constraints on both hubs and inter hub links. The decision variables include capacity levels for both hub and transfer links between hubs. The computational results with modified Correia et al. [7] formulation were provided.

In this paper, we extend the capacitated single allocation *p*-hub median problem (CSA*p*HMP) to determine the capacity decision simultaneously. It is called the capacitated single allocation *p*-hub median problem with multiple capacity levels (CSA*p*HMPMC). It is assumed that there is a set of different sizes available for each candidate hub. We consider the case in which the capacity constraints refer to the incoming flows from non-hub nodes to that hub and flow originated in the selected hub.

To the best of our knowledge, the CSApHMPMC has not been studied in the literature. The problem will be formulated as a mixed integer programming problem that is solved using a Lagrangian relaxation approach. Four sets of benchmark instances from the literature will be tested for the proposed Lagrangian relaxation heuristic and compared with the solutions obtained by the optimization solver Gurobi.

The remainder of the paper is organized as follows. In the next section, the problem is described and the path-based mathematical formulation is proposed for the problem. In section 3, we develop a Lagrangian relaxation scheme of the formulation by relaxing the constraints that link the assignment variables with the path variables. The Lagrangian function decomposes the relaxed problem into two smaller subproblems what can be solved efficiently. Computational results on benchmark instances are provided in section 4. Conclusions and future research directions follow in section 5.

2 Problem Description

Consider a network of n demand nodes and p hubs must be located. Each non-hub node is allocated to a single hub. The flow between an origin/destination (OD) pair (i, j) must be routed through either one or at most two hubs k and l. The cost of transport a unit of flow along the path i-k-l-j is computed as C_{ij}^{kl} (k and l could be the same hub). The transportation cost for an OD (i, j) pair served via hubs k and l includes cost for collection from the origin i to hub k, transfer between hubs k and l, and distribution from hub l to the destination j. The rate for transfer cost between hubs is less than that for collection and distribution discount due to the economics of scale. Usually, the routing cost between two hub nodes is discounted at a rate of α to reflect the savings due to economies of scale. The hub capacities are allowed to take one of Q_k capacity levels at candidate hub node k with corresponding fixed costs f_k^q and the capacity size B_k^q, respectively.

The CSApHMPMC consist in determining the location and capacity size of the selected p hubs as well as the routing of each origin-destination pair through the hub nodes. The objective is to minimize the sum of fixed cost and transportation cost to establish the p hubs. It is also assumed that every pair of origin-destination will contain at least one and at most two hubs.

2.1 Notation

The following notation is used throughout the paper.

Parameters

B_k^q	:	the capacity for a hub at node k with capacity level q
C_{ij}^{kl}	:	the transportation cost of a unit of flow from node i to node j routed via hubs k and l, $C_{ij}^{kl} = \chi \times d_{ik} + \alpha \times d_{kl} + \delta \times d_{lj}$
d_{ij}	:	the distance between nodes i and j
f_k^q	:	fixed cost of locating at node k with capacity level q
N	:	set of nodes
p	:	number of required hubs
Q_k	:	set of available capacity levels of a candidate site k, $Q_k = \{1, ..., r_k\}$
w_{ij}	:	the flow between nodes i and j
α	:	the unit flow costs for transfer
χ	:	the unit flow costs for collection

δ : the unit flow costs for distribution

Decision variables

$$X_{ij}^{kl} = \begin{cases} 1 & \text{if the flow from node } i \text{ to } j \text{ routed via hubs } k \text{ and } l \\ 0 & \text{otherwise} \end{cases}$$

$$Z_{ik}^{q} = \begin{cases} 1 & \text{if node } i \text{ is allocated to hub } k \text{ which has capacity level } q \\ 0 & \text{otherwise} \end{cases}$$

It is assumed that the triangle inequality holds for the distance matrix. The fixed costs also include the operation cost for the hub which dependent on the selected capacity level. Clearly, a necessary condition for the feasibility of the problem is that the total selected capacity should be at least as large as the total flow from each origin.

