Symmetry-breaking and Symmetry-restoring Dynamics of a Mixture of Bose-Einstein Condensates in a Double Well

Indubala I. Satija
George Mason University

Radha Balakrishnan
The Institute of Mathematical Sciences

Phillip Naudus
George Mason University

Jeffrey Heward
Georgia Southern University

Mark Edwards
Georgia Southern University, edwards@georgiasouthern.edu

See next page for additional authors

Follow this and additional works at: https://digitalcommons.georgiasouthern.edu/physics-facpubs

Part of the Physics Commons

Recommended Citation
https://digitalcommons.georgiasouthern.edu/physics-facpubs/17

This article is brought to you for free and open access by the Physics and Astronomy, Department of at Digital Commons@Georgia Southern. It has been accepted for inclusion in Physics and Astronomy Faculty Publications by an authorized administrator of Digital Commons@Georgia Southern. For more information, please contact digitalcommons@georgiasouthern.edu.
Authors
Indubala I. Satija, Radha Balakrishnan, Phillip Naudus, Jeffrey Heward, Mark Edwards, and Charles W. Clark
Symmetry-breaking and symmetry-restoring dynamics of a mixture of Bose-Einstein condensates in a double well

Indubala I. Satija,1,2 Radha Balakrishnan,3 Phillip Naudus,1 Jeffrey Heward,4 Mark Edwards,4,2 and Charles W. Clark2
1Department of Physics, George Mason University, Fairfax, Virginia 22030, USA
2National Institute of Standards and Technology, Gaithersburg, Maryland 20899, USA
3The Institute of Mathematical Sciences, Chennai 600 113, India
4Department of Physics, Georgia Southern University, Statesboro, Georgia 30460-8031, USA

We study the coherent nonlinear tunneling dynamics of a binary mixture of Bose-Einstein condensates in a double-well potential. We demonstrate the existence of a type of mode associated with the “swapping” of the two species in the two wells of the potential. In contrast to the symmetry-breaking macroscopic quantum self-trapping (MQST) solutions, the swapping modes correspond to the tunneling dynamics that preserves the symmetry of the double-well potential. As a consequence of two distinct types of broken-symmetry MQST phases where the two species localize in different potential wells or coexist in the same well, the corresponding symmetry-restoring swapping modes result in dynamics where the two species either avoid or chase each other.

In view of the possibility to control the interaction between the species, the binary mixture offers a very robust system to observe these novel effects as well as the phenomena of Josephson oscillations and π modes.

DOI: 10.1103/PhysRevA.79.033616 PACS number(s): 03.75.Mn, 34.50.–s

I. INTRODUCTION

Ultracold laboratories have had great success in creating Bose-Einstein condensates (BECs) [1] in a variety of atomic gases such as rubidium (Rb), lithium (Li), sodium (Na), and ytterbium (Yb). These quantum fluids exist in various isotopic forms as well as in different hyperfine states. The rapid pace of development in this field has led to condensates which are robust and relatively easy to manipulate experimentally. In particular, the tunability of inter-species and intra-species interactions [2] via magnetic and optical Feshbach resonances makes the BEC mixture a very attractive candidate for exploring new phenomena involving quantum coherence and nonlinearity in a multicomponent system.

The subject of this paper is to investigate the tunneling dynamics of a binary mixture of BECs in a double-well potential. A single species of BEC in a double well is called a bosonic Josephson junction (BJJ), since it is a bosonic analog of the well-known superconducting Josephson junction. In addition to Josephson oscillations (JO), the BJJ exhibits various novel phenomena such as π modes and macroscopic quantum self-trapping (MQST), as predicted theoretically [3,4]. In the JO and the π modes, the condensate oscillates symmetrically about the two wells of the potential. In contrast to this, the MQST dynamics represents a broken-symmetry phase as the tunneling solutions exhibit population imbalance between the two wells of the potential. These various features have been observed experimentally [5]. Our motivation is to explore whether new phenomena arise when there are two interacting condensates trapped in a symmetric double well.

Although our formulation and results are valid for a variety of BEC mixtures, our main focus here is the Rb family of two isotopes, namely, the mixture of 87Rb and 85Rb, motivated by the experimental setup at JILA [6]. The scattering length of 85Rb is known to be 100 a.u. while the interspecies scattering length is 213 a.u. In experiments, the scattering length of 85Rb can be tuned using the Feshbach resonance method [7].

The ability to tune the scattering length of one of the species makes this mixture of isotopes an ideal candidate for studying the coupled BJJ system. First, it opens up the possibility of exploring the parameter space where the 85Rb-85Rb scattering length is equal to the 87Rb-87Rb scattering length. As will be discussed below, this symmetric parameter regime simplifies the theoretical analysis of the system and also captures most of the new phenomena that underlie the dynamics of the binary mixture. Furthermore, the tunability of the 85Rb scattering length can be exploited to study a unique possibility where one of the species has a negative scattering length, a case which strongly favors the π-mode oscillations that have not been observed so far.

