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ON SOME ORDER 6 NON-SYMPLECTIC AUTOMORPHISMS OF ELLIPTIC K3 SURFACES.

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Abstract. We classify non-symplectic automorphisms of order 6 on elliptic K3 surfaces which commute with a given elliptic fibration. We show how their study can be reduced to the study of non-symplectic automorphisms of order 3 and to a local analysis of the fibers. In particular, we determine the possible fixed loci and give their location on the singular fibers. When the Picard lattice is fixed, we show that K3 surfaces come in mirror pairs.

1. Introduction

An automorphism of a K3 surface is called non-symplectic when the induced action on the holomorphic 2-form is non-trivial. The study of non-symplectic automorphisms was pioneered by Nikulin [Nik81] who analyzed the case of involutions. Since then, these automorphisms have been extensively studied by several authors. Let us mention Vorontsov [Vor83], Kondo [Kon86, Kon92], Xiao Gang [Xia96], Machida and Oguiso [MO98], Oguiso and Zhang [OZ98, OZ00], Zhang [Zha07], Artebani and Sarti [AS08], and Artebani, Sarti and Taki [AST11]. From these works, we now know that if a K3 surface admits a non-symplectic automorphism, then the surface is algebraic and the Euler totient function evaluated at the order of the automorphism is at most 66. Moreover, non-symplectic automorphisms of prime order have been classified, a synthetic classification can be found in [AST11], and some authors have started to investigate the simultaneous existence of symplectic and non-symplectic automorphisms [Fra11]. One of the reasons behind the interest in non-symplectic involutions is the mirror

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construction of Borcea \cite{Bor97} and Voisin \cite{Voi93}. They construct Calabi-Yau and an explicit mirror map using, as building blocks K3 surfaces with non-symplectic involutions, and elliptic curves. This construction can be extended to K3 surfaces with non-symplectic automorphisms of order 3, 4 and 6 \cite{Dil06}.

In this paper, using the classification of non-symplectic automorphisms of order 3 \cite{AS08}, we study order 6 automorphisms of elliptic K3 surfaces commuting with the fibration by performing a combinatorial analysis of the action on the fixed locus.

2. Plan

In Section \ref{sec:notation} we define primitive non-symplectic automorphisms and fix the notation for the rest of the paper. In Section \ref{sec:classification} we give the final classification of all possible fixed loci. In Section \ref{sec:fixed_loci} we show that the fixed locus of a primitive non-symplectic automorphism of order 6 consists in a disjoint union of points, rational curves, and possibly one genus one curve. Our cases fall thus in two distinct situations which are analyzed in Sections \ref{sec:isolation} and \ref{sec:smooth_curves}. In the last Section, \ref{sec:special_case}, we focus on the special case where $\zeta$ fixes the Picard lattice.

3. Notation

Let $X$ be a smooth projective K3 surface and $\zeta$ an automorphism of $X$. The induced action of $\zeta$ on $H(X, \Omega^2) \cong \mathbb{C}$ gives rise to a character $\chi$. An automorphism is called symplectic, if it lies in the kernel of $\chi$, and non-symplectic otherwise. If the order of $\zeta$ and $\chi(\zeta)$ agree, then $\zeta$ is called primitive. In the rest of the article, $\zeta$ will be a primitive non-symplectic automorphism of order 6 acting on $X$.

As suggested by Cartan \cite{Car57}, given a fixed point $P$ of $\zeta$, we can linearize the action around it. Since $\zeta$ is of order 6 and primitive, the linearized action can be written as

$$\begin{pmatrix} \xi_6^k & 0 \\ 0 & \xi_6^{k'} \end{pmatrix}$$

where $(k, k') \in \{(0, 1); (2, 5); (3, 4)\}$ and $\xi_6$ is a primitive 6\textsuperscript{th} root of unity. While the first case corresponds to $P$ lying on a fixed smooth curve, the last two options correspond to $P$ being isolated. We will use the standard notation and say that $P$ is of type $\frac{1}{6}(k, k')$. Since $\zeta$ is primitive, its iterates will also be non-symplectic. We will denote their fixed locus by $X[i] = \{x \in X \text{ s.t. } \zeta^i x = x\}$. The components of the $X[i]$ will be described by the following variables:

- $p_{\pm}(k, k')$: number of isolated fixed points of type $\frac{1}{n}(k, k')$ in $X[\frac{n}{6}]$, for $n \in \{6, 3, 2\}$.
- $l[i]$: number of rational curves in $X[i]$.
- $g[i]$: maximal genus among the curves in $X[i]$.
- $g_M = \max\{1, g[1]\}$.

