Theory and Applications of Graphs

Volume 4 | Issue 1 Article 2

2017

An Eternal Domination Problem in Grids

William Klostermeyer
University of North Florida, klostermeyer@hotmail.com

Margaret-Ellen Messinger
mmessinger@mta.ca

Alejandro Angeli Ayello
aloangeli@gmail.com

Recommended Citation

Klostermeyer, William; Messinger, Margaret-Ellen; and Angeli Ayello, Alejandro (2017) "An Eternal Domination Problem in Grids," *Theory and Applications of Graphs*: Vol. 4: Iss. 1, Article 2.
DOI: 10.20429/tag.2017.040102
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol4/iss1/2

An Eternal Domination Problem in Grids

Cover Page Footnote

M.E.Messinger acknowledges research support from NSERC (grant application 356119-2011 and DDG-2016-00017) and Mount Allison University. A. Angeli Ayello acknowledges research support from the Graphs and Games Collaborative Research Group, funded by the Atlantic Association for Research in Mathematical Sciences. The authors also thank the referees for their detailed and thoughtful suggestions.

Abstract

A dynamic domination problem in graphs is considered in which an infinite sequence of attacks occur at vertices with mobile guards; the guard at the attacked vertex is required to vacate the vertex by moving to a neighboring vertex with no guard. Other guards are allowed to move at the same time, and before and after each attack, the vertices containing guards must form a dominating set of the graph. The minimum number of guards that can defend the graph against such an arbitrary sequence of attacks is called the m-eviction number of the graph. In this paper, the m-eviction number is determined exactly for small grids and upper bounds are given for all $m \geq n \geq 8$.

1 Introduction

In this paper, we shall be concerned with defending a finite, undirected graph $G = (V, E)$ against an infinite sequence of attacks that occur, one at a time, at vertices. This is sometimes also called *protecting* the graph. A variety of graph protection problems and models have been considered in the literature of late, see the survey [16]. In the usual protection model, each attack is defended by a mobile guard that is sent to the attacked vertex from a neighboring vertex.

A *dominating set* of graph G is a set $D \subseteq V$ such that for each $u \in V - D$, there exists an $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets of G is the *domination number*, $\gamma(G)$. For any dominating set D and $x \in D$, we say that $v \in V - D$ is an *external private neighbor* of x if v is adjacent to x but to no other vertex in D. A dominating set can be viewed as being able to protect a graph against a single attack at a vertex.

For each $i \geq 1$, let $D_i \subseteq V$ be a dominating set with one *guard* located at each vertex of D_i. A vertex is said to be *occupied* if a guard is located on it and *unoccupied* otherwise. At most one guard can be located on any vertex at any one time. In the *eternal dominating set problem*, we aim to protect a graph against any infinite sequence of attacks at vertices. In this problem, we may assume that each attack occurs at an unoccupied vertex. Following an attack at a vertex $r_i \in V - D_i$, one or several guards (depending on the exact nature of the model) move along edges to adjacent vertices, with one guard moving to r_i, thus occupying the vertices in the set D_{i+1}. This is called *defending* an attack. Once the guards move to configuration D_{i+1}, the next attack, at a vertex $r_{i+1} \in V - D_{i+1}$, occurs and must be defended.

The minimum number of guards required to protect the graph, i.e., to defend it against each attack in any infinite sequence of attacks, is called the *eternal domination number* $\gamma^{\infty}(G)$ (if only one guard is allowed to move at a time), or the m-*eternal domination number* $\gamma_{\mathrm{m}}^{\infty}(G)$ (if any number of guards are allowed to move in response to an attack). The eternal and m-eternal domination problems were introduced in [2] and [9], respectively. One may think of these eternal domination problems as two-player games played between players that alternate turns: a *defender*, who chooses each D_i (and so the defender can be thought of as moving first, in that they choose the initial dominating set), and an *attacker*, who then chooses each r_i. The defender wins the game if they can successfully defend any sequence of attacks and the attacker wins otherwise. For example, one can observe that $\gamma_{\rm m}^{\infty}(C_5) = 2$ and $\gamma^{\infty}(C_5) = 3$. See [7] for the combinatorial game variant of the eternal dominating set problem.

The primary focus of this paper is on a variation of the eternal domination problem known as the *eternal dominating set eviction problem*, or simply the *eternal eviction problem*. In this problem, each attack in the infinite attack sequence occurs at a vertex $r_i \in D_i$, i.e., each attack occurs at an occupied vertex. When a vertex is attacked, one or several guards (again, depending on the nature of the model) move along edges to adjacent vertices, with the guard at r_i moving to an unoccupied neighbor and no guard moving to r_i at the same time, so as to form the guard set D_{i+1}. That is, the guard at an attacked vertex is *evicted* from that vertex.

We emphasize the following three facts in the eviction problem: (i) D_i is a dominating set, (ii) $r_i \in D_i - D_{i+1}$ and (iii) if r_i has no unoccupied neighbor, then no action is taken on the part of the defender. The minimum number of guards that can protect the graph according to this model is the *eternal eviction number* $e^{\infty}(G)$ (if only the guard on r_i moves) or the m-*eternal eviction number* $e_{\rm m}^{\infty}(G)$ (if the guard on r_i moves and all other guards may also move to neighboring vertices, if they so choose). The latter model is sometimes called the *all-guards move* model. In the latter model, each D_i is called an m-*eternal eviction set* of G, or sometimes just an *eviction set*, for short. It is important to remember that in the m-eternal eviction problem, the attacked vertex must remain unoccupied at least until the next vertex is attacked. It is known from [14] that $\gamma(G) \leq e_{\rm m}^{\infty}(G) \leq \alpha(G)$, where $\alpha(G)$ denotes the independence number of G. Further, $\gamma_{\rm m}^{\infty}(G)$ and $e_{\rm m}^{\infty}(G)$ are not directly comparable, as $\gamma_{\rm m}^{\infty}(K_{1,m}) = 2 < e_{\rm m}^{\infty}(K_{1,m}) = m$ when $m > 2$, yet $\gamma_{\rm m}^{\infty}(G) > e_{\rm m}^{\infty}(G)$ for the graph G consisting of $K_{2,5}$ with an edge added between the two vertices in the maximal independent set of cardinality two.

