2014

Analysis of Resource Allocation through Game Theory

Brittney L. Benzio
University Honors Program

Analysis of Resource Allocation through Game Theory

An Honors Thesis submitted in partial fulfillment of the requirements for Honors in Mathematical Science.

By
Brittney Benzio

Under the mentorship of Dr. Hua Wang

ABSTRACT

Game theory presents a set of decision-makers in a model in order to simulate how they will interact according to a set of rules. The game is set up with a set of players, actions, and preferences. The model allows each player to, in some way, be affected by the actions of all players. Nash equilibrium illustrates that the best action for any given player depends on the other players' actions, and so, each player must make some assumption about what the competition will do. The goal of this project is to model situations of different car companies to improve our understanding of how they will allocate their resources. In our case, the players will be two car manufacturers. The actions of the players will be how each company invests its resources with some particular vehicle, make or model, and the preferences will be what each company wants to spend the most resources on. The payoff functions will be generated for each player that will also represent the preferences under given assumptions of each player's activities. By finding the Nash equilibrium of the "game", a stable activity table will be concluded and compared with the manufacturers' choices and gains in reality.

Thesis Mentor:__________________________________

Dr. Hua Wang

Honors Director:________________________________

Dr. Steven Engel

April 2014
Mathematical Science
University Honors Program
Georgia Southern University

Acknowledgments

First, I would like to thank Dr. Hua Wang from the Department of Mathematical Sciences at Georgia Southern University for his time and effort in helping me complete this project. Without his mentoring, my undergraduate thesis would not have been possible.

I want to give my sincerest thanks to Dr. Martha Abell, who has been my advisor, as well as my professor. She has always been more than supportive of both my academic and athletic endeavors. With that said, there are numerous other professors from the Department of Mathematical Science at Georgia Southern University who have contributed to my education and deserve recognition.

I would also like to thank the University Honors Program for the opportunity to be a part of this reputable organization and for the continued support over my four years at Georgia Southern University.

1. Introduction

In the study of economics, one aims to invest their money, time, and resources in general to produce a profit. However, there are many factors that affect an investment: for example, competition within the market, demand of the consumer, and cost of production, which includes labor and materials. With all factors considered, sometimes strategies for producing a profit are not at all obvious. Instead, one must reallocate their resources wisely in order to break even, minimize their losses, or maximize the profit. As an example, car manufacturers choose certain models from their showroom for which they will invest the majority of their resources based on the demands of society, costs, potential competitors, etc.

In finance, the cost-volume formula is applied to determine production cost given a certain number of units. Production costs include any fixed or variable expenses incurred when a business provides goods or services. Fixed costs do not increase or decrease regardless of the output. They are constant and independent of business activity, for example, the lease payment owed on a building.

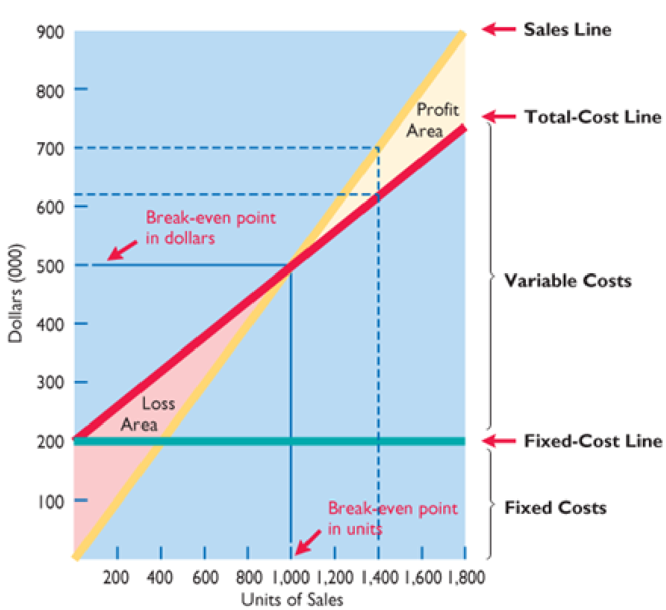

Figure 1: Cost-Volume Formula

Variable costs, on the other hand, fluctuate depending on how many goods are produced or how much service is provided. Materials and labor are examples of variable costs. Together, fixed costs and variable costs make up total production cost. See Figure 1 for an illustration of the cost-volume formula.

Game theory, introduced in 1944 by John von Neumann and Oskar Morgenstern (Ross, 2012), models a situation where two or more decision makers interact. Each strategic game consists of a set of players, a set of actions, and player preferences in regards to their actions. Player is a generic term that can be used to represent firms, political candidates, or prisoners like in the *Prisoner's Dilemma* example explained in the following section. Actions represent the players' decisions or allocation of resources. The preferences, in our setting, are the pay-off function or the profit a player can expect.