When referring to \cite{AS08}, we will use their notation, namely:

- $n$: number of fixed points in $X[2]$ (all are of type $\frac{1}{3}(2, 2)$).
4. Results

Our first result is a global description of the fixed locus of ζ.

**Theorem 4.1.** The fixed locus $X^{[1]}$ consists of one of the two following collections:

1. a smooth genus 1 curve and three isolated fixed points of type $\frac{1}{2}(2, 5)$.
2. a disjoint union of smooth rational curves and points, $C_1 \sqcup \ldots \sqcup C_l \sqcup P_1 \sqcup \ldots \sqcup P_{p_1(3, 4) + p_1(2, 5)}$, satisfying

\[
p_{\frac{1}{2}(3, 4)} + 2p_{\frac{1}{2}(2, 5)} - 6l^{[1]} = 6.
\]

(4.1)

The proof of this Theorem follows from Section 5.

From our analysis in Sections 7 and 6 we obtain the following:

**Classification 4.2.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface and ζ a primitive non-symplectic automorphism of order 6 preserving the elliptic fibration. The fixed locus of ζ is one of the configurations listed in Table 1 or consists of the disjoint union of a genus 1 curve and three isolated points.

Reading Table 1. First note that since ζ preserves the fibration, we have an induced action, $\psi$, on the basis. The order of $\psi$ is either one or two.

In the Table, each row begins by a description of $X^{[2]}$, the fixed locus of $\zeta^2$. After that comes a list of the singular fibers of the fibration ($x$ is the number of fibers of type $X$). Then, we give a description of $X^{[1]}$ when $\psi$ is the identity. Finally, the last two groups refer to the case where $\psi$ is an involution; we list the fibers above the two fixed points, and the components of $X^{[1]}$.

5. Study of the fixed locus

**Lemma 5.1.** The fixed locus $X^{[1]}$ consists of a disjoint union of smooth curves and points

$X^{[1]} = C_0 \sqcup \ldots \sqcup C_m \sqcup P_1 \sqcup \ldots \sqcup P_{p_1(3, 4) + p_1(2, 5)}$

with $g(C_0) \geq 0 = g(C_1) = \ldots = g(C_m)$.

Proof. The first part of the statement follows from the Hodge Index Theorem. The argument is analogue to those found in [Nik81, Voi93, Dil06, AS08].

A disjoint union of smooth curves on a K3 surface can have at most one element with strictly positive self-intersection. By adjunction, that is a curve of genus at least 2.

If a curve has self-intersection 0, then it is an elliptic curve and induces an elliptic fibration $\pi : X \to \mathbb{P}^1$. Since the action is non-symplectic, it descends non-trivially to the base and fixes two points. The fixed locus of ζ is thus a component of the fibers above these two points. One of the fibers is the original fixed curve. The remaining curves of the fixed locus are either a smooth elliptic curve or a disjoint union of rational components of one of Kodaira’s singular fibers. So either the fixed locus is as the one described in the statement, or it consists exactly in the disjoint union of two genus 1 curves. However, if $X^{[1]}$ were to contain two genus 1 curves, then so would $X^{[2]}$ and this option was ruled out in [AS08].