2.2 Mathematical Programming Model

We extend the uncapacitated single assignment p-hub location formulation by Skorin-Kapov et al. [14] to our CSApHMPMC. The model proposed for the problem considers two sets of classical decision variables in single allocation p-hub median problems under path-based formulation as follows.

$$Min \quad \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{l\in N} w_{ij} C_{ij}^{kl} x_{ij}^{kl} + \sum_{k\in N}\sum_{q\in Q_k} f_k^q z_{kk}^q \tag{1}$$

$$S.T. \quad \sum_{k\in N}\sum_{q\in Q_k} z_{kk}^q = p \tag{2}$$

$$z_{ik}^q \le z_{kk}^q \qquad \forall i,k\in N, q\in Q_k \tag{3}$$

$$\sum_{k\in N}\sum_{l\in N} x_{ij}^{kl} = 1 \qquad \forall i,j\in N \tag{4}$$

$$\sum_{l\in N} x_{ij}^{kl} = \sum_{q\in Q_k} z_{ik}^q \qquad \forall i,j,k\in N \tag{5}$$

$$\sum_{k\in N} x_{ij}^{kl} = \sum_{q\in Q_l} z_{jl}^q \qquad \forall i,j,l\in N \tag{6}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij} z_{ik}^q \le B_k^q z_{kk}^q \qquad \forall k\in N, q\in Q_k \tag{7}$$

$$\sum_{q\in Q_k} z_{kk}^q \le 1 \qquad \forall k\in N \tag{8}$$

$$x_{ij}^{kl} \in \{0,1\} \qquad \forall i,j,k,l\in N \tag{9}$$

$$z_{ik}^q \in \{0,1\} \qquad \forall i,k\in N, q\in Q_k \tag{10}$$

The objective function (1) minimizes the total cost which includes the transportation cost and the cost for installing the hubs. Constraint (2) ensures that exactly p hubs are chosen. Constraint (3) assures that node i can be allocated to hub k only when k is selected as a hub. Constraint (4) states that every node is allocated to exactly one hub. Constraint (5) ensures that for every destination j, the total flow from origin i to destination j routed via paths using link i-k will be nonzero only if node i is allocated to hub k with one selected capacity level. Similarly, constraints (6) assures that for every origin i and every hub k, a flow through the path i-k-l-j is feasible only if j is allocated to hub l. Constraint (7) ensures that all the assigned demand to an opened facility must less than or equal to the selected capacity level. Constraint (8) states that every candidate hub k can only selected at most one capacity level. Constraints (9) and (10) are binary integrality constraints. It is noted that when $|Q_k| = 1$ for all candidate hubs, the CSApHMPMC reduces to the classical CSApHMP. The problem can be solved with typical CSApHMP approaches.

3 Solution Methodology

Since the CSApHMP is NP-hard, the studied CSApHMPMC is also NP-hard. Solving the problem requires considerable running time as the size of the instances increase. We propose a Lagrangian relaxation heuristic to solve the problem. In order to simplify the problem, we relax the constraints that link the location/assignment variables with the flow variables (Eqs. (5) and (6)). Dualizing Eqs. (5) and (6) with Lagrangian multiplier vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we obtain the following Lagrangian function $L(\boldsymbol{u}, \boldsymbol{v})$:

$$Min \quad L(u,v) = \sum_{k\in N}\sum_{q\in Q_k} f_k^q z_{kk}^q - \sum_{i\in N}\sum_{j\in N}\sum_{k\in N} u_{ijk} \sum_{q\in Q_k} z_{ik}^q - \sum_{i\in N}\sum_{j\in N}\sum_{l\in N} v_{ijl} \sum_{q\in Q_m} z_{jl}^q + \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{l\in N} (w_{ij}C_{ij}^{kl} + u_{ijk} + v_{ijl}) x_{ij}^{kl} \tag{11}$$

$$S.T \quad \sum_{k\in N}\sum_{q\in Q_k} z_{kk}^q = p \tag{2}$$

$$z_{ik}^q \le z_{kk}^q \qquad \forall i,k \in N, q \in Q_k \tag{3}$$

$$\sum_{k\in N}\sum_{l\in N} x_{ij}^{kl} = 1 \qquad \forall i,j \in N \tag{4}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij} z_{ik}^q \le B_k^q z_{kk}^q \qquad \forall k \in N, q \in Q_k \tag{7}$$

$$\sum_{q\in Q_k} z_{kk}^q \le 1 \qquad \forall k \in N \tag{8}$$

$$x_{ij}^{kl} \in \{0,1\} \qquad \forall i,j,k,l \in N \tag{9}$$

$$z_{ik}^q \in \{0,1\} \qquad \forall i,k \in N, q \in Q_k \tag{10}$$

Considering the independence between the two sets of variables, $\boldsymbol{z}$ and $\boldsymbol{x}$, CSApHMPMC is then decomposed into two sub-problems. The first is a semi-assignment problem in a bipartite graph. The optimal solution is found by searching the minimum cost path among hubs k and l for each origin-destination pair (i, j) and setting the corresponding decision variable to 1 and the rest to 0.