In our exploration of nonlinear tunneling dynamics of coupled BJJ systems, the MQST states are found to be of two types. In the broken-symmetry MQST state, the two components may localize in different wells resulting in a phase separation or they may coexist in the same well and hence coexist. By varying the parameters such as initial conditions, the phase-separated broken-symmetry MQST states can be transformed to a symmetry-restoring phase where the species continually “avoid” each other by swapping places between the two wells. In other words, if the dynamics is initiated with both species in the same potential well, the sustained tunneling oscillations are seen where the two species swap places between the well one and the well two. From the coexisting MQST phase, one can achieve symmetry restoring swapping dynamics by initiating the dynamics with two species in the separate wells. In this case, the emergence of the swapping modes can be interpreted as a phase where the two species “chase” each other.

The paper is organized as follows. In Sec. II, we discuss the model and use the two-mode approximation to the Gross–Pitaevskii (GP) equation to map it to a system of two coupled pendulums with momentum-dependent lengths and...
coupling. Section III discusses the stationary solutions and Section IV discusses their stability. These results enable us to look for various qualitatively different effects without actually solving the GP equations. Section V describes the numerical solutions of the GP equations as various parameters of the system are tuned. Although we have explored the multidimensional parameter space, the novelties attributed to the binary mixture in a double-well trap are presented in a restricted parameter space where the scattering lengths of the two species are equal. Additionally, in our numerical results described here, we fix the ratio of $^{87}\text{Rb} : ^{85}\text{Rb}$ interaction to be 2.13. This restricted parameter space is accessible in the JILA setup and provides a simple means to describe various highlights of the mixture dynamics. Section VI provides additional details of the JILA setup relevant for our investigation. A summary is given in Sec. VII.

II. TWO-MODE GP EQUATION FOR THE BINARY MIXTURE

In the semiclassical regime where the fluctuations around the mean values are small, the two-component BEC is described by the following coupled GP equations for the two condensate wave functions $\Phi_l(x,t)$, with $l=a, b$ representing the two species in the mixture:

$$i\hbar \dot{\Phi}_a = -\frac{\hbar^2}{2m_a}\nabla^2 \Phi_a + V_a + (g_{aa}|\Phi_a|^2 + g_{ab}|\Phi_b|^2)\Phi_a,$$

$$i\hbar \dot{\Phi}_b = -\frac{\hbar^2}{2m_b}\nabla^2 \Phi_b + V_b + (g_{bb}|\Phi_b|^2 + g_{ab}|\Phi_a|^2)\Phi_b.$$

Here, $m_l$, $V_l$, and $g_l=4\pi\hbar^2a_l/m_l$ denote, respectively, the mass, the trapping potential, and the intra-atomic interaction of each species, with $a_l$ as the corresponding scattering length. $g_{ab}=2\pi\hbar^2/(1/m_a+1/m_b)a_{ab}$ is the interspecies interaction, where $a_{ab}$ is the corresponding scattering length. For the JILA experiment, in view of the tight confinement of the condensate transverse to the trap, it is sufficient to consider the corresponding one-dimensional Gross-Pitaevskii equations (GPE).

The condensate wave functions satisfy the normalization conditions

$$\int d^3x |\Phi_l|^2 = N_l.$$

The total number of atoms in the mixture is $N = N_a + N_b$.

Previous studies have investigated the stability of binary BEC mixture characterized by three different scattering lengths [8] and have pointed out the possibility that the coupling parameters may depend upon each other. Our studies investigating the tunneling dynamics of the BEC mixture in a double well will focus on robust behavior that exists in a wide range of parameters and initial conditions and should be observable in experiments that typically have multiple knobs that tune the parameters.

In the weakly linked limit, the dynamical oscillations of the two-component BEC can be described by two wave functions representing the condensate in each trap labeled by $k = 1, 2$, with the spatial and the temporal contributions factored as follows [3,9–11]:

$$\begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = \begin{pmatrix} \chi^a(x)\phi_1(t) \\ \chi^b(x)\phi_2(t) \end{pmatrix} = \begin{pmatrix} \chi^a(x)\phi_1^0(t) \\ \chi^b(x)\phi_2^0(t) \end{pmatrix} + \begin{pmatrix} \chi^a(x)\phi_1(t) \\ \chi^b(x)\phi_2(t) \end{pmatrix}$$

The localized spatial modes $\chi^l(x)$ are computed as sums and differences of the symmetric and antisymmetric solutions of the time-independent coupled GP equations [3,9]. To derive the equations of motion in the two-mode approximation, we introduce $z_l(t)$, the population imbalance, and $\phi_l(t)$, the relative phase of species $l$ between the left and right sides of the double-well potential,

$$z_l(t) = (|\phi_1|^2 - |\phi_2|^2)/N,$$

$$\phi_l(t) = (\dot{\theta}_l - \dot{\theta}_r),$$

where $\phi_l^0(t) = |\phi_l^0(t)| \exp i\theta_l^0$ are the time-dependent coefficients in the two-mode equations.

Following the methodology of previous works related to single-component BEC [3,9] we substitute Eq. (2) into the coupled GP equations. Using the orthogonality and the definite parity of the spatial modes, we integrate out the spatial degrees of freedom [9]. We would like to emphasize that we follow the so-called “improved-two-mode” approach [9] where we retain all overlap integrals. In the case of single species, this has been shown to be an improvement over the previous approach [3] where the overlap integrals involving the spatial modes $\chi_1$ and $\chi_2$ were neglected. In other words, given the ansatz (2), we average over the spatial part from the coupled partial differential equations (coupled GPE), resulting in coupled ordinary differential equations. The spatial averaging leads to the renormalization of the bare coupling parameters $g_l$.