As another simple example, observe that

$$e_{\rm m}^{\infty}(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \text{ when } n \not\equiv 0 \pmod 3 \tag{1}$$

and

$$\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil < e_{\rm m}^{\infty}(P_n) = \left\lceil \frac{n}{3} \right\rceil + 1 \text{ when } n \equiv 0 \pmod 3. \tag{2}$$

The eternal eviction problems were introduced in [14] and further studied in [15]. Motivated by the study of domination in grid graphs in, for example, [1, 6, 11, 13] and eternal domination in grid graphs, see [8, 10], we consider the m-eternal eviction problem in grid graphs. Recall that the $m \times n$ grid graph is the Cartesian product of paths on m and n vertices, which we denote as $P_m \square P_n$.

2 Eviction on $2 \times n$ and $4 \times n$ grids

In this section, we shall determine the m-eternal eviction number of $2 \times n$ and $4 \times n$ grids. In achieving these results, it was discovered that one of the proofs used in describing the domination numbers of $4 \times n$ grids in [13] is incorrect, though the result is correct. For completeness, we present a corrected proof in the appendix.

A *total-switch* of a dominating set D into a dominating set D' is a simultaneous replacement of all vertices in D, where each vertex $v_i \in D$ is replaced by a neighbor $z_i \in D', z_i \in N(v_i)$ such that $|D| = |D'|$ and $D \cap D' = \emptyset$. Total-switches were introduced in [3] in the context of independent sets, rather than dominating sets. Observe that for a total switch to occur, it must be that D and D' are disjoint dominating sets with a perfect matching between them. Though disjoint dominating sets have been studied in the literature, see for example [12], it appears the concept of having a matching between them has not been studied. Let $DD_{\mathrm{m}}(G)$ denote the size of smallest disjoint dominating sets D and D' such that there is a perfect matching between them. If G has no such sets, take $K_{1,m}$, $m \geq 2$, for example, then define $DD_{\mathrm{m}}(G) = \infty$. It is obvious that $e^{\infty}_{\mathrm{m}}(G) \leq DD_{\mathrm{m}}(G)$, for all graphs G. Note that $DD_{\mathrm{m}}(G)$ and $\alpha(G)$ are, in general, not comparable, since there exist graphs for which each exceeds the other. The graph $G = C_4$ is an example of a graph where $e^{\infty}_{\mathrm{m}}(G) = DD_{\mathrm{m}}(G)$. An example of a graph G with $e^{\infty}_{\mathrm{m}}(G) = 3 < 4 = DD_{\mathrm{m}}(G)$ is given in Figure 1. It can be shown using the concepts of [14, 15] that for any tree T, if $DD_{\mathrm{m}}(T)$ is finite (which it is not always), then $e^{\infty}_{\mathrm{m}}(T) = DD_{\mathrm{m}}(T)$ (the proof of this requires details beyond the scope of this paper). In Section 5, we ask whether this is the case for all grid graphs.

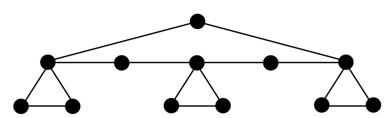


Figure 1: Graph G with $e^{\infty}_{\mathrm{m}}(G) = 3 < 4 = DD_{\mathrm{m}}(G)$.

Theorem 1 *For all* $n \in \mathbb{Z}^+$, $e^{\infty}_{\mathrm{m}}(P_2 \square P_n) = \gamma(P_2 \square P_n) = \lceil \frac{n+1}{2} \rceil$.

Proof: From [13], we know $\gamma(P_2 \square P_n) = \lceil \frac{n+1}{2} \rceil$. Place γ guards on vertices of $P_2 \square P_n$ as indicated by the squares in Figure 2. When a vertex with a guard is attacked, the guards perform a total-switch: guards in row 1 move to row 2 and vice versa.

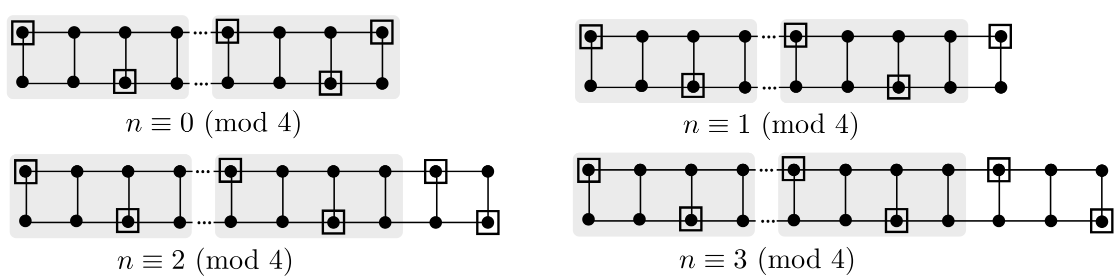


Figure 2: Eviction sets on $2 \times n$ grids.

■

It follows that $e^{\infty}_{\mathrm{m}}(P_2 \square P_n) = DD_{\mathrm{m}}(P_2 \square P_n)$ for all $n \geq 1$.

The value of $\gamma(P_4 \square P_n)$ is known to be $n+1$ for $n \in \{1, 2, 3, 5, 6, 9\}$ and n otherwise, see either [6, 11, 13] or the appendix. Table 1 in [1] is a convenient presentation of $\gamma(P_m \square P_n)$ for small values of m, n.