2. Introduction to Game Theory

The *Prisoner's Dilemma* is one of the simplest examples of a two-player game (see, for instance, Osborne, 2000). For this game, there are two suspects being held in separate cells. There is enough evidence against each suspect to convict them of a minor offense, but the only way to convict either one of a major crime is if the other suspect acts as an informer and 'finks' to the authorities. Under this situation, both prisoners serve a minor sentence if neither finks; but finking results in a reduction penalty; staying quiet, however, has the risk of receiving the maximum penalty if the other prisoner finks. The table below illustrates the game between the two players:

Table 1. The *Prisoner's Dilemma*

		Suspect 2 Quiet	Suspect 2 Fink
Suspect 1	**Quiet**	2,2	0,3
	Fink	3,0	1,1

In Table 1, one player chooses from the row actions (Q/F) and the other from the columns. The numerical values correspond to the preferences of each player under each scenario with higher values representing a higher pay-off, which in our case is less prison time. For example, from the entry [3,0], suspect 1 finks and suspect 2 remains quiet, which would result in a higher pay-off for suspect 1 and maximum prison time for suspect 2, consequently, the minimum pay-off.

3. Nash Equilibrium

Nash equilibrium is a concept of game theory where the optimal outcome for the model is one where no player has an incentive to change their decision after considering the decision of their opponent. Thus, the player would gain no advantage from having deviated from their original action assuming their opponent's strategy was held constant. Any particular game could have several Nash equilibria or none at all. This concept was named after its creator, Nobel Prize winner John Forbes Nash, Jr., who published an article in 1950 introducing this notion of equilibrium for *n*-person games. The concept has been applied in both economics and behavioral sciences and has since become the centerpiece of modern game theory (Holt, 2004, 3999-4002).

In the *Prisoner's Dilemma* example in Table 2, Nash equilibrium can be accomplished when both players 'fink' represented by the last cell, (1,1). For example, referring to the bottom row of Table 1, say player 1 chooses to 'fink'; player 2 is better off also choosing 'fink' to yield the highest possible payoff given the first player's action. Given that player 2 chooses 'fink', we see from the table, player 1 would also choose to 'fink' between a payoff of zero for remaining quiet and a payoff of one for finking. Analyzing the first cell, (2,2) meaning 'quiet', 'quiet', both players will want to deviate from their action to receive a higher payoff knowing that their opponent chose 'quiet'. Consequently, the only Nash equilibrium of the *Prisoner's Dilemma* is when both players 'fink' because the incentive to go free eliminates the mutually desirable outcome (2,2) where both players remain 'quiet' (see, for instance, Osborne, 2000).

4. Competing Car Manufacturers

In this note, we seek to propose a model that illustrates the competition between two car manufacturers, Toyota and Ford, and how they allocate their resources according to their preferences and the restrictions of the market.

Table 2 below shows the data collected. This includes the unit prices and sales statistics for the two manufacturers for each of the three models under consideration. The unit price recorded represents the most generic version of each model. Also, the sales statistics were divided into three four-month periods for the year for simpler calculations.

Table 2.

			Amount Sold Yearly in the US		
Make	**Model**	**Starting Price**		**2012**	
			Jan-Apr	**May-Aug**	**Sept-Dec**
Toyota	**Corolla**	$16,800.00	90176	106445	91270
Ford	**Focus**	$16,605.00	85468	81482	78972
Toyota	**Camry**	$22,235.00	142224	138311	124350
Ford	**Fusion**	$21,900.00	85559	96306	59398
Toyota	**Sienna**	$26,585.00	33648	44634	36440
Ford	**Explorer**	$29,600.00	47731	56023	60433

First we consider the simple model where we let X_p be the percent of the price that indicates profit. We will let X_p be fixed and equal 2/3. We then find the amount of profit each manufacturer would earn from selling one model. For example, the starting price for a Toyota Corolla is $16,800.00, and two-thirds of this is $11,200.

Table 3.

		Ford		
		Focus	Fusion	Explorer
	Corolla	11200, 11070	11200, 14600	11200, 19733.33
Toyota	Camry	14823.33, 11070	14823.33, 14600	14823.33, 19733.33
	Sienna	17723.33, 11070	17723.33, 14600	17723.33, 19733.33

From Table 3, when only considering the price and profit ratio, it would indicate that both companies should invest their resources in selling SUVs. The last entry indicates that Nash equilibrium can be achieved, but using what we know about the market, this would not make sense. This is actually an invalid conclusion because there is not infinite demand for SUVs due to gas prices and whether or not the vehicle is for a family or an individual. Also, if both companies channeled all resources into manufacturing and selling SUVs there would be greater competition to appeal to the fixed number of consumers looking to buy an SUV, meaning both companies would miss out on another population entirely that is interested in a coupe or a sedan.