$$Min \quad L_x(u,v) = \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{m\in N} (w_{ij}C_{ij}^{kl} + u_{ijk} + v_{ijl}) x_{ij}^{kl} \tag{12}$$

$$S.T \quad \sum_{k\in N}\sum_{l\in N} x_{ij}^{kl} = 1 \qquad \forall i,j \in N \tag{4}$$

$$x_{ij}^{kl} \in \{0,1\} \qquad \forall i,j,k,l \in N \tag{9}$$

The second sub-problem separated into sets of smaller problems, one for each candidate hub. In an attempt to strengthen the lower bound given by Lagrangian relaxations, we add a valid inequality which might be redundant for the formulation. This inequality is to ensure that the selected capacity levels of the opened facilities should provide enough capacity for all demand as in eq. (14).

$$\text{Min} \quad L_z(u,v) = \sum_{k\in N}\sum_{q\in Q_k} f_k^q z_{kk}^q - \sum_{i\in N}\sum_{j\in N}\sum_{k\in N} u_{ijk}\sum_{q\in Q_k} z_{ik}^q - \sum_{i\in N}\sum_{j\in N}\sum_{l\in N} v_{ijl}\sum_{q\in Q_l} z_{jl}^q \tag{13}$$

$$= \sum_{k\in N}\sum_{q\in Q_k} f_k^q z_{kk}^q - \sum_{i\in N}\sum_{k\in N}\sum_{j\in N} (u_{ijk} + v_{jik})\sum_{q\in Q_k} z_{ik}^q$$

S.T

$$\sum_{k\in N}\sum_{q\in Q_k} z_{kk}^q = p \tag{2}$$

$$z_{ik}^q \le z_{kk}^q \qquad \forall i,k \in N, q\in Q_k \tag{3}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij}\sum_{q\in Q_k} z_{ik}^q \le \sum_{q\in Q_k} B_k^q z_{kk}^q \qquad \forall k\in N \tag{7}$$

$$\sum_{q\in Q_k} z_{kk}^q \le 1 \qquad \forall k\in N \tag{8}$$

$$\sum_{k\in N}\sum_{q\in Q_k} B_k^q z_{kk}^q \ge \sum_{i\in N}\sum_{j\in N} w_{ij} \tag{14}$$

$$z_{ik}^q \in \{0,1\} \qquad \forall i,k \in N, q\in Q_k \tag{10}$$

We apply subgradient optimization technique for solving the $L(u, v)$. For a given vector ($\boldsymbol{u}$, $\boldsymbol{v}$), the Lagrangian relaxation processes are as follows. The output of the algorithm is a best lower bound *LB* and *UB* denotes a best upper bound on the optimal value of the original problem. The best lower bound corresponds to the optimal multipliers to the dual Lagrangian problem. The step size parameter ω^m is halved if lower bound has not improved in a given number e of consecutive iterations.

Step1. $\boldsymbol{u}^0 = 0$, $\boldsymbol{v}^0 = 0$, $\omega^1 = 2$, $UB = \infty$, $LB = -\infty$, $m = 1$.

Step 2. Use optimization software Gurobi to solve $L_z(u,v)$ for selecting the p locations.

Step 3. Find the assignment for non-hub nodes based on the selected p locations found in step 2 as follow.

$$x_{ij}^{kl} = \begin{cases} 1 & \min\{w_{ij}C_{ij}^{kl} + u_{ijk} + v_{ijl} \mid k,l \in N\} \\ 0 & \text{otherwise} \end{cases} \qquad ,\forall i,j\in N$$

Step 4. Compute Lagrangian objective function (L^m), if $L^m > LB$ then $LB = L^m$.

Step 5. Set $z_{ik}^q = 0, \forall i,k (i\ne k), q = 0$.