□

**Lemma 5.2.** The components of $X^{[1]}$ satisfy

\[
p_{\frac{1}{2}(3, 4)} + 2p_{\frac{1}{2}(2, 5)} - 6l^{[1]} + 6g_M = 12.
\]

(5.1)
### Table 1: Fixed locus when $X$ is elliptic

<table>
<thead>
<tr>
<th>$g$</th>
<th>$n$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>36</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>48</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>72</td>
<td>1</td>
<td>12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>84</td>
<td>1</td>
<td>14</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>96</td>
<td>1</td>
<td>16</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>108</td>
<td>1</td>
<td>18</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>120</td>
<td>1</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>132</td>
<td>1</td>
<td>22</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>144</td>
<td>1</td>
<td>24</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>156</td>
<td>1</td>
<td>26</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>168</td>
<td>1</td>
<td>28</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: The table lists the fixed locus for $X$ being elliptic, with $g$, $n$, and $k$ as the parameters and $\lambda_1$ and $\lambda_2$ as the invariant quantities.
Proof. The formula is simply the Lefschetz holomorphic formula, see [TT75], applied to $\zeta$.

We will use the classification, determined by Artebani and Sarti [AS08], of non-symplectic automorphism of order 3 to find information on $X[2]$, which in turn will yield us data on the nature of $X[1]$. We first recapitulate the results of [AS08], which we will use later, and then relate the fixed loci $X[1]$ and $X[2]$.

**Theorem 5.3.** [AS08 Proposition 4.2] Let $\sigma$ be a non-symplectic automorphism of order 3 acting on a K3 surface $X$. If the fixed locus of $\sigma$ contains two or more curves, then $X$ is isomorphic to an elliptic K3 surface whose Weierstrass equation is

$$y^2 = x^3 + p_{12}(t)$$

and on which $\sigma$ acts as $(x, y, t) \mapsto (\zeta^2 x, y, t)$.

**Proposition 5.4.** [AS08 Corollary 4.3] Let $\sigma$ and $X$ be as in the statement of Theorem 5.3. If $X[1]$ contains a curve $C$ of positive genus, then $C$ is a double section of the Weierstrass fibration, i.e. $C$ is hyperelliptic.

**Lemma 5.5.** If $P \in X[1]$ is of type $16(2, 5)$, then it is also an isolated point in $X[2]$. If $P \in X[1]$ is of type $16(3, 4)$, then it lies on a smooth curve in $X[2]$. Moreover, one has the following inequalities $p_{1/2}(2, 2) \geq p_{1/3}(2, 5)$ and $l[2] \geq l[1]$.

**Proof.** The first two statements are obvious after one takes the square of the matrix giving the localized action of $\zeta$ at $P$. The inequalities ensue.

**Corollary 5.6.** If $X[1]$ contains at least two distinct curves or a curve and an isolated point of type $16(3, 4)$, or more generally, if $X[2]$ contains at least two distinct curves, then $X$ is isomorphic to an elliptic K3 surface whose Weierstrass equation is

$$y^2 = x^3 + p_{12}(t).$$

**Proof.** Lemma 5.5 implies that $X[2]$ contains at least two distinct curves. The first part of the statement follows thus directly from Theorem 5.3.

In the rest of the paper, we will focus on elliptic K3 surfaces. The following two Lemmas show that this is a somewhat mild restriction.

**Lemma 5.7.** Let $X$ be a K3 surface and $\tau$ a non-symplectic automorphism of order 3 which preserves the fibration. Then there exists a primitive non-symplectic automorphism $\zeta$ of $X$ such that $\tau = \zeta^2$ and $X$ is $\zeta$-elliptic.

**Proof.** We know from 5.3 that $X$ is of the form $y^2 = x^3 + p_{12}(t)$ and $\tau$ acts as $(x, y, t) \mapsto (\xi x, y, t)$. It is easy to see that $\sigma : (x, y, t) \mapsto (\xi x, -y, t)$ acts on $X$ and has the required properties.

**Lemma 5.8.** If the fixed locus of a non-symplectic automorphism of order 6 contains a rational curve then $X$ is elliptic.