Building on the previous table, we analyze price with number of models sold to determine the price to profit ratio for Table 4. For example, for a Toyota Corolla, we multiply starting price by the number of Corollas sold between January and April of 2012. From the data collected in Table 2, the starting price is $16,800, and the amount sold is 90,176, which yields 1,514,956,800. This introduces the demand for a particular model into the table.

Table 4.

		Focus	Ford Fusion	Explorer
	Corolla	1514956800, 1419196140	1514956800, 1873742100	1514956800, 1412837600
Toyota	**Camry**	3162350640, 1419196140	3162350640, 1873742100	3162350640, 1412837600
	Sienna	894532080, 1419196140	894532080, 1873742100	894532080, 1412837600

Table 4 indicates that the consideration of demand for any one model forces the Nash equilibrium to the center, the entry "Camry, Fusion", (3162350640, 1873742100). This makes more sense in reality than the afore mentioned Nash equilibrium because selling one SUV compared to one sedan would always be more profitable due to the higher price. However, from what we know about the market, more consumers are buying these sedans, and in the long run, the larger number of sales overcomes the greater unit price. All things considered, if both companies invest all their resources into a mid-sized sedan, they will be competing for one population of consumers, again one company will control the market and the other will be missing out on the population in demand of a larger vehicle or the market will be divided and neither company will return a significant profit.

For more practical purposes, Table 5 introduces the demand of the entire market as opposed to the demand for just a particular vehicle. Thus, we are calculating the price to profit ratio similar to Table 4, but are considering the consumers who purchased both the comparable Ford vehicle and Toyota vehicle. Recall that, we multiplied the starting price by the amount sold for Corollas between January and April of 2012, which yielded 1,514,956,800. Now, we divide

1,514,956,800 by the sum of 90,176 and 85,468, which represents the number of consumers who purchased both Toyota Corollas and Ford Focuses, because the Focus is the most comparable model to the Corolla in the Ford showroom.

Table 5.

		Ford		
		Focus	**Fusion**	**Explorer**
	Corolla	8625.155, 8079.958	8625.155, 8225.996	8625.155, 17361.206
Toyota	**Camry**	13883.172, 8079.958	13883.172, 8225.996	13883.172, 17361.206
	Sienna	10992.173, 8079.958	10992.173, 8225.996	10992.173, 17361.206

In this most complex model thus far, the Nash equilibrium is the cell for Camry, Explorer, (13883.172, 17361.206). This indicates that Toyota profits more from the section of the market represented by the mid-sized sedan and that Ford controls the sports-utility sector, suggesting that Toyota invests the majority of its resources in production of the Camry and Ford invests the majority of its resources in production of the Explorer. We find this to be true for today's market. Once again, this is more realistic then the Nash equilibrium from Table 4 that suggested both companies invested in their competitive mid-sized sedans. Our new, modified Nash equilibrium for Table 5 allows both companies to profit from different sectors of the market.

5. Mixed Strategy

As one can see, when we progressively consider the factors from reality, different Nash equilibriums appear in different models, resulting in different preferences between the two manufacturers. However, maximizing the profit, as one can imagine, relies on not investing all resources in the same model. And here is where the more advanced game theoretical study, in terms of mixed strategies, comes into play.

A mixed strategy Nash equilibrium where each player chooses his or her actions probabilistically is referred to as stochastic. This particular action profile requires a steady state in which every player's behavior is constant or the behavior pattern remains constant each time the game is played, and no player wishes to deviate knowing another player's action. The following is an example of a stochastic steady state (Osborne, 98).

Table 6. ***Matching Pennies***

		Player 2	
		Head	**Tail**
Player 1	**Head**	\$1, -\$1	-\$1, \$1
	Tail	-\$1, \$1	\$1, -\$1

This game is setup with two players who will simultaneously show either the head side or the tail side of a penny. When both players show the same side, player

2 must pay player 1 a dollar; player 1 will pay player 2 a dollar if the sides are different. The payoff for this game will be the amount of money involved. And in regards to player preferences, player 1 wants to make the same action as player 2, and player 2 hopes to have the opposite action of player 1. Thus, player 1 and player 2's preferences will always conflict.