Step 5.1. For each non-hub node i, find the nearest hub node ($z_{kk}^{\hat{q}} = 1$),

$$\min_k = \{d_{ik} \mid z_{kk}^{\hat{q}} = 1\}\ \forall i\in N.$$

Step 5.2 Set $z_{ik}^{\hat{q}} = 1$.

Step 6. If $z_{kk}^{\hat{q}} = 1$ and $\sum_{i\in N}\sum_{j\in N} w_{ij} z_{ik}^{\hat{q}} > B_k^{\hat{q}} z_{kk}^{\hat{q}}$ then

Step 6.1 Find q such that $B_k^{q-1} < \sum_{i\in N}\sum_{j\in N} w_{ij} z_{ik}^{\hat{q}} \le B_k^q$, $\hat{q} = q$.

Step 6.2 Set $z_{kk}^{\hat{q}} = 1$.

Step 7. Set $x_{ij}^{kl} = z_{ik}^{\hat{q}} z_{jl}^{\hat{q}}, \quad i, j, k, l \in N$

Step 8. Computing Eq. (1) objective function (Obj^m), if $Obj^m < UB$ then $UB = Obj^m$

Step 9. Compute step size t^m using Eq. (15)

Step 10. Update Lagrangian multipliers, ***u*** and ***v***, using Eqs. (16) and (17)

Step 11. $m = m+1$ repeat Steps 2-10 until the stopping criterion is met.

$$t^m = \frac{\omega^m (UB - L^m)}{\sum_{i\in N}\sum_{j\in N}\sum_{k\in N}(\sum_{l\in N} x_{ij}^{kl} - \sum_{q\in Q_k} z_{ik}^q)^2 + \sum_{i\in N}\sum_{j\in N}\sum_{l\in N}(\sum_{k\in N} x_{ij}^{kl} - \sum_{q\in Q_l} z_{jl}^q)^2} \tag{15}$$

$$u_{ijk}^{m+1} = u_{ijk}^m + t^m (\sum_{l\in N} x_{ij}^{kl} - \sum_{q\in Q_k} z_{ik}^q) \tag{16}$$

$$v_{ijl}^{m+1} = v_{ijl}^m + t^m (\sum_{k\in N} x_{ij}^{kl} - \sum_{q\in Q_l} z_{jl}^q) \tag{17}$$

For stopping criterion in Lagrangian relaxation heuristic, in our experiment we implement 3 stopping criterion, the algorithm terminates when one of the following condition is met.

1. The given maximum number of iterations $Iter_{\max}$ is reached.
2. The step size parameter ω^n is less than a threshold value.
3. The lower bound equals the best upper bound or is close to upper bound below a threshold value ($|UB - L^n| < \varepsilon$).

4 Computational Tests

For testing, we use the AP data set introduced by Ernst and Krishnamoorthy [10] to build the test instances for the CSApHMPMC. In the case of the CSApHMPMC there are capacity levels for each candidate hub. The data contains up to 200 nodes of the Austria Post locations and is available in the OR-Library (http://people.brunel.ac.uk/~mastjjb/jeb/info.html). The instances with 10, 20, 25, 40, 50, 75 and 100 nodes were considered. Ernst and Krishnamoorhy [10] also used a combination of two types of fixed cost, tight (T) and loose (L), and two types of capacities, tight (T) and loose (L) for each problem size. For every problem size the four instances correspond to one of the four possible combinations, LL, LT, TL, and TT. The number of hubs p in tested instances is between 2 and 5. For the tight capacity, the number of hubs must be larger than 3. We extend the data set by defining five capacity levels for all candidate hubs: 1, 2, 3, 4 and 5. Thus, 40 possible combinations exist using the number of nodes and the number of capacity levels. Totally, 102 instances are tested by the Lagrangian relaxation heuristic. The cost parameters are $\lambda = 3$, $\alpha = 0.75$, and $\delta = 2$.

In order to evaluate the proposed formulations, the general solver Gurobi 4.5.2 was used with the running time of two hours. No change was made in the default values of the solver parameters apart from the time limit which was set to 2 hours. The Lagrangian relaxation heuristic was coded in Microsoft Visual Studio 2010 C++. The tests were run on a PC with an Intel Core2 Duo 3.0GHz processor and 2.0 GB RAM under Windows 7 operation systems.

