Theory and Applications of Graphs

Volume 4 | Issue 1 Article 5

2017

# Note On 6-regular Graphs On The Klein Bottle

Michiko Kasai
mikko2046@gmail.com

Naoki Matsumoto
*Seikei University*, naoki.matsumo10@gmail.com

Atsuhiro Nakamoto
*Yokohama National University*, nakamoto@ynu.ac.jp

Takayuki Nozawa
takayuki.nozawa0117@gmail.com

Hiroki Seno
hiroki.seno@gmail.com

*See next page for additional authors*



## Recommended Citation

Kasai, Michiko; Matsumoto, Naoki; Nakamoto, Atsuhiro; Nozawa, Takayuki; Seno, Hiroki; and Takiguchi, Yosuke (2017) "Note On 6-regular Graphs On The Klein Bottle," *Theory and Applications of Graphs*: Vol. 4: Iss. 1, Article 5.
DOI: 10.20429/tag.2017.040105
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol4/iss1/5

**Abstract**

Altshuler [1] classified 6-regular graphs on the torus, but Thomassen [11] and Negami [7] gave different classifications for 6-regular graphs on the Klein bottle. In this note, we unify those two classifications, pointing out their difference and similarity. In particular, we reduce five types of Thomassen's classification to three types which are classified by Negami.


**Keyword:** 6-regular graphs, triangulations, Klein bottle

In this note, we only deal with a finite simple undirected graph $G$ on a surface, which is called a *map* on the surface. By Euler's formula, the average degree of $G$ on the torus or the Klein bottle is at most 6, and the equality holds when $G$ triangulates the surfaces. Actually, both the surfaces admit infinitely many 6-regular graphs, which must be triangulations.

In 1973, Altshuler [1] characterized the 6-regular graphs on the torus by using three parameters, as follows: For positive integers $p$ and $q$, let $A_{p,q}$ be the map on the annulus with $p \times (q+1)$ vertices shown in Figure 1, where we identify the top and the bottom of the rectangle. For $u_{i,j} \in V(A_{p,q})$, we take the first subscripts of $u$ modulo $q+1$ and the second modulo $p$. For an integer $r$ with $0 \leq r \leq p-1$, let $G[p \times q, r]$ denote the 6-regular graph on the torus with $p \times q$ vertices obtained from $A_{p,q}$ by identifying $u_{0,j}$ with $u_{q,j+r}$ for each $j = 0, 1, \ldots, p-1$.

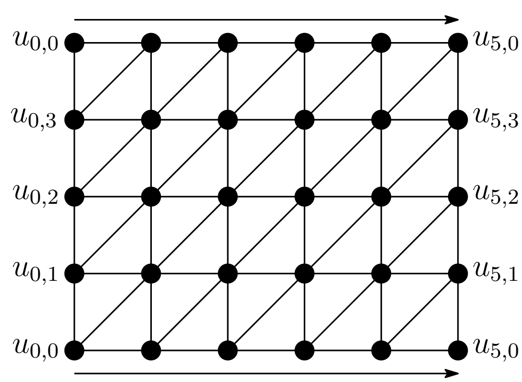


Figure 1: The graph $A_{p,q}$ where $p = 4, q = 5$

**Theorem 0.1** (Altshuler [1]). *Every 6-regular graph on the torus is isomorphic to $G[p \times q, r]$ for some integers $p \geq 3, q \geq 1$ and $r \geq 0$.*

Although Thomassen [11] referred to Theorem 0.1 in his paper, he gave another, but more complicated, classification of 6-regular toroidal graphs. Consequently, in the literature, several results on 6-regular toroidal graphs have been established, using Theorem 0.1, for example, results in [2, 9].

On the other hand, for 6-regular graphs on the Klein bottle, Thomassen [11] and Negami [7] have given different classifications, where the former consists of five types but the latter three. We can find several results on 6-regular graphs on the Klein bottle, and all of them rely on their classifications. For example, the papers [3, 5] have used Thomassen's result (Theorem 0.2). On the other hand, the papers [4, 10] have used Negami's one (Theorem 0.3).