Lemma 2 *For all $n = 8, 10, 11, 12$ and $n \geq 14$, $\gamma(P_4 \square P_n) = e_{\mathrm{m}}^{\infty}(P_4 \square P_n)$.*

Proof: From [13], we know $\gamma(P_4 \square P_n) = n$ for $n = 8$ and $n \geq 10$. Suppose $n \equiv 0 \pmod 4$: For each 4×4 sub-grid, place guards on vertices as shown in Figure 3 (a); then four guards dominate each 4×4 sub-grid. If a vertex is attacked, the guards move as illustrated by arrows in Figure 3 (a): a total-switch is performed. Note that a total-switch of the entire $4 \times n$ grid is also possible.

Suppose $n \equiv 2 \pmod 4$: Let $n = 4k + 2$ and consider a decomposition of the $4 \times n$ grid into one 2×4 sub-grid and k 4×4 sub-grids where the 2×4 sub-grid is adjacent to two 4×4 sub-grids. In each 4×4 sub-grid, place guards on vertices as shown in Figure 3 (a) and in the 2×4 sub-grid, place guards on vertices as shown in Figure 3 (b). Note that each sub-grid is dominated. If a vertex is attacked, then all guards move according to the arrows shown in Figure 3 (b); performing a total-switch of the grid.

Suppose $n \equiv 3 \pmod 4$: This is similar to the case when $n \equiv 2 \pmod 4$ and the guard shift pattern is shown in Figure 3 (c). Observe that we have three guards in the middle 4×3 sub-grid although $\gamma(P_3 \square P_4) = 4$. Thus when a vertex is attacked, we perform a total-switch on the entire $4 \times n$ grid using the pattern shown in Figure 3 (c). Upon the next attack, we perform a total-switch back to the initial guard configuration.

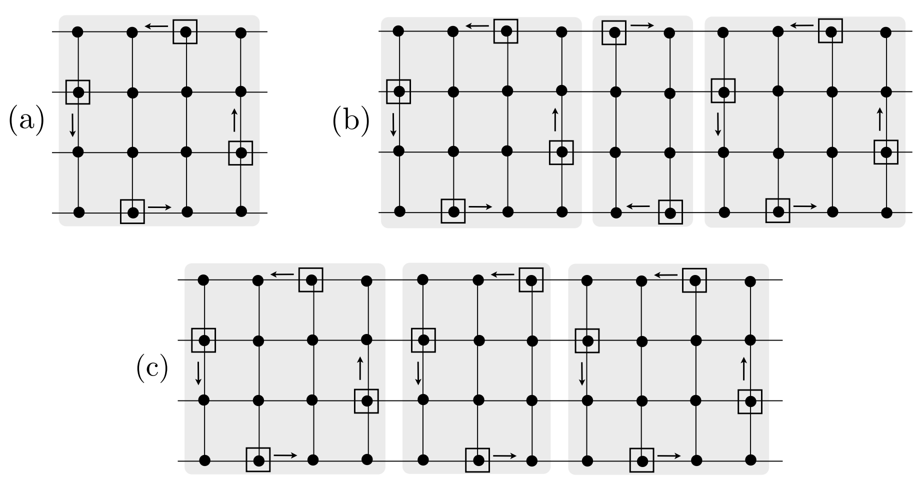


Figure 3: Guard configurations for $n \equiv 0, 2, 3 \pmod 4$ for the $4 \times n$ grid.

Suppose $n \equiv 1 \pmod 4$: This is similar to the case when $n \equiv 2 \pmod 4$, noting that we configure the "middle" sub-grid as a 4×9 sub-grid as shown in Figure 4. Observe that we locate nine guards at vertices in the middle 4×9 sub-grid. Thus when a vertex is attacked, we perform a total-switch on the entire $4 \times n$ grid using the pattern shown in Figure 4. Upon the next attack, we perform a total-switch back to the initial guard configuration.

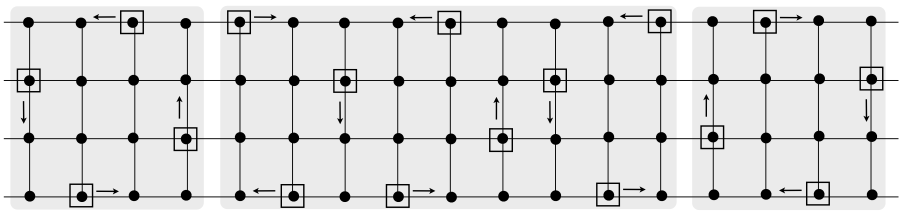


Figure 4: Guard configuration for $n \equiv 1 \pmod 4$ for the $4 \times n$ grid.

■

To complete the analysis of $4 \times n$ grids, we must consider the cases when $n \leq 7$, $n = 9$ and $n = 13$. The cases when $n \leq 2$ are trivial. The case when $n = 3$ can be handled by partitioning the 4×3 grid into two 2×3 grids and using two guards in each. The case when $n = 4$ is identical to the $n \equiv 0 \pmod 4$ case above.

When $n = 5$, partition the grid into a 4×4 and a 4×1 grid, using four and two guards in each, respectively.

When $n = 6$, partition the grid into a 4×4 and a 4×2 grid, using four and three guards in each, respectively.

When $n = 7$, $P_4 \square P_7$ has two disjoint dominating sets of cardinality seven that are joined by a perfect matching (vertices 2 and 6 on row 1, vertex 4 on row 2, vertices 1 and 7 on row 3, and vertices 3 and 5 on row 4 are one of the sets and the other can be obtained by a horizontal reflection).

When $n = 9$, partition the grid into two 4×4 grids and one 4×1 grid, using four guards in the 4×4 grids and two guards in the 4×1 grid.

When $n = 13$, $P_4 \square P_{13}$ has two disjoint dominating sets of cardinality 13 that are joined by a perfect matching (see the first 13 columns of the $4 \times n$ grid shown in Figure 3.1 of [4] and its horizontal reflection).

Thus we have the following result.

Theorem 3 $e_{\mathrm{m}}^{\infty}(P_4 \square P_n) = DD_{\mathrm{m}}(P_4 \square P_n) = \gamma(P_4 \square P_n) = \begin{cases} n+1 & \text{if } n \in \{1,2,3,5,6,9\} \\ n & \text{otherwise.} \end{cases}$

3 Eviction on $3 \times n$ grids

The exact value of the m-eternal domination number has yet to be determined exactly for all $3 \times n$ grids, see [8, 10, 16]. Thus it is not surprising that the analysis of the m-eternal eviction number of $3 \times n$ grids is slightly more involved than that seen in the previous section.