The number of capacity levels available for each candidate hub was set the same for all nodes. The capacity levels for each candidate location were defined by Correia et al. [7] as follow.

$$B_k^{r_k} = B_k \text{ and } B_k^q = 0.7 \times B_k^{q+1},\ q = 1,...,r_k - 1,\ k \in N \tag{18}$$

where B_k denotes the "tight" capacity for hub k in the corresponding AP instance. This means the largest capacity level was set equal to the tight capacity of the corresponding node in the AP data set instance. The fixed cost for each potential hub was defined as eq. (19). The fixed cost of a hub at its highest capacity level was set equal to the fixed cost of the same potential hub in the corresponding AP instance. Then, an increase defined by the factor ρ (= 1.1 in this paper) was assumed for the unitary capacity cost when the capacity level decreased. The value means a 10% increase is considered for the unitary capacity cost when the capacity level decreases.

$$f_k^{r_k} = f_k \text{ and } f_k^q = \rho \times B_k^q \times \frac{f_k^{q+1}}{B_k^{q+1}},\ q = 1,...,r_k - 1,\ k \in N \tag{19}$$

After a preliminary test, we set the following parameter values: e =4, $\omega^1 = 2$, $\varepsilon = 0.001$, and $Iter_{max} = 1000$. In this paper, we report two different capacity levels: $r_k = 1$ and $r_k = 5$. It is noted that the former value corresponds to the classical CSApHMP. For the small size instances, we compare the results with Gurobi solutions, while the larger instances we compare with the solutions of CSPHLP for $r_k = 1$.

Table 1 shows the results of small size instances of $r_k = 1$. In the table, the first column gives the instance name based on the fixed cost type, capacity type, number of nodes n and number of medians p. Each instance name has a suffix composed of two letters in the set {T, L}. For example, LL 10-2 represents the loose fixed cost and loose capacity type instance with 10 nodes and 2 medians. The next two columns are the optimal solution and the CPU time in seconds provided by Gurobi. The next two columns are the percentage gap with respect to the optimal solution and the CPU time provided by LR heuristic. The column headings for the next five columns have the same meanings as the previous five columns. The percentage gap in the table is computed as eq. (20).

$$gap = \frac{\text{LR solution - Opt.}}{\text{Opt.}} \times 100\% \tag{20}$$

In Table 1, we can observe that our LR can obtain the optimal solutions for all loose fixed cost instances except 4 instances. The average gap of these instances 0.42%. The computational time of LR decreases when the number of medians increases for given number of nodes. However, the average computational time is longer than that of Gurobi. The possible reason might be that we solve the location decision subproblem (**z** variable) by Gurobi instead of a simple heuristic. Solve the location subproblem with other heuristic could be one of the future research directions.

The results for tight fixed cost type instances are not as good as those for loose fixed cost type instances. The LR cannot obtain optimal solutions in six instances. However, the computational time by LR is much smaller than that by Gurobi. Another observation is that the tight capacity type instance is more difficult to solve than the loose capacity type ones as mentioned in Ernst and Krishnamoorhy [10]. The average gap for tight capacity type instances is larger than that of loose capacity type instances. The computational time for tight capacity type instances by Gurobi is much larger than that for loose capacity type ones.

Table 2 provides the results for the medium and large size instances of r_k = 1. These instances cannot be solved by the Gurobi due to the two-hour time limit or out of memory. We compare the results with the solutions provided by Ernst and Krishnamoorhy [10] for the capacitated hub location problem. Our LR can obtain the optimal solutions for most of the loose capacity instances with shorter CPU time. The average gap for loose capacity type instances is smaller than those for tight capacity type instances.