The four coupled nonlinear ordinary differential equations which we refer to as the “two-mode” model are

$$\dot{Z}_a = -\vec{K}_a\sqrt{1-Z_a}\sin \phi_a,$$

$$\dot{Z}_b = -\vec{K}_b\sqrt{1-Z_b}\sin \phi_b,$$

$$\dot{\phi}_a = \vec{\Lambda}_a Z_a + \vec{\Lambda}_{ab} Z_b + \vec{\Lambda}_a \frac{Z_a}{\sqrt{1-Z_a}} \cos \phi_a,$$

$$\dot{\phi}_b = \vec{\Lambda}_b Z_b + \vec{\Lambda}_{ab} Z_a + \vec{\Lambda}_b \frac{Z_b}{\sqrt{1-Z_b}} \cos \phi_b,$$

where $Z_l = z_l/f_l$. In the above equations, $f_l = N_l/N$ denotes the fraction of atoms of species $l$, while the renormalized parameters $\vec{K}_l$ and $\vec{\Lambda}_l$ are given by

$$\vec{K}_a = K_a - 2f_a C_a \sqrt{1-Z_a^2} \cos \phi_a + f_a D_{ab} \sqrt{1-Z_b^2} \cos \phi_b,$$

$$\vec{K}_b = K_b - 2f_b C_b \sqrt{1-Z_b^2} \cos \phi_b + f_b D_{ba} \sqrt{1-Z_a^2} \cos \phi_a,$$

$$\vec{\Lambda}_a = \Lambda_a + \frac{\Lambda_{ab} Z_b}{\sqrt{1-Z_a}},$$

$$\vec{\Lambda}_b = \Lambda_b + \frac{\Lambda_{ab} Z_a}{\sqrt{1-Z_b}}.$$
The spin mapping provides an alternative means to visualize the effective interaction between the two species during the tunneling. If we ignore the spatial overlap integrals between the localized modes in two wells, \((C_i=0, D_{ab}=0)\), the binary mixture of condensates in two-mode approximation maps to two Ising-type spins in a transverse magnetic field. The full two-mode variable tunneling feature induces \(XY\)-like spin interaction.

In this paper, we find it convenient to exploit mapping to the coupled pendulums for exploring tunneling dynamics in the coupled BJJ. Although we have explored the full two-mode variable tunneling model, we will only discuss the constant tunneling case \((K_i \text{ replaced by } K_i)\) and \((\Lambda_i \text{ replaced by } \Lambda_i)\) as the overlap integrals are small and the various novel effects of the mixture described here are found to be robust and unaffected by the variable tunneling parameters.

### III. STATIONARY SOLUTIONS: FIXED POINTS

The solutions of the coupled system are characterized by the interactions \(\Lambda_i\), the ratio of the tunneling amplitude for the two species, \(K_a/K_b\), which we denote by \(R\) as well as the initial phase difference \(\phi_i(t=0)\), and the initial population imbalance \(Z_i(t=0)\). In the multidimensional parameter space the equilibrium or fixed-point solutions, in which the right-hand-sides of Eq. (8) are zero, provide an effective tool to classify different categories of behavior of the system.

In general, these fixed-point equations are transcendental and have to be solved numerically. However, in the symmetric case where \(\Lambda_a=\Lambda_b=\Lambda, K_a=K_b=K\), the fixed-point equations can be tackled analytically. Further, as can be seen from Eq. (8), the parameter \(K\) can be eliminated in this case by rescaling \(t(t\rightarrow Kt)\) and redefining \(\Lambda_x\) as \(\Lambda_x=\Lambda_x/K\). Our detailed analysis shows that this special case captures many relevant phenomena characterizing the binary mixture in a double well. In this case, the fixed points belong to two broad categories as stated below, resulting in two types of small amplitude oscillations about these fixed points:

(i) Zero-mode fixed points \((\phi^*_a=\phi^*_b=0)\),

\[
Z^*_a=Z^*_b=0,
\]

(ii) \(\pi\)-mode fixed points \((\phi^*_a=\phi^*_b=\pi)\),

\[
Z^*_a=Z^*_b=0,
\]

\[
Z^*_a=Z^*_b=\frac{c}{|\Lambda_a+\Lambda_b|}.
\]

It should be noted that the mixed-mode fixed points, \(\phi^*_a=\pi\) and \(\phi^*_b=\pi\), \(Z^*_a=Z^*_b=0\), are unstable for the restricted parameter regime we are considering here and hence will not be discussed.

The small oscillations about the fixed-point \((Z^*=0, \phi^*=0)\) result in zero mode while small oscillations about \((Z^*=0, \phi^*\neq\pi)\) lead to \(\pi\) mode. The oscillation frequencies are in Sec. IV.

The nontrivial fixed points \((Z^*\neq0)\) result in solutions with population imbalance and lead to tunneling dynamics with macroscopic quantum self-trapping or the MQST. In view of the \(a-b\) symmetry, we have two sets of stationary solutions: \(Z^*_a=Z^*_b\), \((x=\alpha, \beta)\), which suggests the possibility of modes where each species oscillates about the binary fixed...
points, going back and forth between the two wells. Unlike MQST, these modes will preserve the symmetry of the double well. However, in contrast to zero modes, these modes are nonlinear and give rise to "swapping phase" that will be discussed later.