Table 1 gives the values of domination parameters for $3 \times n$ grids for some small values of n. The values for e_{m}^{∞} for $n = 1, 2, 3$ are trivial to determine while the values of e_{m}^{∞} for $n \geq 4$ are determined in this section. The values for $\gamma_{\mathrm{m}}^{\infty}$ are taken from [8, 10]. Note that Table 1 illustrates that there are some values of n for which $\gamma \neq \gamma_{\mathrm{m}}^{\infty} \neq e_{\mathrm{m}}^{\infty}$.

$3 \times n$	γ	γ_m^∞	e_m^∞
3×1	1	2	2
3×2	2	2	2
3×3	3	3	3
3×4	4	4	4
3×5	4	5	5
3×6	5	6	6
3×7	6	7	6

$3 \times n$	γ	γ_m^∞	e_m^∞
3×8	7	8	7
3×9	7	8	8
3×10	8	9	9
3×11	9	10	10
3×12	10	11	10
3×13	10	12	11

Table 1: Domination parameters for small n.

Theorem 4 *For $n \geq 4$ and $n \equiv 0 \pmod 4$, $e_m^\infty(P_3 \square P_n) = \gamma(P_3 \square P_n) = \lceil \frac{3n+1}{4} \rceil$.*

Proof: Let $n = 4k$ for $k \in \mathbb{Z}^+$. From [13], $\gamma(P_3 \square P_{4k}) = \lceil \frac{3(4k)+1}{4} \rceil = 3k + 1$. Consider the configuration of guards indicated by squares in Figure 5(a). The graph $P_3 \square P_{4k}$ is decomposed into k vertex disjoint 3×4 sub-grids and each 3×4 sub-grid is assigned 3 guards, with the exception of the right-most 3×4 sub-grid, which is assigned 4 guards. Clearly the guards form a dominating set on $P_3 \square P_{4k}$. In (a), given an attack at any vertex with a guard, the guards can move to (b). As the set of vertices with guards in Figure 5(a) and Figure 5(b) are disjoint, each attack can be handled by a total-switch. Thus, $e_m^\infty(P_3 \square P_{4k}) \leq 3k + 1 = \gamma(P_3 \square P_{4k})$. ■

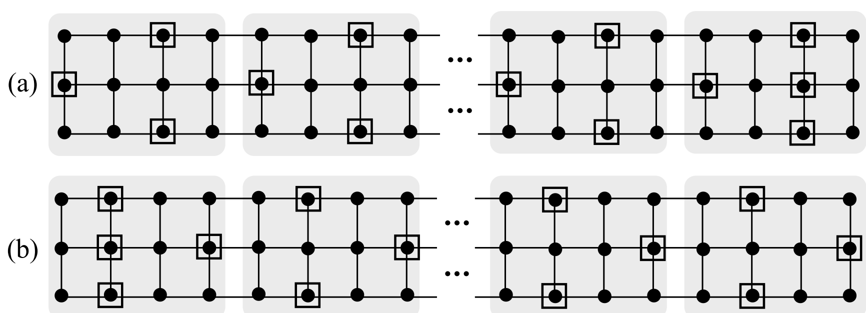


Figure 5: A total switch on the $3 \times 4k$ grid.

Lemma 5 *For $n \geq 5$ and $n \equiv 1 \pmod 4$, $e_m^\infty(P_3 \square P_n) \leq \gamma(P_3 \square P_n) + 1$.*

Proof: Let $n = 4k + 1$ for $k \in \mathbb{Z}^+$. From [13], $\gamma(P_3 \square P_{4k+1}) + 1 = \lceil \frac{3(4k+1)+1}{4} \rceil + 1 = 3k + 2$. Consider the configuration of $3k + 2$ guards indicated by squares in Figure 6(a). Given an attack at any vertex with a guard in (a), observe that in response, the guards can move to the configuration in (b) or a reflection (over the horizontal axis) of (b). The intersection between the dominating sets given in (a) and (b) contains only one vertex: the top vertex in column 3 of the second block. If that vertex is attacked, then the guards can move from (a) to a reflection (over the horizontal axis) of (b), instead of moving to (b). Otherwise, guards can move from (a) to (b). Consequently, $e_m^\infty(P_3 \square P_{4k+1}) \leq 3k + 2 = \gamma(P_3 \square P_{4k+1}) + 1$. ■

DOI: 10.20429/tag.2017.040102

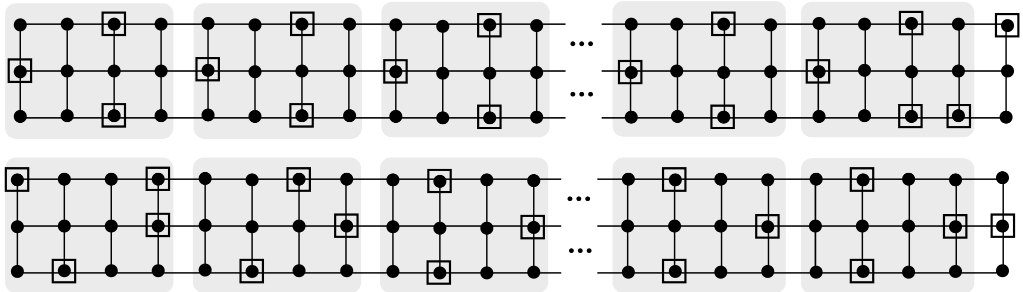


Figure 6: Eviction sets on the $3 \times (4k+1)$ grid.