Table 1. The results for the small size instances with loose capacity of r_k = 1

Instance	Gurobi		LR		*Instance*				
	Opt.	time	gap	Time		Opt.	time	gap	Time
LL 10-2	230008.5	1.14	0.00	25.28	TL 10-2	264544.0	1.08	0.00	12.59
LL 10-3	224250.1	1.21	0.00	2.32	TL 10-3	263399.9	1.22	0.00	2.44
LL 10-4	229172.6	1.32	0.00	2.31	TL 10-4	269074.3	1.06	0.00	0.72
LL 10-5	239292.3	1.21	0.00	1.28	TL 10-5	281327.3	1.09	0.00	0.30
LL 20-2	234691.0	18.13	0.00	79.23	TL 20-2	271128.2	17.19	0.00	54.09
LL 20-3	239444.2	20.56	0.00	35.54	TL 20-3	281304.8	18.61	0.43	29.51
LL 20-4	251939.7	20.91	0.00	38.80	TL 20-4	295223.9	18.86	0.00	22.98
LL 20-5	266745.2	19.00	0.00	15.18	TL 20-5	310122.8	16.85	0.00	3.50
LL 25-2	238978.0	87.19	0.00	190.37	TL 25-2	310317.6	61.15	0.00	113.42
LL 25-3	242437.2	88.64	0.55	132.29	TL 25-3	328092.6	215.45	0.00	90.56
LL 25-4	252716.6	123.20	0.00	64.91	TL 25-4	347416.8	122.83	0.00	74.59
LL 25-5	263518.3	59.97	2.87	74.68	TL 25-5	368288.6	46.82	0.07	31.63
LT 10-2	256048.6	1.17	0.00	46.87	TT 10-2	264544.0	1.01	0.00	2.22
LT 10-3	252973.6	1.05	0.00	137.65	TT 10-3	2633 99.9	1.12	0.00	2.25
LT 10-4	250992.3	1.03	0.00	22.91	TT 10-4	269074.3	1.19	0.00	0.48
LT 10-5	261451.2	1.01	0.00	38.59	TT 10-5	281327.3	1.06	0.00	0.28
LT 20-2	253517.4	639.65	2.99	316.97	TT 20-2	329068.6	1849.93	9.80	54.97
LT 20-3	257247.7	2079.15	3.18	18.39	TT 20-3	296035.4	24.96	0.00	70.67
LT 20-4	260678.9	269.24	0.00	280.05	TT 20-4	306296.3	14.88	0.00	9.25
LT 20-5	274975.4	291.50	0.00	217.75	TT 20-5	325568.5	14.57	0.58	10.97
LT 25-3	276372.5	751.32	0.00	1494.30	TT 25-3	348369.1	442.95	1.51	99.59
LT 25-4	278235.0	271.99	0.00	1862.89	TT 25-4	369576.8	1364.41	0.00	67.05
LT 25-5	284952.6	122.83	0.00	377.67	TT 25-5	391996.5	357.94	1.16	38.47
Average		211.84	0.42	238.10			199.84	0.59	34.46

Note that the LR is able to obtain optimal solutions in most of the loose capacity type (LL and TL) instances. However, the performance of our LR heuristic does not provide good solutions in tight capacity type instances. These results indicate that the type of configuration for the fixed costs and capacities affect the performance of the LR heuristic.

Table 2. The results for medium and large size instances of r_k = 1

Instance	E&K	LR		*Instance*	E&K	LR	
	BKS	gap	Time		BKS	gap	Time
LL 40-2	241955.71	0.00	18.97	TL 40-2	298919.01	0.00	231.11
LL 50-2	238520.59	0.00	564.40	TL 50-2	319015.77	0.00	381.79
LL 75-2	238024.22	0.27	750.33	TL 75-2	303363.55	0.00	687.53
LL 100-3	246713.97	2.78	2470.97	TL 100-3	362950.09	9.11	2430.92
LT 40-3	272218.32[a]	5.36	289.24	TT 40-3	354874.10	4.66	189.44
LT 50-4	272897.49	6.30	49.98	TT 50-4	417440.99	10.11	57.30
LT 75-3	256188.12	1.38	650.61	TT 75-4	347189.81	0.75	629.81
LT 100-3	256638.38	9.07	2523.17	TT 100-4	474680.32	6.21	2947.70
Average		3.15	914.71			3.86	944.45

To our knowledge, this paper is the first one to study the capacitated single allocation p-hub median problem with multiple capacity levels. There are no results available to compare with our results with multiple capacity levels. We only report the small size instances in this paper for r_k = 5. The results of the comparison between the Gurobi and LR heuristic for small size instances with loose and tight capacity type of r_k = 5 are provided in Table 3. The headings of the column are the same as those in table 1.

The results presented in table 3 show that similar results can be obtained on instances with r_k =5. Observe that the LR cannot solve more instances for r_k =5 than that of r_k =1. The average gaps for loose fixed cost type and tight fixed cost type instances are 0.42% and 1.04%, while the computational times are 98.97 seconds and 126.47 seconds, respectively. The computational time by LR is smaller than that by Gurobi. The tight capacity type instances are more difficult to solve due to more decision variables are considered. There are more instances that cannot be solved by LR in tight capacity type instances that loose capacity type ones.