The emergence of fixed points with opposite signs for the two species ($Z_a^*=-Z_b^*$) in the zero mode phase suggests that MQST in zero mode is accompanied by phase separation of the two species. In contrast, in the $\pi$-mode MQST phase, the two species could coexist in the same potential well as ($Z_a^*=Z_b^*$). Therefore, the fixed-point equations suggest that $\pi$ modes mimic attractive interaction between the two species.

The onset from oscillatory to MQST phase corresponds to the values of the parameters where the nontrivial fixed points move from the complex to the real plane. Alternatively, the condition for the broken-symmetry phase can be obtained by linear stability analysis of the fixed-point equations. This is discussed in Sec. IV.

In the asymmetric case when $\Lambda_a \neq \Lambda_b$, the fixed points are obtained by solving the coupled transcendental equations

$$(-1)^p \frac{K_a Z_a}{\sqrt{1-Z_a^2}} + \frac{1}{2} (\Lambda Z_a^* + \Lambda_{ab} Z_b^*) = 0,$$

$$(-1)^p \frac{K_b Z_b}{\sqrt{1-Z_b^2}} + \frac{1}{2} (\Lambda Z_b^* + \Lambda_{ab} Z_a^*) = 0,$$

where $p=0(1)$ for $\delta^p=0(\pi)$ and $\phi^p=0(\pi)$. Analogous to the symmetric case, both the zero and the $\pi$-mode solutions including those corresponding to MQST can be found numerically. As expected, for the MQST fixed points $Z_a^*\neq-Z_b^*$ in the zero mode and $Z_a^*\neq-Z_b^*$ in the $\pi$ mode and we do not have the permutation symmetry or the $a-b$ symmetry. However, unlike the symmetric case, $K_i$'s do not scale time $t$ and the parameters and hence the ratio $K = \frac{K_a}{K_b}$ emerge as a new parameter.

IV. NORMAL MODES: LINEAR STABILITY ANALYSIS OF FIXED POINTS

Frequencies of small amplitude oscillations about ($Z^*=0, \delta^p=0$) and ($Z^*=0, \delta^p=\pi$), respectively, referred to as the zero mode or the $\pi$ modes, are given by

$$\omega^2 = \frac{1}{2} (K_a \Lambda_a^* + K_b \Lambda_b^*)$$

$$+ \frac{1}{2} \pm \sqrt{(K_a \Lambda_a^* - K_b \Lambda_b^*)^2 + 4K_a K_b \Lambda_{ab}},$$

where

$$\Lambda_a^* = (-1)^p K_a + \Lambda_a,$$

$$\Lambda_b^* = (-1)^p K_b + \Lambda_b,$$

where $p=0$ for the zero mode and $p=1$ for the $\pi$ mode. In the symmetric case, with $\Lambda_a=\Lambda_b$ and $f_a=f_b$, the normal-mode frequencies $\omega_b$ and $\omega_{\pi}$ simplify to

$$\omega_b^2 = K^2 + K(\Lambda \pm \Lambda_{ab})/2,$$

FIG. 1. (Color online) The upper (black) and the lower (yellow) shaded regime corresponds to the parameter values for the existence of stable zero mode and $\pi$ mode with $R=K_a/K_b=1$.

$$\omega_{\pi}^2 = K^2 - K(\Lambda \pm \Lambda_{ab})/2.$$

The condition for the instability of the fixed point is determined when one of the normal-mode frequencies becomes complex. This gives rise to new fixed points where $Z^*_i \neq 0$ resulting in MQST phase where there is a population imbalance between the two wells of the double-well potential for each species. The condition for the existence of MQST is given by

$$f_{ab} \Lambda_{ab}^2 \equiv \Lambda_a^* \Lambda_b^*.$$  

In the zero-mode case, this inequality gives the condition for phase separation (see Sec. V C for further details) of the two species as $Z_a^*=-Z_b^*$. It should be noted that in the weak tunneling limit when $K_i<\Lambda_i$, the above equation is reminiscent of the phase-separation condition, $g_{ab}^2 \equiv g_{a}g_{b}$, obtained using Thomas-Fermi approximation. Our analysis is in fact valid only in the weak tunneling limit, when the renormalized parameters $\Lambda_i$ can be assumed to be linearly related to the bare parameters, namely, the scattering lengths $g_i$ (see the Appendix). Emergence of this important relationship from two independent approaches and approximations strengthens the validity of the criterion for phase separation.

For the parameter values where both the zero and the $\pi$ modes coexist, $\pi$-mode frequencies are smaller than the zero-mode frequencies. Figure 1 shows the values of $\Lambda_a, \Lambda_b$, where the tunneling is governed by the zero mode and the $\pi$ mode. For $\Lambda_b>0$, the regime where the $\pi$ modes exist is small but finite. However, by tuning $\Lambda_b$ to negative values, the $\pi$ modes that have not been seen in earlier studies can be observed. Variation with the parameter $R$, the tunneling ratio for the two species, leads to similar results, with the parameter space for the existence of $\pi$ mode increasing slightly with $R$. The unshaded regime corresponds to MQST phase.