Lemma 6 *Let D be a dominating set of $P_3 \square P_n$. For any $\ell \in \{1, 2, \ldots, \lfloor n/4 \rfloor\}$, there are at least 3ℓ guards in the first (and last) 4ℓ columns.*

Proof: Let D be a dominating set of $P_3 \square P_n$ and let $\ell \in \{1, 2, \ldots, \lfloor n/4 \rfloor\}$. A guard is located at each vertex of D. Consider the first 4ℓ columns of $P_3 \square P_n$. Since the vertices of column 4ℓ could be dominated by guards in column $4\ell+1$, there must be at least $\gamma(P_3 \square P_{4\ell-1})$ guards in the first 4ℓ columns of $P_3 \square P_n$ in order for D to form a dominating set. From [13], $\gamma(P_3 \square P_{4\ell-1}) = \lceil \frac{3(4\ell-1)+1}{4} \rceil = 3\ell$ and the result follows. ■

Observation 7 *Let H_3 be a subgraph induced from $P_3 \square P_3$ by the deletion of a vertex of degree 2 or 3. Then $\gamma(H_3) \geq 3$.*

The two possibilities for graph H_3 are given in (a) and (b) of Figure 7. The result can be easily verified by inspection.

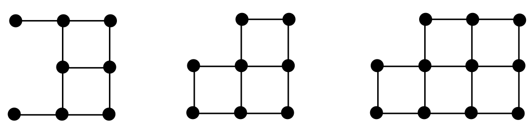


Figure 7: The graphs H_3 and H_4.

Lemma 8 *Let H_4 be the subgraph induced from $P_3 \square P_4$ by the deletion of a vertex of degree 2, as shown in Figure 7(c). Then $\gamma(H_4) \geq 4$.*

Proof: Since $\Delta(H_4) = 4$, it follows that $\gamma(H_4) \geq 3$. Suppose to the contrary that D is a dominating set of H_4 with $|D| = 3$.

If column 1 contains no vertex of D, then both the middle and bottom vertices of column 2 must be in D. This leaves one vertex of D to dominate both the top vertex of column 3 and the bottom vertex of column 4, which is not possible.

If column 1 contains 1 vertex of D, then by Observation 7, at least 3 vertices of D must be in columns 2, 3 and 4, which contradicts $|D| = 3$.

If column 1 contains 2 vertices of D, then one vertex of D must dominate both the top vertex of column 3 and the bottom vertex of column 4, which is not possible. ■

Lemma 9 *Let D be a minimum dominating set of $P_3 \square P_n$. If for some $i \geq 1$, at most one vertex in column $4i+1$ is occupied by a guard and there are exactly $3i$ guards in the first $4i$ columns, then a guard must occupy the middle vertex in column $4i+1$.*

Proof: Let D be a minimum dominating set of $P_3 \square P_n$. We locate one guard at each vertex of D. Suppose that for some $i \geq 1$, there is at most one guard in column $4i+1$ and exactly $3i$ guards in the first $4i$ columns. We first observe there must be a guard in column $4i+1$. Otherwise, because $\gamma(P_3 \square P_{4i}) = \lceil \frac{3(4i)+1}{4} \rceil = 3i+1$ by [13], we have a contradiction as there are at most $3i$ guards in the first $4i$ columns. Thus, there is exactly one guard in column $4i+1$.

Suppose $i > 1$ (the $i = 1$ case is considered later) and for a contradiction, assume the guard in column $4i+1$ is not located at the middle vertex; w.l.o.g. the guard is located at the top vertex. By Lemma 6, there are at least $3(i-1)$ guards in the first $4(i-1)$ columns. This leaves at most 3 guards in columns $4i-3, 4i-2, 4i-1, 4i$. It is easy to see that 2 guards are not sufficient to dominate the middle and bottom vertices of column $4i$ as well as the three vertices in columns $4i-1$ and $4i-2$. Thus, there must be 3 guards in columns $4i-3, 4i-2, 4i-1, 4i$. Since one guard cannot dominate the middle and bottom vertices in both columns $4i-1$ and $4i$, there can be at most one guard in column $4i-3$. Finally, we observe there cannot be a guard located at the middle vertex of column $4i-3$ because two guards cannot dominate the top and middle vertices of column $4i-2$, the three vertices of column $4i-1$ and the middle and bottom vertices of column $4i$.

Consequently, there are exactly $3(i-1)$ guards in the first $4(i-1)$ columns and there is exactly one guard in column $4(i-1)+1 = 4i-3$ and the guard is not located at the middle vertex. Applying the argument repeatedly, we conclude there are exactly 3 guards in the first 4 columns and there is exactly one guard in column 5 and the guard in column 5 is not located at the middle vertex. By Lemma 8, it is not possible for 3 guards to dominate the vertices in the first 3 columns as well as the middle and w.l.o.g. bottom vertices of column 4. Therefore, our assumption that the guard in column $4i+1$ was not located at the middle vertex was false. ∎

Lemma 10 *For $n \geq 5$ and $n \equiv 1 \pmod 4$, $e^{\infty}_{\mathrm{m}}(P_3 \square P_n) > \gamma(P_3 \square P_n)$.*

Proof: Let $n = 4k+1$ for some $k \in \mathbb{Z}^+$. Assume to the contrary that $e^{\infty}_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n)$. By [13], $\gamma(P_3 \square P_n) = \lceil \frac{3n+1}{4} \rceil = 3k+1$. Consider $3k+1$ guards placed at the vertices of a minimum dominating set of $P_3 \square P_n$.

By Lemma 6 (applied from right-to-left), there must be at least $3k$ guards in the last $4k$ columns. This leaves at most one guard in column 1.

First, suppose there are no guards in column 1. Then there must be three guards in column 2. By Lemma 6, there are at least $3k-3$ guards in the last $4k-4 = n-5$ columns. This leaves at most one guard in columns $3, 4, 5$. Obviously then, one guard must be located at the middle vertex of column 4 and exactly $3k-3$ guards must be located in the last $4k-4 = n-5$ columns. Consequently, there must be a guard at the top and bottom vertices of column 6, else the vertices of column 5 are not all dominated. This forces at least six guards to be located in the first six columns.