**Proof.** When the fixed locus contains at least one curve, formula 5.1 reduces to $p_{1/2}(3, 4) + 2p_{1/2}(2, 5) - 6l[1] = 6$. If $p_{1/2}(3, 4)$ is strictly positive, then Corollary 5.6 implies that $X$ is elliptic. Otherwise, $p_{1/2}(2, 5) \geq 6$ and thus $n$ is an odd number greater than or equal to 6. From [AS08 Table 2] one can see that all cases where $X[2]$ contains a rational curve and where $n$ is an odd number larger than 6 are elliptic.
While Lemma 5.7 shows that any automorphism of order 3 preserving the fibration factors through an automorphism of order 6, this automorphism does not have to be unique. It is possible that a generic automorphism of order 6 when applied twice gives an automorphism of order 3 commuting with the fibration. From now on we will focus on automorphisms of order 6 that actually do preserve the elliptic fibration:

**Definition 5.9.** If $X, \zeta$ are as in the statement of Corollary 5.6, we will say that $X$ is $\zeta$-elliptic if $\zeta$ preserves the elliptic fibration.

In this situation, we have an induced action on the basis which we will denote by $\psi = \pi \circ \zeta \circ \pi^{(-1)}$. Indeed, the action of $\zeta$ preserves the fibration, i.e. $\pi \circ \zeta \circ \pi^{(-1)}$ is well defined, and $\zeta^2$ acts as $(x, y, t) \mapsto (\zeta^2 x, y, t)$. In particular, this implies that the induced action on the base is at most of order 2 and if this induced action is trivial, then $\zeta$ restricts to an action of order 6 on each fiber.

An important property of automorphisms which preserve fibrations is given by the following statement:

**Lemma 5.10.** If $X$ is $\zeta$-elliptic than $X[i]$ does not contain curves of strictly positive genus.

**Proof.** Assume that $X[i]$ contains a curve $C_0$ which is not rational. Proposition 5.4 tells us that $C_0$ is a double section of the fibration and therefore, the action induced on the base is trivial. We are then in the situation where $\zeta$ induces an automorphism of order 6 on each fiber. Moreover, since $\zeta$ fixes at least two points per fiber, the points of intersection with $C_0$, it ought to be the identity: a contradiction. \qed

**Remark.** If $X$ is $\zeta$-elliptic, Lemma 5.10 tells us that the fixed locus of $\zeta$ contains no curves of positive genus. However, one could have an automorphism of order 6 commuting with an elliptic fibration and fixing a curve of genus one. Indeed, the proof of the Lemma does not exclude the fixed curve to be a fiber itself.

We can thus conclude

**Conclusion 5.11.** The fixed locus of $\zeta$ consists either of a disjoint union of smooth rational curves and points or of a configuration containing possibly one elliptic curve. In Section 6.2, we show that the genus one situation is actually unique; all other cases, which correspond to $X \not\in\zeta$-elliptic, are discussed in Section 7.

### 6. The case where the fixed locus contains a genus 1 curve

**Lemma 6.1.** If $g(C_0) = 1$ then $p_{1/2}(3,4) = \ell[1] = 0$, $p_{1/2}(2,5) = 3$ and $\ell[2] = 0$, $p_{1/2}(2,2) = 3$.

**Proof.** If $\ell[1]$ or $p_{1/2}(3,4)$ were to be strictly positive, Corollary 5.6 would imply that $X$ is $\zeta$-elliptic contradicting Lemma 5.10. Formula 5.1 gives us the value of $p_{1/2}(2,5)$. Similarly, the case $\ell[2] > 0$ is excluded as we would reach a similar contradiction. Finally, the value for $p_{1/2}(2,2) = 3$ can be found in [AS08, Table 1]. \qed

A non-symplectic automorphism of order 6 which fixes a smooth elliptic curve fixes thus also three isolated points and nothing else. Actually,
Proposition 6.2. If the fixed locus of a non-symplectic automorphism contains an elliptic curve $C_0$, then the action is defined uniquely i.e., the fixed loci of $\zeta$, $\zeta^2$ and $\zeta^3$ are determined uniquely.

- The fixed locus of $\zeta$ and $\zeta^2$ are identical: $X^{[1]}$ and $X^{[2]}$ consist of $C_0$ and three isolated points – as described in the previous Lemma.
- The fixed locus of $\zeta^3$ is a superset of the previous fixed loci: it consists of $C_0$ and a second smooth elliptic curve $C_1$.