V. TUNNELING DYNAMICS WITH $\Lambda_a=\Lambda_b$

We now describe the numerical solution of the tunneling equations, solved using the standard numerical (sixth order)
achieved by first tuning the 

However, as we discuss below, in a binary mixture, we can 

SYMMETRY-BREAKING AND SYMMETRY-RESTORING

/G9011

/h0011

As seen in the Fig. 3, the motion is in phase

initial population imbalance is small

To achieve this, the system executes a coherent oscillatory dynam-

In our numerics, we set $\Lambda_{ab}=2.13\Lambda$ and study the dynamics for different values of $\Lambda$. These conditions can be achieved by first tuning the $g_0$ via a Feshbach resonance so that $\Lambda_\sigma=\Lambda_\rho$. The variation of $\Lambda$ corresponds to varying the number of atoms in the double-well trap. As already mentioned, $K$ can be eliminated by using $t\rightarrow Kt$ and $\Lambda\rightarrow \Lambda/K$. The dynamics is governed by $\Lambda$ and the initial conditions $Z_a(0), Z_b(0), \phi_a(0),$ and $\phi_b(0).

As we discuss below, tunneling solutions belong to three broad categories:

(i) zero-phase mode, characterized by $\langle \phi_b \rangle=0$;
(ii) $\pi$-phase mode characterized by $\langle \phi_b \rangle=\pi$;
(iii) “running-phase mode” characterized by $\langle \phi_b \rangle$ proportional to $t$, where $\langle A \rangle$ represents the time average of $A$. In the single species case, $\langle \phi_b \rangle=0$ also corresponds to $\langle Z_b \rangle=0$. However, as we discuss below, in a binary mixture, we can have $\langle \phi_b \rangle=0$ but $\langle Z_b \rangle \neq 0$. This gives rise to a broken-symmetry MQST phase in zero modes as well.

A. Zero modes

For $\Lambda<\Lambda_c^0=1.77$, $\phi_a(0)\phi_b(0)=0$ and $|Z_b(0)|<1$, both species execute small amplitude oscillations (such as oscillations of a nonrigid pendulum) with $\langle Z_a(t) \rangle=0$ and $\langle \phi_a(t) \rangle=0$ as shown in Fig. 2. Such modes exhibit quasiperiodic dynamics characterized by superposition of sinusoidal modes with two competing frequencies. As $Z_b(0)$ increases, we see large amplitude nonsinusoidal oscillations. Therefore, in spite of the repulsive interaction between the two conden-

Runge-Kutta method. For small population imbalance, we confirm the dynamics predicted by the fixed points as discussed above. However, numerical solutions also illustrate nonlinear modes not described by the fixed-point analysis. The fact that new features continue to exist in the nonlinear regime assures their robustness.

In our numerics, we set $\Lambda_{ab}=2.13\Lambda$ and study the dynamics for different values of $\Lambda$. These conditions can be achieved by first tuning the $g_0$ via a Feshbach resonance so that $\Lambda_\sigma=\Lambda_\rho$. The variation of $\Lambda$ corresponds to varying the number of atoms in the double-well trap. As already mentioned, $K$ can be eliminated by using $t\rightarrow Kt$ and $\Lambda\rightarrow \Lambda/K$. The dynamics is governed by $\Lambda$ and the initial conditions $Z_a(0), Z_b(0), \phi_a(0),$ and $\phi_b(0).

As we discuss below, tunneling solutions belong to three broad categories:

(i) zero-phase mode, characterized by $\langle \phi_b \rangle=0$;
(ii) $\pi$-phase mode characterized by $\langle \phi_b \rangle=\pi$;
(iii) “running-phase mode” characterized by $\langle \phi_b \rangle$ proportional to $t$, where $\langle A \rangle$ represents the time average of $A$. In the single species case, $\langle \phi_b \rangle=0$ also corresponds to $\langle Z_b \rangle=0$. However, as we discuss below, in a binary mixture, we can have $\langle \phi_b \rangle=0$ but $\langle Z_b \rangle \neq 0$. This gives rise to a broken-symmetry MQST phase in zero modes as well.

A. Zero modes

For $\Lambda<\Lambda_c^0=1.77$, $\phi_a(0)\phi_b(0)=0$ and $|Z_b(0)|<1$, both species execute small amplitude oscillations (such as oscillations of a nonrigid pendulum) with $\langle Z_a(t) \rangle=0$ and $\langle \phi_a(t) \rangle=0$ as shown in Fig. 2. Such modes exhibit quasiperiodic dynamics characterized by superposition of sinusoidal modes with two competing frequencies. As $Z_b(0)$ increases, we see large amplitude nonsinusoidal oscillations. Therefore, in spite of the repulsive interaction between the two conden-

Runge-Kutta method. For small population imbalance, we confirm the dynamics predicted by the fixed points as discussed above. However, numerical solutions also illustrate nonlinear modes not described by the fixed-point analysis. The fact that new features continue to exist in the nonlinear regime assures their robustness.

In our numerics, we set $\Lambda_{ab}=2.13\Lambda$ and study the dynamics for different values of $\Lambda$. These conditions can be achieved by first tuning the $g_0$ via a Feshbach resonance so that $\Lambda_\sigma=\Lambda_\rho$. The variation of $\Lambda$ corresponds to varying the number of atoms in the double-well trap. As already mentioned, $K$ can be eliminated by using $t\rightarrow Kt$ and $\Lambda\rightarrow \Lambda/K$. The dynamics is governed by $\Lambda$ and the initial conditions $Z_a(0), Z_b(0), \phi_a(0),$ and $\phi_b(0).