By Lemma 6, there are at least $3k-6$ guards in the last $4k-8=n-9$ columns. Having seven guards in the first six columns would result in no guards being in each of columns $7,8,9$, which cannot happen. Therefore, there are exactly six guards in the first six columns, exactly $3k-6$ guards in the last $n-9$ columns and exactly one guard in columns $7,8,9$. Obviously, that guard must be located at the middle vertex of column 8. So there is no guard in column 9. Therefore, the $3k-6$ guards in the last $n-9$ columns must dominate all vertices in the last $n-9$ columns. As $\gamma(P_3 \square P_{n-9}) = \gamma(P_3 \square P_{4k-8}) = (4k-8) - \lfloor \frac{4k-9}{4} \rfloor = (4k-8)-(k-3) = 3k-5$, this cannot happen. Therefore, there must be a guard in column 1.

By Lemma 9, the guard in column 1 is located at the middle vertex. By attacking that vertex, the guard moves away and we are left without a dominating set. ■

From Lemma 10 and Lemma 5, we immediately get the following.

Corollary 11 *For $n \geq 5$ and $n \equiv 1 \pmod 4$, $e_{\mathrm{m}}^{\infty}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1 = \lceil \frac{3n+5}{4} \rceil$.*

By adding a guard in the same location in the second block of graphs in Figures 6 (a) and 6 (b), adjacent to the (unique) guard the two configurations have in common, one can see that the following is true.

Observation 12 *$DD_{\mathrm{m}}(P_3 \square P_n) \leq \gamma(P_3 \square P_n) + 2$ when $n \equiv 1 \pmod 4$.*

The following observation was stated in [10] for the eternal domination problem, but clearly applies to the eviction problem as well.

Observation 13 *If $e_{\mathrm{m}}^{\infty}(P_3 \square P_n) \leq t$ and $e_{\mathrm{m}}^{\infty}(P_3 \square P_m) \leq r$, then $e_{\mathrm{m}}^{\infty}(P_3 \square P_{m+n}) \leq t + r$.*

Lemma 14 *For $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$, $e_{\mathrm{m}}^{\infty}(P_3 \square P_n) \leq \gamma(P_3 \square P_n) + 1$.*

Proof: Suppose $n = 4k+2$ for some integer $k \in \mathbb{Z}^+$. Then

$$\begin{aligned}
e_{\mathrm{m}}^{\infty}(P_3 \square P_{4k+2}) &\leq e_{\mathrm{m}}^{\infty}(P_3 \square P_{4k}) + e_{\mathrm{m}}^{\infty}(P_3 \square P_2) && \text{by Observation 13} \\
&\leq (3k+1) + 2 && \text{by the proof of Theorem 4 and Table 1} \\
&= \gamma(P_3 \square P_{4k+2}) + 1 && \text{from [13].}
\end{aligned}$$

Suppose $n = 4k+3$ for some integer $k \in \mathbb{Z}^+$. Using Observation 13, the proof of Theorem 4, Table 1 and [13], the desired result can be obtained for $e_{\mathrm{m}}^{\infty}(P_3 \square P_{4k+3})$. ■

Lemma 15 *For $n \geq 6$ and $n \equiv 2 \pmod 4$, $e_{\mathrm{m}}^{\infty}(P_3 \square P_n) > \gamma(P_3 \square P_n)$.*

Proof: Let $n = 4k+2$ for some $k \in \mathbb{Z}^+$. Suppose, by way of contradiction, that $e_{\mathrm{m}}^{\infty}(P_3 \square P_{4k+2}) = \gamma(P_3 \square P_{4k+2})$. By [13], $\gamma(P_3 \square P_{4k+2}) = 3k+2$. We will show that every minimum dominating set on $P_3 \square P_{4k+2}$ contains the middle vertex of column 1. Then in the eviction problem, we simply attack the middle vertex of column 1 to show $e_{\mathrm{m}}^{\infty}(P_3 \square P_{4k+2}) > \gamma(P_3 \square P_{4k+2})$.

Suppose there exists a minimum dominating set D of $P_3 \square P_{4k+2}$ that does not include the middle vertex of column 1. We locate one guard at each vertex of D. Since there is no guard located at the middle vertex of column 1, we observe there must be at least two guards located at vertices in the first two columns. By Lemma 6, there are at least $3k$ guards in the last $4k$ columns, which leaves at most two guards in columns 1 and 2. Thus in D, there are exactly two guards in columns 1 and 2. Observe that at least one guard must be located in column 1 (otherwise a vertex of column 1 is not dominated). This leaves at most one guard in column 2. Applying Lemma 9 from the right-to-left forces a guard to be located at the middle of column 2. Thus, there is one guard located at either the top or bottom vertex of column 1, one guard located at the middle vertex of column 2 and $3k$ guards located in the last $4k$ columns. Observe that either the top vertex or the bottom vertex of column 1 is not dominated.

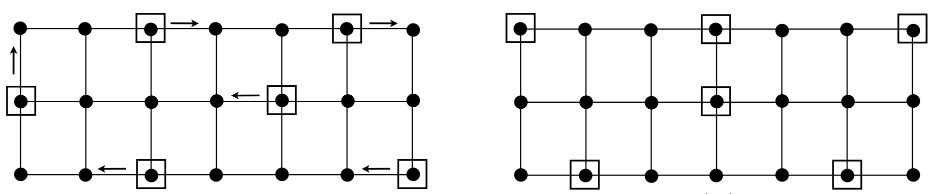
Thus, a guard must be located at the middle vertex of column 1 in every minimum dominating set. If $e^{\infty}_{\mathrm{m}}(P_3 \square P_{4k+2}) = \gamma(P_3 \square P_{4k+2})$, then any minimum eviction set contains the middle vertex of column 1. Attacking that vertex yields a contradiction. ■

Corollary 16 *For $n \geq 6$ and $n \equiv 2 \pmod 4$, $e^{\infty}_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1 = \lceil \frac{3n+5}{4} \rceil$.*

Based on the previous, the following is easy to see. For the upper bound, note that we can perform a total-switch in the first $n-2$ columns and a total-switch in the last two columns.