Let's assume player 2 chooses *Head* with probability q, then he or she chooses *Tail* by $1-q$. Say player 1 chooses *Head* with probability p; player 1 will earn \$1 with probability $pq + (1 - p)(1 - q)$ denoting that the outcome is either (*Head*, *Head*) or (*Tail*, *Tail*). Player 2 earns \$1 by $(1 - p)q + p(1 - q)$, an outcome of either (*Head*, *Tail*) or (*Tail*, *Head*). After distributing and combining like terms the first probability becomes $1 - q + p(2q - 1)$; the second simplifies to $q + p(1 - 2q)$. If we let the value of q be less than ½, then player 2 chooses *Head* less than half of the times the game is played. From the two equations, we see that the first payoff function is decreasing in the variable p and the second is increasing in p. Then, the value of p that gives player 1 the best opportunity to win is zero. In conclusion, if player 2 chooses *Head* with a probability less than ½, then player 1 should choose *Tail* with certainty and similarly, if player 2 chooses *Head* by more than ½, player 1 should choose *Head* on every play (Osborne, 98-99).

This particular game has no Nash equilibrium, but in a stochastic steady state, each player chooses his or her action with probability ½. To see this, suppose that our first player chooses each of his actions with probability ½. If our second player chooses *Head* with probability p, he chooses *Tail* with probability $1-p$. This means the outcomes (*Head*, *Head*) and (*Head*, *Tail*) each occur by probability $\frac{1}{2}p$, and the outcomes (*Tail*, *Tail*) and (*Tail*, *Head*) happen with probability ½(1 – p). Applying basic algebra we determine player 2 wins $1 with the probability ½$p$ + ½(1 – p), which simplifies to ½. Thus, player 2 also loses $1 with probability ½, and can do no better than choose either *Head* or *Tail* with probability ½, indicating every value of p is optimal. Switching the roles of player 1 and player 2, player 1 would also choose *Head* or *Tail* with probability ½, so we have a stochastic steady state for *Matching Pennies* when the players choose their actions by ½ (Osborne, 98).

6. Dynamic Model

A cube root function can depict the relationship between the quantity of vehicles sold and the profit returned for a particular make and model. The curve of a cube root approaches a horizontal asymptote from either direction and changes concavity. Our relationship can be illustrated by the following function:

$$\boldsymbol{f(x) = 2(x + a)^{1/3} + b}$$

$$\boldsymbol{f(0) = 0}$$

$$\boldsymbol{f(s) = f(p)}$$

where s denotes the quantity sold and p denotes profit.

We must include the technical coefficient two for the cube root so that there is always a solution to the equation. This function best fits the data set because the more vehicles produced will return a greater profit up until a given point, where the profit increases at a decreasing rate, eventually leveling off. We will let our breakeven point, the point on the curve where there is a change in concavity, be determined by the historical data.

Referring back to Table 2, we see that between January and April 2012, Toyota sold 90,176 Corollas. Utilizing our fixed proportion X_p, we multiply the price by 2/3 to obtain $11,200, which recall is the profit from selling one Corolla, and then multiplying that by our total quantity sold, 90,176, gives us $100.9 million. Our breakeven point for the graph modeling the relationship between number of vehicles sold and profit for a Toyota Corolla has the ordered pair, (90176 , 100.9). We use the same procedure to obtain the breakeven point for Ford Focus, which we find to be (85468, 946.1). The graphs for both models are displayed below.