Proof. Consider the elliptic fibration given by the linear system $|C_0|$. Since $C_0$ is in the fixed locus, the induced action on the base is of order 6, i.e. it is a cyclic action with two fixed points: the image of $C_0$ and some additional point $Q$. Since the Euler characteristic of a K3 surface is 24, the Euler characteristic of the fiber above $Q$ is a multiple of 6. From Kodaira's classification of the possible singular fibers, the fiber above $Q$ is of the type $I_6N$ or $I^*_{6N}$. However, as will follow from Section 7.1, only in the case $I_0$ does $\zeta$ not fix any rational curves. The fiber above $Q$ is thus smooth and the fixed loci of $\zeta$ and its powers are readily found. □

Remark. An example of a K3 with a primitive non-symplectic automorphism of order 6 fixing an elliptic curve is given by the surface $y^2 = x^3 + (t^6 - 1)^2$, where the action is $\zeta : (x, y, t) \mapsto (x, y, \xi t)$. The volume form $\omega = dx \wedge dt$ gets mapped to $\zeta^* \omega = \xi_0 \omega$.

7. Elliptic case

In this Section we consider $X$ to be $\zeta$-elliptic. The induced automorphism, $\psi$, on $\mathbb{P}^1$, is either trivial or an involution. The two cases are analyzed respectively in Sections 7.2 and 7.3. Our discussion begins in Section 7.1 where we analyze how $\zeta$ acts on the fibers of $\pi$.

7.1. Local analysis. Let $X$ be a K3 surface. The Gram graph of $X$ is the incidence graph of the effective smooth rational curves on $X$. E.g., when the Picard lattice of $X$ is isomorphic to $\text{U} \oplus E_8^2$ of $S_X$, then the Gram graph is as in figure 1. Let $D$ be an effective divisor on $X$ and $\zeta$ an automorphism of $X$. We call $D$ stable if $\zeta(D) = D$, and we say that $D$ is fixed if $\zeta|_D = \text{id}$.

Figure 1. Gram graph of $U \oplus E_8^2$.

Lemma 7.1. Consider a tree of rational curves on a surface $X$ which are stable componentwise under the action of a primitive non-symplectic automorphism of order 6. Then, the points of intersection of the rational curves are fixed and the action at one fixed point determines the action on the whole tree.

Proof. The key in this proof is to realize that the action of the automorphism on a given rational component and the action on a fixed point of this curve determine each other completely. Recall that an action of $\mathbb{C}$ will be of the form $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$ under suitable coordinates. Now, $\lambda$ is nothing but the eigenvalue associated to the fixed point of coordinate 0, or the inverse of the eigenvalue associated to the fixed
point at infinity. Conversely, if one knows one eigenvalue of the automorphism localized at a point, then one knows the full action at that point. First, the eigendirections correspond to the components of the tree passing through the point. Second, since the three types of points, $\frac{1}{6}(3,4)$, $\frac{1}{6}(2,5)$ and $\frac{1}{6}(1,0)$, all have distinct eigenvalues it is clear to which eigenvalue corresponds each direction.

\begin{remark}
It follows from the proof of the previous lemma that if we look at the types of points of intersection on a chain of smooth rational curves, these will embed in the following periodic sequence:

$$\ldots, \frac{1}{6}(2,5), \frac{1}{6}(3,4), \frac{1}{6}(3,4), \frac{1}{6}(2,5), \frac{1}{6}(1,0), \frac{1}{6}(1,0), \ldots$$

\end{remark}

**Example 7.2.** Consider a type $IV^*$ configuration of rational curves which is stable under the action of $\zeta$, a non-symplectic automorphism of order 6. Moreover, assume that it contains on one of the weight 1 curves, $L$, a point $P$ of type $\frac{1}{6}(3,4)$, such that the eigendirection corresponding to the eigenvalue $-1$ is transversal to the $L$.