As we discuss below, tunneling solutions belong to three broad categories:

(i) zero-phase mode, characterized by $\langle \phi_b \rangle=0$;
(ii) $\pi$-phase mode characterized by $\langle \phi_b \rangle=\pi$;
(iii) “running-phase mode” characterized by $\langle \phi_b \rangle$ proportional to $t$, where $\langle A \rangle$ represents the time average of $A$. In the single species case, $\langle \phi_b \rangle=0$ also corresponds to $\langle Z_b \rangle=0$. However, as we discuss below, in a binary mixture, we can have $\langle \phi_b \rangle=0$ but $\langle Z_b \rangle \neq 0$. This gives rise to a broken-symmetry MQST phase in zero modes as well.

A. Zero modes

For $\Lambda<\Lambda_c^0=1.77$, $\phi_a(0)\phi_b(0)=0$ and $|Z_b(0)|<1$, both species execute small amplitude oscillations (such as oscillations of a nonrigid pendulum) with $\langle Z_a(t) \rangle=0$ and $\langle \phi_a(t) \rangle=0$ as shown in Fig. 2. Such modes exhibit quasiperiodic dynamics characterized by superposition of sinusoidal modes with two competing frequencies. As $Z_b(0)$ increases, we see large amplitude nonsinusoidal oscillations. Therefore, in spite of the repulsive interaction between the two conden-

Runge-Kutta method. For small population imbalance, we confirm the dynamics predicted by the fixed points as discussed above. However, numerical solutions also illustrate nonlinear modes not described by the fixed-point analysis. The fact that new features continue to exist in the nonlinear regime assures their robustness.

In our numerics, we set $\Lambda_{ab}=2.13\Lambda$ and study the dynamics for different values of $\Lambda$. These conditions can be achieved by first tuning the $g_0$ via a Feshbach resonance so that $\Lambda_\sigma=\Lambda_\rho$. The variation of $\Lambda$ corresponds to varying the number of atoms in the double-well trap. As already mentioned, $K$ can be eliminated by using $t\rightarrow Kt$ and $\Lambda\rightarrow \Lambda/K$. The dynamics is governed by $\Lambda$ and the initial conditions $Z_a(0), Z_b(0), \phi_a(0),$ and $\phi_b(0).

As we discuss below, tunneling solutions belong to three broad categories:

(i) zero-phase mode, characterized by $\langle \phi_b \rangle=0$;
(ii) $\pi$-phase mode characterized by $\langle \phi_b \rangle=\pi$;
(iii) “running-phase mode” characterized by $\langle \phi_b \rangle$ proportional to $t$, where $\langle A \rangle$ represents the time average of $A$. In the single species case, $\langle \phi_b \rangle=0$ also corresponds to $\langle Z_b \rangle=0$. However, as we discuss below, in a binary mixture, we can have $\langle \phi_b \rangle=0$ but $\langle Z_b \rangle \neq 0$. This gives rise to a broken-symmetry MQST phase in zero modes as well.

A. Zero modes

For $\Lambda<\Lambda_c^0=1.77$, $\phi_a(0)\phi_b(0)=0$ and $|Z_b(0)|<1$, both species execute small amplitude oscillations (such as oscillations of a nonrigid pendulum) with $\langle Z_a(t) \rangle=0$ and $\langle \phi_a(t) \rangle=0$ as shown in Fig. 2. Such modes exhibit quasiperiodic dynamics characterized by superposition of sinusoidal modes with two competing frequencies. As $Z_b(0)$ increases, we see large amplitude nonsinusoidal oscillations. Therefore, in spite of the repulsive interaction between the two conden-

Runge-Kutta method. For small population imbalance, we confirm the dynamics predicted by the fixed points as discussed above. However, numerical solutions also illustrate nonlinear modes not described by the fixed-point analysis. The fact that new features continue to exist in the nonlinear regime assures their robustness.

In our numerics, we set $\Lambda_{ab}=2.13\Lambda$ and study the dynamics for different values of $\Lambda$. These conditions can be achieved by first tuning the $g_0$ via a Feshbach resonance so that $\Lambda_\sigma=\Lambda_\rho$. The variation of $\Lambda$ corresponds to varying the number of atoms in the double-well trap. As already mentioned, $K$ can be eliminated by using $t\rightarrow Kt$ and $\Lambda\rightarrow \Lambda/K$. The dynamics is governed by $\Lambda$ and the initial conditions $Z_a(0), Z_b(0), \phi_a(0),$ and $\phi_b(0).

As we discuss below, tunneling solutions belong to three broad categories:

(i) zero-phase mode, characterized by $\langle \phi_b \rangle=0$;
(ii) $\pi$-phase mode characterized by $\langle \phi_b \rangle=\pi$;
(iii) “running-phase mode” characterized by $\langle \phi_b \rangle$ proportional to $t$, where $\langle A \rangle$ represents the time average of $A$. In the single species case, $\langle \phi_b \rangle=0$ also corresponds to $\langle Z_b \rangle=0$. However, as we discuss below, in a binary mixture, we can have $\langle \phi_b \rangle=0$ but $\langle Z_b \rangle \neq 0$. This gives rise to a broken-symmetry MQST phase in zero modes as well.