5 Conclusion

In this paper, we extend the capacitated single allocation p-hub median problem to capacitated single allocation *p*-hub median problem with multiple capacity levels (CSA*p*HMPMC). The locations and their capacity levels are not determined priori, and the amount of flow collected in the hub is limited. Since this problem is NP-hard, we propose a Lagrangian relaxation heuristic to solve the problem. The LR heuristic decomposes the problem into two smaller subproblems that can be solved efficiently. Four sets of benchmark instances from AP hub data set with five capacity levels are tested for our Lagrangian relaxation heuristic. The results are also compared with the Gurobi optimizer.

The LR heuristic can obtain most of the optimal solutions in small size instances but cannot provide similar results for large size instances. The present study shows that the LR heuristic is an effective method to solve CSApHMPMC, but the computational time could be reduced by applying simple heuristic instead of the optimization solve for the location subproblem. In the future, we could further develop simple heuristic on determining the feasible hub median locations and also add local search to the solution found by LR to obtain better solution. Other research direction might to develop metaheuristic algorithms, such as GRASP or ACO, to solve the problem.

Table 3. The results for the small size instances with loose capacity of $r_k = 5$

Instance	Gurobi		LR		*Instance*	Gurobi		LR	
	Opt.	time	gap	Time		Opt.	time	gap	Time
LL 10-2	202892.36	2.84	0.00	6.42	TL 10-2	234234.24	13.20	0.00	14.97
LL 10-3	177657.01	2.76	0.00	12.87	TL 10-3	213146.64	6.86	0.00	42.50
LL 10-4	160975.34	2.69	0.00	41.94	TL 10-4	194630.17	11.59	0.00	41.81
LL 10-5	144001.59	1.28	0.00	4.89	TL 10-5	179449.64	1.42	0.00	5.10
LL 20-2	215263.24	152.64	0.00	33.79	TL 20-2	243691.58	275.53	0.00	35.03
LL 20-3	202177.31	402.55	0.00	167.13	TL 20-3	231017.92	1547.12	0.00	148.42
LL 20-4	196803.30	983.70	0.00	130.52	TL 20-4	220852.24	355.57	0.00	104.29
LL 20-5	190808.20	1861.44	0.20	74.42	TL 20-5	217086.21	66.62	0.00	44.39
LL 25-2	232315.89	3520.85	0.00	437.70	TL 25-2	280261.05	778.73	1.98	535.04
LL 25-3	210233.16	1864.67	0.67	254.84	TL 25-3	263045.49	473.02	2.02	400.84
LL 25-4	195607.23	85.89	3.54	225.30	TL 25-4	261269.52	4874.66	0.00	354.39
LL 25-5	192471.63	101.67	0.00	173.49	TL 25-5	260850.78	2122.74	0.00	252.86
LT 10-2	250459.25	3.29	0.00	1.99	TT 10-2	259071.83	3.44	0.00	2.88
LT 10-3	226992.74	8.78	0.00	17.93	TT 10-3	236063.72	6.81	0.00	14.42
LT 10-4	202365.75	4.11	0.00	1.24	TT 10-4	214508.00	5.45	1.30	10.34
LT 10-5	194710.39	19.06	0.00	1.72	TT 10-5	209460.47	16.24	0.05	5.91
LT 20-2	253517.40	45.25	2.99	46.73	TT 20-2	329068.57	645.64	9.80	55.44
LT 20-3	240502.81	1293.07	0.00	78.91	TT 20-3	281852.11	285.94	2.02	109.17
LT 20-4	224068.68	263.18	0.00	51.11	TT 20-4	278674.41	1542.03	0.58	67.20
LT 20-5	219489.74	960.00	0.00	78.42	TT 20-5	271849.26	995.04	1.08	102.86
LT 25-3	270793.41	1541.18	1.00	120.52	TT 25-3	334695.31	1020.55	1.57	220.39
LT 25-4	263599.35	1052.89	0.60	200.57	TT 25-4	331964.31	1668.61	2.24	148.49
LT 25-5	256303.85	906.88	0.65	113.90	TT 25-5	331515.10	316.74	1.19	192.10
Average		655.68	0.42	98.97			740.59	1.04	126.47

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