Since the appearance of the two classifications looks different, we are afraid that one of the classifications is wrong, which might destroy the reliability of some of the results on 6-regular graphs on the Klein bottle. Therefore, in this note, by clarifying the difference and similarity of Thomassen's and Negami's classifications, we would like to unify them.

Thomassen actually gave a classification of 3-regular maps on the Klein bottle with each face hexagon into the five types, $H_{k,m,a}, H_{k,m,b}, H_{k,m,c}, H_{k,d}$ and $H_{k,m,f}$, but in order to deal with 6-regular triangulations in this paper, we introduce their duals $H^*_{k,m,a}, H^*_{k,m,b}, H^*_{k,m,c}, H^*_{k,d}$ and $H^*_{k,m,f}$, respectively.

For any $k \geq 3$ and $m \geq 2$, let $H^*_{k,m,a}$ denote the map obtained from $A_{k,m+1}$ by $u_{0,0} = u_{m+1,0}$, $u_{0,k-1} = u_{m+1,1}$, $u_{0,k-2} = u_{m+1,2}$, $\ldots$, $u_{0,1} = u_{m+1,k-1}$. (See the left-hand side of Figure 2.) When $k \geq 4$ is even and $m \geq 2$, let $H^*_{k,m,b}$ denote the map obtained from $A_{k,m+1}$ by identifying $u_{0,0} = u_{m+1,k-1}$, $u_{0,k-1} = u_{m+1,0}$, $u_{0,k-2} = u_{m+1,1}$, $\ldots$, $u_{0,1} = u_{m+1,k-2}$. (See the right-hand side of Figure 2.)

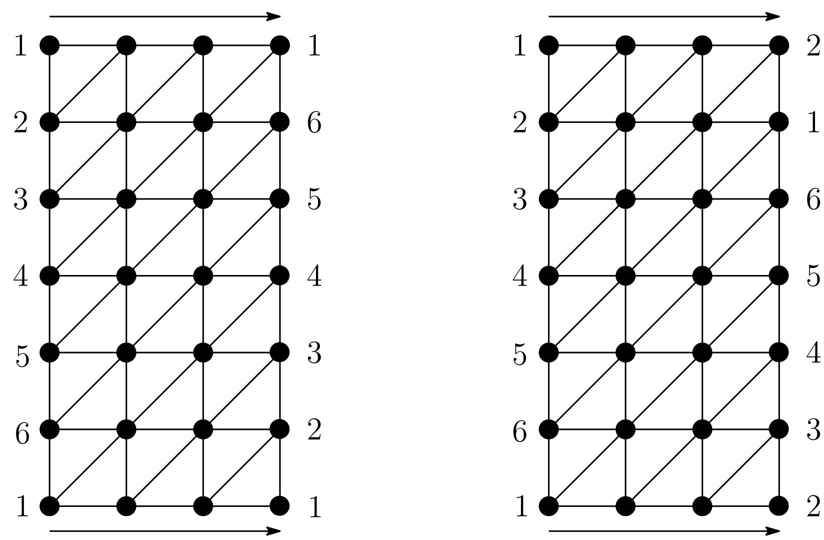


Figure 2: The graphs $H^*_{k,m,a}$ and $H^*_{k,m,b}$ with $k = 6, m = 2$, where we identify the vertices with the same number.

Suppose that $k \geq 5$ is even and $m \geq 1$. Let $H^*_{k,m,c}$ denote the map obtained from $A_{k,m+1}$ by identifying $u_{0,j}$ and $u_{0,j+k/2}$, and $u_{m+1,j}$ and $u_{m+1,j+k/2}$, respectively, for $i = 0, 1, \ldots, k/2 - 1$. When $k = 2t + 1$ is odd ($t \geq 2$) and $m \geq 1$, let $H^*_{k,m,f}$ denote the map obtained from $A_{k,m+1}$, as follows: First we attach a Möbius band to each of the two boundary component of $A_{k,m+1}$, and for each $j$, join $u_{0,j}$ to $u_{0,j+t}$ and $u_{0,j+t+1}$ and join $u_{q-1,j}$ to $u_{q-1,j+m}$ and $u_{q-1,j+m+1}$. In particular, let $H^*_{k,d} = H^*_{k,0,f}$. (Hence, taking $m \geq 0$ in $H^*_{k,m,f}$, we can deal with the type $H^*_{k,d}$ as one in $H^*_{k,m,f}$ with $m \geq 0$.)