A. Zero modes

For $\Lambda<\Lambda_c^0=1.77$, $\phi_a(0)\phi_b(0)=0$ and $|Z_b(0)|<1$, both species execute small amplitude oscillations (such as oscillations of a nonrigid pendulum) with $\langle Z_a(t) \rangle=0$ and $\langle \phi_a(t) \rangle=0$ as shown in Fig. 2. Such modes exhibit quasiperiodic dynamics characterized by superposition of sinusoidal modes with two competing frequencies. As $Z_b(0)$ increases, we see large amplitude nonsinusoidal oscillations. Therefore, in spite of the repulsive interaction between the two conden-
wells but remain phase separated as shown in Fig. 4. As seen in the figure (at \(t=0\)), the dynamics is initiated with positive population imbalance of both species. However, the resulting dynamics corresponds to back and forth motion where the two species swap places between the two wells. In contrast to MQST, the swapping dynamics restores the symmetry of the tunneling solution in the double well. However, the two species remain mostly phase separated, avoiding each other by swapping.

In other words, the swapping phase is characterized by

\[
\langle Z_a(t) \rangle = -\langle Z_b(t) \rangle = 0, \text{ but } \langle Z_a(t) Z_b(t) \rangle \neq 0.
\]

That is, at a given instant of time, the two species are more likely to be found in separate wells. Thus in the swapping mode, the two species oscillate back and forth between the two wells and still manage to avoid each other. The swapping is found to occur in the nonlinear zero mode as well as in the running mode. Furthermore, a transition from MQST to swapping phase can be achieved either by varying \(\Lambda\) (Fig. 5) or by varying the initial conditions (Fig. 6).

### E. Symmetry breaking in \(\pi\) modes: Coexistence phase

For \(\Lambda < \Lambda_c = 0.67\), \(\phi_a(t=0) = \phi_b(t=0) = \pi\) and \(\langle Z_a(t=0) \rangle \ll 1\), both species execute small amplitude oscillations with \(\langle Z_a(t) \rangle = 0\rangle\) and \(\langle \phi(t) \rangle = \pi\), as shown in Fig. 7. Such modes are characterized by superposition of sinusoidal modes with two competing frequencies and the resulting dynamics is in general quasiperiodic. As expected from the fixed-point analysis, the two species with both interspecies and intraspecies repulsive interactions can self-trap in the same well. That is, we have MQST where the species coexist in the same potential well, in spite of repulsive interaction among them.

### F. Swapping in \(\pi\) modes

As illustrated in Fig. 8, within the \(\pi\)-mode phase, if the dynamics of the two species is initiated in separate wells, that is, \(Z_a(t=0)\) and \(Z_b(t=0)\) have opposite signs, the MQST phase can be destroyed when the initial population imbalance increases beyond a critical value. The tunneling solutions become symmetric as MQST is replaced by swapping modes. In this case the swapping can be viewed as the two species “chasing” each other.

It should be noted that the swapping dynamics in the zero and the \(\pi\) modes is very similar. However, swapping in the zero mode corresponds to two species avoiding each other while swapping in the \(\pi\) mode corresponds to one component chasing the other. This is because in the zero mode, species prefer residing in the separate wells while in the \(\pi\)
mode, they like to stay in the same well. This unique type of coherence between the two different species is one of the most fascinating aspects of the binary mixture dynamics in double-well potential.

VI. EXPERIMENTAL REALIZATION

The effects described in this paper should be realizable for condensate mixtures that already exist in the laboratory. One example in particular is a mixture of $^{85}$Rb and $^{87}$Rb atoms that has been created in several recent experiments at JILA [7,12]. This system is relevant to the analysis in this paper because the scattering length, $a_{85-85}$, that characterizes the interaction between $^{85}$Rb atoms is tunable by an external magnetic field via a Feshbach resonance centered at approximately 155 [13]. Additionally, the interspecies scattering length, $a_{85-87}$, is also tunable with two Feshbach resonances (for a $|2\rangle_{85}/|1\rangle_{87}$ collision) located at approximately $B=267$ G and $B=356$ G.

In the most recent experiment [12], a $^{85}$Rb/$^{87}$Rb BEC mixture was produced by trapping a thermal-gas sample of the mixture and performing evaporative cooling on the $^{85}$Rb which sympathetically cools the $^{87}$Rb. The cold gas mixture is then transferred to an optical trap that provides tight confinement transverse to the trapping beam and loose confinement along the beam. If an additional pair of beams were applied along this direction as was done in the Albiez experiment [5], it would create a setup to which the analysis in this paper would apply.

VII. SUMMARY

Existence of a variety of BEC species with tunable inter- and intraspecies scattering lengths makes BEC mixtures one of the most attractive candidates for exploring novel phenomena involving quantum coherence and nonlinearity. Our analysis, based on the two-mode GP equation for the two interacting species of BEC in a double-well trap unveils a variety of phenomena describing broken symmetry as well as subsequent restoration of symmetry, as we change the parameters or the initial conditions. Such coherence is found to exist over a broad range of parameters, establishing the robustness of the effects.