Observation 17 *For $n \geq 6$ and $n \equiv 2 \pmod 4$, $DD_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1$.*

From [13], we know $\gamma(P_3 \square P_7) = 6$. Consider the dominating set shown in Figure 8 (a). A total switch can be performed (indicated by the arrows) resulting in the dominating set shown in Figure 8 (b). Thus, the following observation can be made.



(a) (b)

Figure 8: A total-switch for $P_3 \square P_7$.

Observation 18 $e^{\infty}_{\mathrm{m}}(P_3 \square P_7) = DD_{\mathrm{m}}(P_3 \square P_7) = \gamma(P_3 \square P_7) = 6$.

Lemma 19 *For $n \geq 11$ and $n \equiv 3 \pmod 4$, $e^{\infty}_{\mathrm{m}}(P_3 \square P_n) > \gamma(P_3 \square P_n)$.*

Proof: Let $n = 4k+3$ for some integer $k \geq 2$. Suppose, by way of contradiction, that $e^{\infty}_{\mathrm{m}}(P_3 \square P_{4k+3}) = \gamma(P_3 \square P_{4k+3})$; by [13], $\gamma(P_3 \square P_{4k+3}) = 3k+3$. We aim to show that if $e^{\infty}_{\mathrm{m}}(P_3 \square P_{4k+3}) = 3k+3$, then a guard must be located at the middle vertex of column 1 in every minimum eviction set. An attack at the middle vertex of column 1 will then yield a contradiction.

Assume there exists a minimum eviction set D of $P_3 \square P_{4k+3}$ that does not include the middle vertex of column 1. By Lemma 6, there are at least $3k$ guards in the last $4k$ columns, so there are either 2 or 3 guards in the first three columns of D.

First, suppose there are two guards in the first three columns. Obviously, a guard must be located at a vertex of column 1, w.l.o.g., a guard is located at the top vertex of column 1. Then a guard must be located at the bottom vertex of column 1 or column 2. If this guard is on the bottom vertex of column 1, then the middle vertex on column 2 is not dominated (since there are only two guards in the first three columns). Therefore, assume there is a guard on the bottom vertex of column 2. As only one vertex of column 3 is dominated by the guards in the first two columns, guards must be located at the top and middle vertices of column 4 (to dominate vertices in column 3). By Lemma 6, there are at least $3k-3$ guards in the last $4k-4$ columns, which implies there are at most 4 guards in columns $4, 5, 6, 7$. Observe there must be exactly 4 guards in columns $4, 5, 6, 7$ as one guard cannot dominate both the bottom vertex of column 5 and the top vertex of column 6. As a result, there are exactly $3k-3$ guards in the last $4k-4$ columns and there is at most one guard in column 7 (otherwise, the bottom vertex of column 5 is not dominated). By Lemma 9, a guard must be located at the middle vertex of column 7 (see Figure 9 (a)). However, this leaves only one guard to dominate both the bottom vertex of column 5 and the top vertex of column 6, which is not possible.

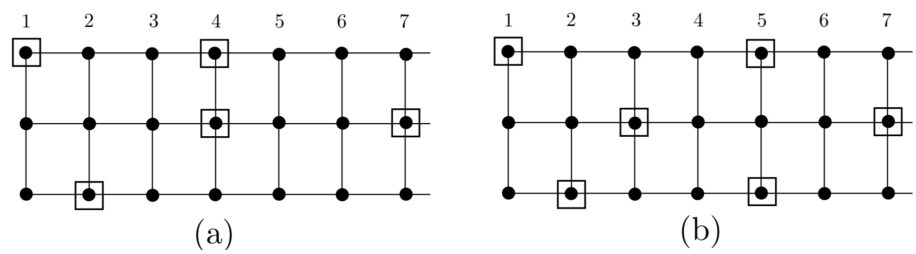


Figure 9: A guard at the top vertex of column 1 in the $3 \times 4k+3$ grid.

Therefore, there must be 3 guards in the first 3 columns. Observe in this case that there is at most 1 guard in column 3 (otherwise, not all vertices in column 1 are dominated). By Lemma 9, applied from right-to-left, a guard must be located at the middle of column 3. As there are 3 guards in the first 3 columns and at least $3k-3$ guards in the last $4k-4$ columns, there are at most 3 guards in columns 4, 5, 6, 7. Further, as 2 vertices of column 4 and 3 vertices in each of columns 5 and 6 must be dominated, exactly 3 guards are located in columns 4, 5, 6, 7 by Observation 7. There is at most one guard in column 7, otherwise one guard must dominate 2 vertices in column 4 and 3 vertices in column 5, which is not possible. By Lemma 9, applied from right-to-left, a guard is located at the middle of column 7. This forces the remaining guards to be located at the top and bottom of vertices in column 5 as shown in Figure 9 (b).

We now consider an attack at the middle vertex of column 7. If the guard at the middle vertex of column 7 in D moves to the top vertex of column 7, the bottom vertex of column 7 or the middle vertex of column 6, then another guard must move to column 7 (otherwise, a guard must be located at the middle vertex of column 7 by Lemma 9). However, this is not possible as there are no guards in column 6 in D and no guard can move from column

8 to column 7 (as there must be at least $3k-3$ guards in the last $4k-4$ columns). Thus, the guard in column 7 in D moves to column 8.

However, such a move leaves the middle vertex of column 6 not dominated. Consequently, a guard in column 5 in D must move to the middle of column 5 or to a vertex of column 6 in order to dominate the middle vertex of column 6. Suppose, w.l.o.g., the guard at the top vertex of column 5 in D moves to the middle vertex of column 5. This leaves the top vertex of column 6 not dominated and no guard can move to a neighbor of the top vertex of column 6. Therefore, w.l.o.g., the guard from the top vertex of column 5 moves to the top vertex of column 6. Observe the guard at the bottom vertex of column 5 must move to column 4. Otherwise, by Lemma 9, the guard must move to the middle of column 5 (as there are exactly 3 guards in the first 4 columns and at most one guard in column 5). But this leaves the bottom vertex of column 6 not dominated. Thus, the guard moves from the bottom of column 5 to column 4. However, this again leaves the bottom vertex of column 6 not dominated.