**Theorem 0.2** (Thomassen [11]). *Every* 6*-regular graph on the Klein bottle is isomorphic to one of*

$$H^*_{k,m,a}, H^*_{k,m,b}\ (k \geq 3, m \geq 2),\ H^*_{k,m,c}, H^*_{k,m,f}\ (k \geq 5, m \geq 1) \text{ and } H^*_{k,d}(= H^*_{k,0,f}).$$

Now we proceed to Negami's classification. Let $Kh(p,q)$ denote the one obtained from $A_{p,q}$ by identifying each $u_{0,j}$ with $u_{q,-j}$, and we call it the *handle type*. Let $Kc_e(p,q)$ denote the one obtained from $A_{p,q}$ with $p \geq 6$ even and $q \geq 2$ by identifying $u_{0,j}$ with $u_{0,j+p/2}$, and $u_{q,j}$ with $u_{q,j+p/2}$ for each $j$, respectively, and we call it the *crosscap even type*. Let $Kc_o(p,q)$ denote the one obtained from $A_{p,q-1}$ with $p$ odd and $q \geq 2$, say $p = 2m+1$ for $m \geq 2$, by pasting a Möbius band to each boundary component of $A_{p,q-1}$, and for each $j$, joining $u_{0,j}$ to $u_{0,j+m}$ and $u_{0,j+m+1}$, and joining $u_{q-1,j}$ to $u_{q-1,j+m}$ and $u_{q-1,j+m+1}$ on the added crosscaps. We call it the *crosscap odd type*. (In Negami's original classification, he did not distinguish $Kc_e(p,q)$ and $Kc_o(p,q)$, because they are distinguishable by the parity of $p$. However, our paper deals with them as distinct types for easily comparing them with Thomassen's classification.)

**Theorem 0.3** (Negami [7]). *Every 6-regular graph on the Klein bottle is isomorphic to either $Kh(p,q)$ for some integers $p \geq 3$ and $q \geq 3$, $Kc_e(p,q)$ or $Kc_o(p,q)$ for some integers $p \geq 5$ and $q \geq 2$.*

By the constructions of Thomassen's and Negami's standard forms, we immediately have the following fact:

**Proposition 0.1.** *For any integers $p \geq 3$ and $q \geq 3$, we have;*

$$Kh(p,q) = H^*_{p,q-1,a},$$

*and for $p \geq 5$ and $q \geq 2$, we have;*

$$Kc_o(p,q) = H^*_{p,q-2,f} \quad \textit{and} \quad Kc_e(p,q) = H^*_{p,q-1,c},$$

*where we put $H^*_{k,d} = H^*_{k,0,f}$.*

By $H^*_{k,d} = H^*_{k,0,f}$, we see that the Thomassen's classification has essentially four types, $H^*_{k,m,a}$, $H^*_{k,m,b}$, $H^*_{k,m,c}$ and $H^*_{k,m,f}$. On the other hand, by Proposition 0.1, $H^*_{k,m,a}$, $H^*_{k,m,c}$ and $H^*_{k,m,f}$ coincide with the three types in Negami's classification, respectively. Hence we consider whether $H^*_{k,m,b}$ is necessary for the classification of 6-regular graphs on the Klein bottle. Therefore, we conclude the following, which points out that the essential four types in the Thomassen's classification has a duplication.

**Theorem 0.4.** *For any even integer $k \geq 3$ and any integer $m \geq 2$, the 6-regular graph $H^*_{k,m,b}$ is isomorphic to one expressed by $H^*_{k,m,a}$.*

*Proof.* For the graph $H^*_{k,m,b}$, we put labels $r_0, r_1, r_2, \ldots, r_{k-1}$ to the vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,k-1}, u_{1,0}$, respectively, as shown on the left-hand side of Figure 3. Then, cutting $H^*_{k,m,b}$ along the cycle $C = r_0 r_1 r_2 \cdots r_{k-1}$, we get the annulus $A_{k,m+1}$ in which $C$ appears in both boundary components. See the center of Figure 3. In the figure, taking the horizontal path through $r_0$, we get $H^*_{k,m,a}$, as shown on the right-hand side of Figure 3. □