To make direct comparison with experiments, we need to solve the coupled GP equations to obtain various parameters of the effective coupled pendulum system in terms of the microscopic parameters of the system and work in this direction is in progress. Furthermore, by quantizing the Hamiltonian (coupled pendulum or the spin Hamiltonian), we hope to study quantum dynamics of number fluctuations that may code the emergence of new quantum phases in the system.

ACKNOWLEDGMENTS

R. B. thanks the council of Scientific and Industrial Research, India for financial support.

APPENDIX: TWO-MODE EQUATION PARAMETERS

We follow the improved two-mode approximation [9] where the localized spatial modes $\chi_{1(2)}^{(a)}$ in each well of the double well are constructed by using $\pm$ combinations of the symmetric ($\chi_s$) and the antisymmetric ($\chi_a$) functions, $\chi_{1,2}^{(a)} \equiv \chi_{s}^{(a)} \pm \chi_{a}^{(a)}$. Here $\chi_{s}^{(a)}(x) = \chi_{a}^{(a)}(-x)$ are the solutions of the time-independent coupled GPE equations

$$\mu_j^{(a)} \chi_j^{(a)}(x) = -\frac{k^2}{2m_a} \frac{\partial^2 \chi_j^{(a)}}{\partial x^2} + V_a(x) \chi_j^{(a)} + g_{aa} N_{a} |\chi_j^{(a)}|^2 \chi_j^{(a)} + g_{ab} N_{a} |\chi_j^{(b)}|^2 \chi_j^{(a)}.$$

(A1)
\[
\mu^{(b)}_j \chi^{(b)}_j(x) = -\frac{\hbar^2}{2m_b} \frac{\partial^2 \chi^{(b)}_j}{\partial x^2} + V_b(x) \chi^{(b)}_j + g_{ab} \chi^{(a)}_j \chi^{(b)}_j \chi^{(b)}_j + g_{ba} \chi^{(a)}_j \chi^{(b)}_j \chi^{(b)}_j,
\]

where \( j = \pm \) and labels the ground and the first-excited state of the system.

Thus \( \chi^2(x^2) \) is localized in the left (right) well of the double-well potential. We define \( g_\alpha = g_N \hbar/(x=\alpha, b, ab) \) and follow the methodology described in the earlier studies for the single-component problem \(^9\) obtaining the following parameters that determine the tunneling dynamics for the binary mixture:

\[
\Lambda_\alpha = g_{ab} \int (2(\chi^{(a)}_\alpha x^{(b)}_\alpha)^2 - 1/8 [(\chi^{(a)}_\alpha)^2 - (\chi^{(b)}_\alpha)^2] \text{d}r,
\]

\[
\Lambda_{ab} = 2g_{ab} \int (\chi^{(a)}_\alpha \chi^{(b)}_\alpha \chi^{(b)}_\alpha \chi^{(b)}_\alpha \text{d}r,
\]

\[
\gamma^\pm_{ab} = g_{ab} \int [(\chi^{(b)}_\alpha)^2] \text{d}r,
\]

\[
\gamma_{al}(b) = g_{ab} \int [(\chi^{(a)}_\alpha)^2] \text{d}r,
\]

\[
\Delta \gamma_{al}(b) = \gamma_{al}(b) - \gamma_{al}(b),
\]

\[
\Delta \gamma_{ab} = g_{ab} \int [(\chi^{(a)}_\alpha \chi^{(b)}_\alpha)^2 - (\chi^{(b)}_\alpha \chi^{(b)}_\alpha)^2] \text{d}r,
\]

\[
\Delta \gamma_{ab} = g_{ab} \int [(\chi^{(a)}_\alpha \chi^{(b)}_\alpha)^2 - (\chi^{(b)}_\alpha \chi^{(b)}_\alpha)^2] \text{d}r,
\]

\[
K_a = [\Delta E_a - f_a \Delta \gamma_a - f_a D_{ab}] / \hbar,
\]

\[
K_b = [\Delta E_b - f_b \Delta \gamma_b - f_b D_{ab}] / \hbar,
\]

\[
C_a = (\gamma_a^+ + \gamma_a^- - 2 \tilde{\gamma}_a) / 2 \hbar,
\]

\[
C_b = (\gamma_b^+ + \gamma_b^- - 2 \tilde{\gamma}_b) / 2 \hbar,
\]

\[
D_{ab} = (\Delta \gamma_{ab} - \Delta \gamma_{ab}) / 2 \hbar.
\]

Here \( \Delta E = \mu^+ - \mu^- \) represents the difference in the chemical potential between the symmetric ground and (antisymmetric) first-excited state of the coupled time-independent GPE equations. It should be noted that within the two-mode ansatz, the tunneling Eqs. (5)–(8) with dressed parameters as given above are exact.

The above equations signify the importance of spatial modes in determining the temporal dynamics of the two species in the double-well potential. The \( \Lambda_\alpha \) describes the renormalization of the bare coupling parameters \( g \), due to averaging over the spatial degrees of freedom. As expected, the renormalized couplings \( \Lambda_{ab} \) depend upon the corresponding spatial mode \( \chi_{ab} \) while the coupling \( \Lambda_{ab} \) is determined by the overlap integral between the two species. The tunneling parameters \( K \) that depend only on \( \Delta E \) for noninteracting systems are appropriately affected by the interaction between the species.