Using Lemma 7.1 we can determine the action on the entire configuration. This action is illustrated in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Study of the action on a type $IV^*$ configuration.}
\end{figure}

From now on, we will focus only on fibers of type $I_0$, $II$, $II^*$, $IV$ and $IV^*$. We will denote by $ii$ the number of type $II$ fibers, $ii^*$ the number of type $II^*$ fibers, etc. We focus our attention on these fibers because of the following theorem:

**Proposition 7.3.** [AS08, Proposition 4.2] Let $X$ be $\zeta$-elliptic, then the numbers $(n,k)$ determine uniquely $ii$, $ii^*$, $iv$ and $iv^*$. More precisely, $\pi$ has

1. $n$ type $IV$ fibers if $k = 2$.
2. $n-3$ type $IV$ fibers and 1 type $IV^*$ fiber if $k = 3$.
3. $n-4$ type $IV$ fibers and 1 type $II^*$ fiber if $k = 4$.
4. $n-7$ type $IV$ fibers, 1 type $IV^*$ fiber and 1 type $II^*$ fiber if $k = 5$.
5. $n-8$ type $IV$ and 2 type $II^*$ fibers if $k = 6$.

**Lemma 7.4.** Let $X$ be $\zeta$-elliptic, $\psi$ trivial, and assume that $X$ has a fiber of type $II$, $IV$, $II^*$ or $IV^*$. When restricted to those fibers, $\zeta$

1. fixes 1 point of type $\frac{1}{6}(3,4)$, namely the cuspidal point of the fiber. (Fiber of type $II$)
2. fixes 3 points of type $\frac{1}{6}(3,4)$, 4 points of type $\frac{1}{6}(2,5)$ and 1 rational curve. (Fiber of type $II^*$)
3. fixes 1 point of type $\frac{1}{6}(2,5)$, namely the common intersection point. (Fiber of type $IV$)
4. fixes 2 points of type $\frac{1}{6}(3,4)$ and 1 point of type $\frac{1}{6}(2,5)$. (Fiber of type $IV^*$)
Moreover, \( \pi \) has also a section fixed by \( \zeta \) and it is the only part of \( X^{[1]} \) not completely included in the fibers.

**Proof.** Since \( \psi \) is trivial, the fibers are preserved by \( \zeta \). Thus either \( \zeta \) fixes the zero section \( \sigma_0 \), or there is another section \( \sigma_1 \) and \( \zeta \) permutes the two. Assume \( \sigma_0 \) is not fixed. Pick a smooth fiber \( F \) of \( \pi \). The automorphism \( \zeta^2 \) is of order 3 on \( F \) and fixes the 2 points of intersection with the two sections \( \sigma_0 \) and \( \sigma_1 \). Therefore, there is a third fixed point. Since \( \zeta \) permutes the first two, it fixes the third one. Since \( \psi \) is trivial, this point is of type \( \frac{1}{6}(1,0) \) and there is a fixed section passing through that point.

Let us describe the action on the fibers explicitly.

1. \((II)\) The point of intersection with the fixed section is not the node, as the section intersects the fiber with multiplicity 1, and is of type \( \frac{1}{6}(0,1) \). On the other hand, the other fixed point, which ought to be the node, is of type \( \frac{1}{6}(3,4) \).
2. \((II^*)\) Since the Gram graph of this fiber has no non-trivial \( \mathbb{Z}/2\mathbb{Z} \) automorphism, the curve of weight 6 is fixed. The remaining fixed points can be found using Lemma 7.1.
3. \((IV)\) The section of the Weierstrass fibration is fixed and intersects the fiber at the curve of weight 1. Using Lemma 7.1 we see that there is a unique possible action, namely the one permuting the two other branches.
4. \((IV^*)\) The action on the fiber follows from lemma 7.1 and is described in figure 3. The black dot corresponds to a point of type \( \frac{1}{6}(2,5) \) and the two white dots to points of type \( \frac{1}{6}(3,4) \).