# References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973), 201–217.

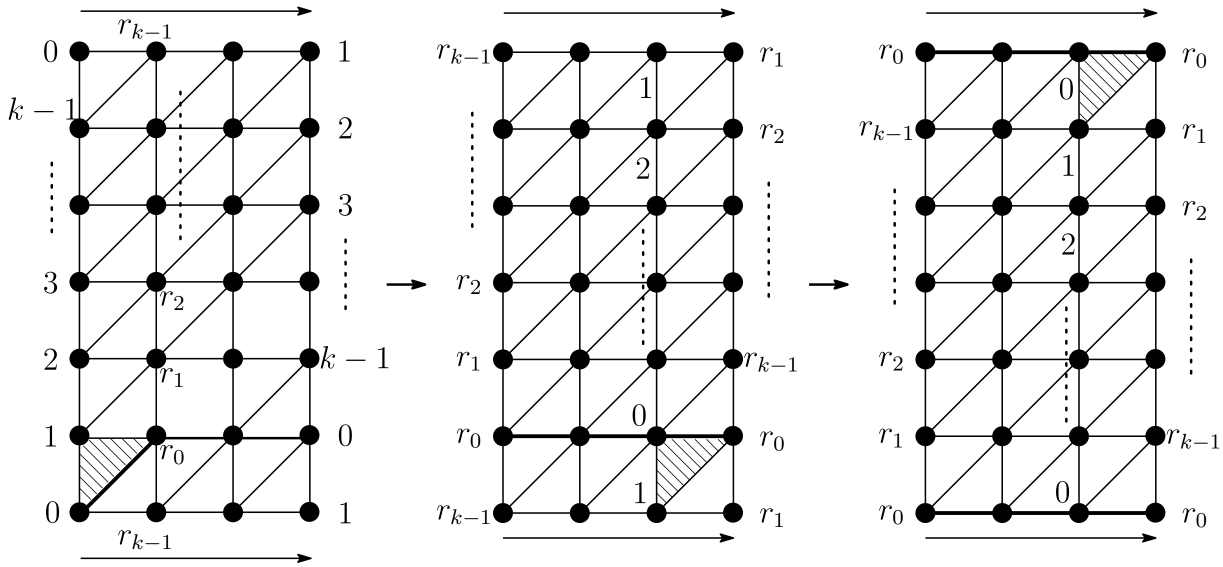

Figure 3: Transfoming $H^*_{k,m,b}$ into $H^*_{k,m,a}$ by cut and paste


[2] K.L. Collins and J.P. Hutchinson, Four-coloring six-regular graphs on the torus, *CRM Proc. Lecture Notes* **23** (1999), 21–34.

[3] S. Jendrol and H.-J. Voss, Subgraphs with restricted degrees of their vertices in polyhedral maps on compact 2-manifolds, *Annals Comb.* **5** (2001), 211–226.

[4] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Comb. Theory B* **70** (1997), 265–291.

[5] Q. Li, S. Liu and H. Zhang, 2-extendability and $k$-resonance of non-bipartite Klein-bottle polyhexes, *Discrete Appl. Math.* **159** (2011), 800–811.

[6] A. Nakamoto and N. Sasanuma, Chromatic numbers of 6-regular graphs on the Klein bottle, *Australasian J. Comb.* **45** (2009), 73–85.

[7] S. Negami, Classification of 6-regular Klein-bottlal graphs, *Res. Rep. Inf. Sci. T.I.T.* **A-96** (1984).

[8] M.D. Plummer and X. Zha, On the connectivity of graphs embedded in surfaces, *J. Comb. Theory B* **72** (1998), 208–228.

[9] J. Preen, Largest 6-regular toroidal graphs for a given diameter, *Australasian J. Comb.* **47** (2010), 53–57.

[10] T. Sulanke and F.H. Lutz, Isomorphism-free lexicographic enumeration of triangulated surfaces and 3-manifolds, *European J. Comb.* **30** (2009), 1965–1979.

[11] C. Thomassen, Tiling of the torus and the Klein Bottle and vertex-transitive on a fixed surface, *Trans. Amer. Math. Soc.* **323** (1991), 605–635.



DOI: 10.20429/tag.2017.040105