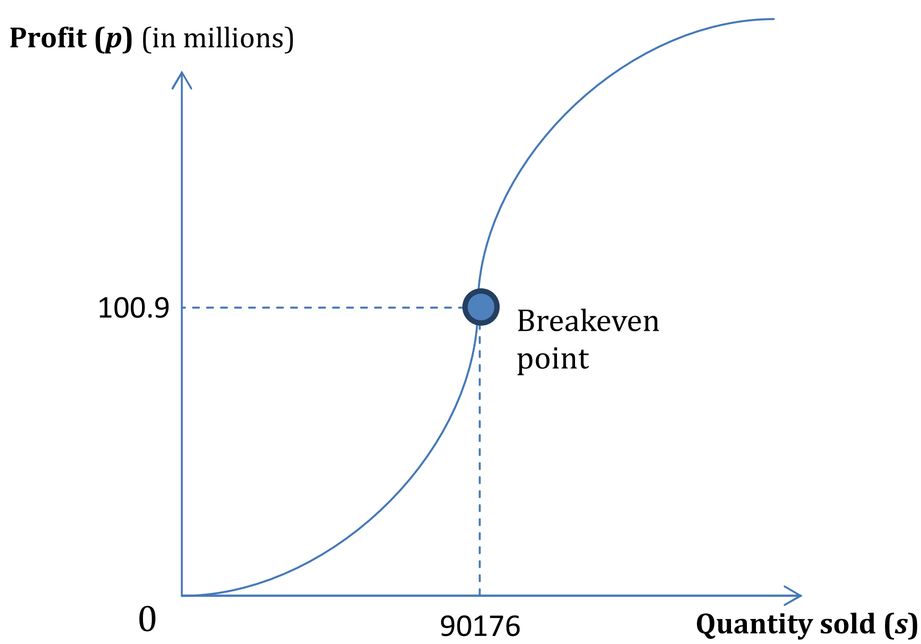

Figure 2: Profit versus Quantity Sold for Toyota Corollas

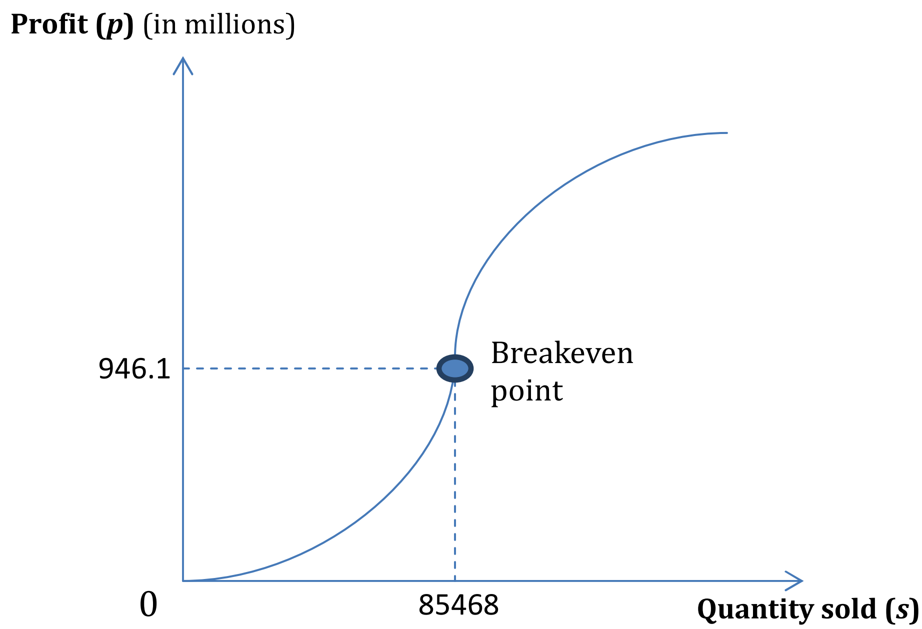

Figure 3: Profit versus Quantity Sold for Ford Focuses

Notice the curves for the two vehicles are not identical. However, any change in either function directly affects its competitor. The slope and altitude of the curve is constantly changing and consequently, the breakeven point also moves along the curve.

This relationship, when considered in a dynamic setting, introduces more mathematical components into the game theoretical model. If we let our function be modeled by the derivative:

$$\boldsymbol{f'(a)} = \frac{\boldsymbol{F}}{\boldsymbol{C + F}}$$

Where F denotes the sales for Ford Focus and C denotes the sales for Toyota Corolla. Then the curve is the integral of that function:

$$\int \boldsymbol{f}'(\boldsymbol{a}) = \int \frac{\boldsymbol{F}}{\boldsymbol{C}+\boldsymbol{F}}$$

From here, it is obvious that any increase or decrease in the sales of Corollas will change the dynamic of the graph for Ford Focus and vice versa.

7. Concluding Remarks

In our game theory model, our players are the two car manufacturers Ford and Toyota. Their actions are defined by how they invested their resources in a certain make or model. By finding the Nash equilibrium of the three "games" we proposed, a stable activity table was concluded and compared with the manufacturers' choices and gains in reality. We further established the groundwork of the dynamic modeling system that considers the constant changes of each player's choices and preferences.

The models and methodologies presented here are not limited to the study of manufacturers or companies in general. Any financial scenario between multiple competitors can be examined through our approach.

References

Holt, Charles A. and Roth, Alvin E., “The Nash Equilibrium: a perspective”, *Proceedings of the National Academy of Sciences of the United States of America* (vol. 101 no. 12), Vernon L. Smith (ed.), URL = <http://www.pnas.org/content/101/12/3999.full?tab=author-info>.

Osborne, Martin J., “An Introduction to Game Theory”, (2000).

Ross, Don, "Game Theory", *The Stanford Encyclopedia of Philosophy* (Winter 2012 Edition), Edward N. Zalta (ed.), URL=<http://plato.stanford.edu/archives/win2012/entries/game-theory/>.