**Figure 3.** Action on a type \( IV^* \) fiber when \( \psi \) is trivial.

\( \square \)

**Lemma 7.5.** Let \( X \) be \( \zeta \)-elliptic and assume \( \psi \) is an involution. Let \( F \) be a fiber preserved by \( \zeta \), i.e. \( \zeta(F) = F \). \( F \) is of type \( I_0 \), \( IV \) or \( IV^* \). Moreover, \( \zeta \)

1. fixes 3 points of type \( \frac{1}{6}(3,4) \), when \( F \) is smooth.
2. fixes 1 point of type \( \frac{1}{6}(3,4) \) and 1 point of type \( \frac{1}{6}(2,5) \), when \( F \) is of type \( IV \). This case is depicted in Example 7.2.
3. fixes 3 points of type \( \frac{1}{6}(3,4) \), 3 points of type \( \frac{1}{6}(2,5) \) and 1 rational curve, when \( F \) is of type \( IV^* \).

Moreover, every component of \( X^{[1]} \) lies in one of those fibers.

**Proof.** Without loss of generality, we can assume that \( \psi \) is of the form \( [x_0 : x_1] \mapsto [-x_0 : x_1] \), or \( t \mapsto -t \). Since, the Weierstrass equation \( y^2 = x^3 + p_{12}(t) \) is invariant under \( \psi \), this implies that that the roots of \( p_{12} \) are double at 0 and \( \infty \). The fibers which correspond to double roots are those of type \( I_0 \), \( IV \) and \( IV^* \). Alternatively, one can perform a local analysis on the fibers, and see that these are the only
possibilities. This analysis will also give us the exact nature of the fixed locus on each fiber. Take a fixed point \( P \) at the intersection of the section of \( \pi \) and a fiber \( F \). Since \( \psi \) is an involution, the eigenvalue corresponding to the direction of the section is \(-1\). Using Lemma 7.1 we can describe the local action in each case:

1. (I\(_0\)) There are 3 fixed points of type \( \frac{1}{6}(3,4) \).
2. (IV) There is 1 fixed point of type \( \frac{1}{6}(3,4) \), and one of type \( \frac{1}{6}(2,5) \).
3. (IV\(^*\)) There are 3 points of type \( \frac{1}{6}(3,4) \), 3 points of type \( \frac{1}{6}(2,5) \) and 1 rational curve.

\[ \square \]

7.2. Induced action on the base is trivial.

**Lemma 7.6.** Let \( X \) be \( \zeta \)-elliptic, \( \psi \) trivial. Then \( X^{[2]} \) determines \( X^{[1]} \). The possibilities are listed in Table 1.

**Proof.** From Proposition 7.3 we know that \((n,k)\) determines the types of fibers of \( \pi \). Since, Lemma 7.4 tells us that the action of \( \zeta \) on each fiber is unique, it follows that \( X^{[1]} \) is completely determined by \( X^{[2]} \). \( \square \)

**Remark.** Unfortunately, the converse is not true: the simple combinatorial data describing \( X^{[1]} \) does not determine uniquely \( X \) or \( X^{[2]} \). See examples 4 and 10 in Table 1.

Finally, the existence of all the examples in Table 1 follows from Lemma 5.7.

7.3. Induced action on the base is an involution.

**Lemma 7.7.** The fixed locus \( X^{[1]} \) is contained in 2 fibers of \( \pi \).

**Proof.** Since \( \psi \) is an involution, it has two fixed points on \( \mathbb{P}^1 \), say 0 and \( \infty \). Since all the fibers not above these points are permuted, \( X^{[1]} \) is a subset of the fibers \( F_0 \) and \( F_\infty \) (\( F_i = \pi^{-1}(i) \)). \( \square \)

**Lemma 7.8.** Let \( X \) be a K3 surface and \( \tau \) a non-symplectic automorphism of order 3 which is \( \tau \)-elliptic. Call \( \pi \) the associated fibration. Assume that the multiset \( X_\pi \) of singular fibers of \( \pi \) can be decomposed \( F \sqcup M \) with \( F \) a multiset of cardinality 2 whose elements come from \( \{I_0, IV, IV^*\} \) and where each element of \( M \) has even multiplicity. Then there exists a pair \((X', \tau')\) consisting of a K3 surface and a non-symplectic automorphism of order 3 such that \( X' \) is \( \tau' \)-elliptic, \( X_\pi = X'_{\pi'} \), and \( X^\tau = X'^{\tau'} \). Moreover, \( \tau' \) factors as \( \tau' = \zeta^2 \) where \( \zeta \) is a primitive non-symplectic automorphism of order 6 commuting with \( \pi' \).

**Proof.** This follows from the local analysis in Lemma 7.3 or from the fact that the only singular fibers corresponding to double roots of \( p_{12}(t) \) are those of \( I_0 \), \( IV \), and \( IV^* \). \( \square \)

Since the action on \( F_0 \) and \( F_\infty \) is determined by Lemma 7.3, we simply list all possibilities in Table 1.

A special case of the above classification consists of analysing only those automorphisms which fix the Picard group. Although this can be recovered from the previous sections, we will try to analyse the case separately to make the analogy with automorphisms of order 2 and 3 as studied by [Nik81] and [AS08].

Recall that for a K3 surface, \( X \), the cohomology \( H^2(X, \mathbb{Z}) \) is a unimodular lattice of signature \( (3, 19) \) i.e., it is isomorphic to \( U \oplus E_8^2 \). Also, it decomposes into the Picard lattice, \( S_X \), and the transcendental lattice \( T_X \):

\[
H^2(X, \mathbb{Z}) \cong S_X \oplus T_X.
\]

Given a lattice \( A \), we will write \( A^\perp \) for its orthogonal complement and \( A^* \) for its dual \( \text{Hom}(A, \mathbb{Z}) \). We say that a lattice is \( p \)-elementary when \( A^*/A \cong (\mathbb{Z}/p\mathbb{Z})^k \) for some \( k \in \mathbb{N} \).

**Lemma 8.1.** Let \( \zeta \) be a primitive non-symplectic automorphism of order 6 of \( X \) which preserves the Picard lattice, then the Picard lattice \( S_X = H^2(X, \mathbb{Z})^\zeta \) is a unimodular.

**Proof.** Fix \( p \in \{2, 3\} \), and let \( \zeta_p = (\zeta^*)^b/p \). The quotients \( S_X^b/S_X \) and \( (S_X^b)^*/S_X^b = T_X^b/T_X \) are isomorphic. Hence, \( p = 1 + \zeta_p + ... + (\zeta_p)^{p-1} = 0 \) on \( T_X \) and \( pT_X^* \subset T_X \).

Since \( S_X \) is both 2 and 3 elementary, it is unimodular. \( \square \)

**Corollary 8.2.** The Picard lattice \( S_X \) is isomorphic to \( U, U \oplus E_8 \) or \( U \oplus E_8^2 \).

**Proof.** By the Hodge index theorem, \( S_X \) is of signature \((1, \ast)\). By adjunction, the lattice is even. Using the classification of even unimodular lattices, e.g. in [Ser70], we get the desired result. \( \square \)

Since in the three cases \( S_X \) decomposes as the direct sum of \( U \) with a negative definite lattice, it is easy to see that we fall everytime in the elliptic case. Moreover, using the Lefschetz topological formula or the fact that only the Picard lattice is fixed, one can see that there are no other irreducible fibers except for the given \( E_8 \) fibers generating part of the Picard lattice. Note that using lemma [7,1] one can see that in the case of \( \text{rk} S_X > 2 \) only the section and the rational lines of degree 3 are fixed.

Recall the following definition, due to Dolgachev [Dol96], of mirror pairs for K3 surfaces.

**Definition 8.3.** The K3 surfaces \((M, W)\) form a mirror pair whenever \( S_M^\perp = S_W \oplus U \).

When applied to the case of unimodular Picard lattices, we see that K3 surfaces form a pair when their Picard groups are respectively \( U^i \oplus E_8^j \) and \( U^{2-i} \oplus E_8^{2-j} \). I.e. the surfaces with Picard groups \( U \) and \( U \oplus E_8^2 \) are dual to one another while the surfaces with Picard group \( U \oplus E_8 \) are self-dual. This confirms the diagrams obtained for automorphisms of order 2 and 3 showing that mirror symmetry is a natural transformation preserving symmetries.

**References**