Consequently, a guard is located at the middle vertex of column 1 in every minimum eviction set. ■

Corollary 20 *For $n \geq 11$ and $n \equiv 3 \pmod 4$, $e^{\infty}_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1 = \lceil \frac{3n+5}{4} \rceil$.*

Observation 21 *$DD_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1$ when $n \geq 11$ and $n \equiv 3 \pmod 4$.*

Based on the previous lemma, both Corollary 20 and Observation 21 are easy to see (since DD_m forms an upper bound for e^{∞}_{m}). For the upper bound of Observation 21, let $n = 4k+3 \geq 11$ and note that we can perform a total-switch in $P_3 \square P_{4k}$ by the proof of Theorem 4 and it is trivial to observe that we can perform a total-switch in $P_3 \square P_3$. Thus, we can perform a total-switch in $P_3 \square P_{4k+3}$ using $3k+3 = \gamma(P_3 \square P_{4k+3})$ guards.

4 Eviction on $m \times n$ grids

In [4], it was shown that for $m \geq n \geq 8$,

$$\gamma(P_m \square P_n) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4. \tag{3}$$

For $n \geq m \geq 16$, equality in this bound was shown in [11]. A convenient table with the values of $\gamma(P_m \square P_n)$ for small values of m, n can be found in [1]. The domination number of $P_m \square P_n$ is a trivial lower bound for $e^{\infty}_{\mathrm{m}}(P_m \square P_n)$. In Theorem 22, we provide an upper bound on $e^{\infty}_{\mathrm{m}}(P_m \square P_n)$ which differs from the lower bound by approximately $n/5$.

A *perfect dominating set* is a set $S \subseteq V$ such that for all $v \in V$, $|N[v] \cap S| = 1$. We use the description given in [4] of a perfect dominating set on an infinite grid graph where the vertices are labeled according to their Cartesian coordinates: for any $t \in \{0, 1, 2, 3, 4\}$, the vertices in the perfect dominating set are given by the set

$$S_t = \left\{ (x, y) \mid y = \frac{1}{2}x + \frac{5}{2}s + t \text{ and } x, y, s \in \mathbb{Z} \right\}. \tag{4}$$

For $t = 0$, the vertices of a perfect dominating set of the infinite grid graph are indicated in Figure 10 (increasing the value of t simply translates the dominating set).

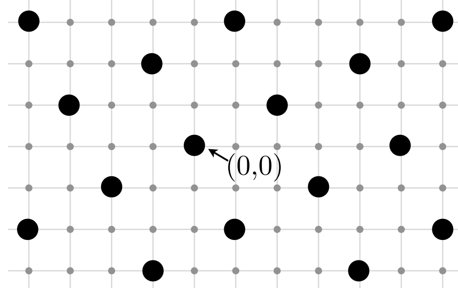


Figure 10: A perfect dominating set on the infinite grid.

Theorem 22 *For $m \geq n \geq 8$,*

$$e_{\mathrm{m}}^{\infty}(P_m \Box P_n) \leq \left\lfloor \frac{(n+2)(m+3)}{5} \right\rfloor - 4.$$

Proof: To obtain the result, we begin by noting the upper bound of (3), initially given by [4], for $m \geq n \geq 8$. In the initial part of the proof, we set out to show for $m \geq n \geq 8$,

$$e_{\mathrm{m}}^{\infty}(P_m \Box P_n) \leq \left\lfloor \frac{(n+2)(m+3)}{5} \right\rfloor.$$

We later improve that bound by four. First, we describe the location of the guards that form the initial dominating set, i.e, the eviction set, and then show that these guards can defend against any sequence of attacks.

Consider the sub-grid $P_m \Box P_n$ of the infinite grid graph, induced by vertices

$$\Big\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\Big\}.$$

For $i \in [m]$ and $j \in [n]$, if vertex (i,j) is in the perfect dominating set of the infinite grid graph defined by S_t in (4), then a guard initially occupies vertex (i,j) in $P_m \Box P_n$. Label these guards as follows: guards at vertices of the form (m,j) for $j \in [n]$ of $P_m \Box P_n$ will be referred to as *white guards* and will be colored white in figures. All other guards that have been placed on vertices of $P_m \Box P_n$ so far will be referred to as *core guards*, also known as *black guards*, and will be colored black in figures. Observe that no two core guards are adjacent.

In the remainder of the proof, we will refer to vertex (i,j) as being in *column* i and *row* j (again, following the Cartesian coordinate convention). Some vertices in row 1 and row n of $P_m \Box P_n$ are not yet dominated. For $i \in [m]$, if vertex $(i, n+1)$ (or $(i, 0)$) in the infinite grid graph is in the perfect dominating set S_t defined in (4), then we place a core (black) guard at vertex (i,n) (or $(i,1)$) in $P_m \Box P_n$. An example of this is shown in Figure 11 (a).

Some vertices in column 1 and column m of $P_m \Box P_n$ are not yet dominated. For $j \in [n]$, if vertex $(0,j)$ in the infinite grid graph is in the perfect dominating set S_t defined by (4), then we place a *white guard* at vertex $(1,j)$ in $P_m \Box P_n$ (provided no guard has already been placed on that vertex in $P_m \Box P_n$). For $j \in [n]$, if vertex $(-1,j)$ or $(m+1,j)$ in the infinite grid graph is in the perfect dominating set S_t defined by (4), then we place a *grey guard* at vertex $(1, j)$ or (m, j) respectively in $P_m \square P_n$ (provided no guard has already been placed on that vertex in $P_m \square P_n$). An example of the placement of black, white and grey guards is illustrated in Figure 11 (a). Note that the arrows in this and subsequent figures do not indicate guard movements, rather, the arrows indicate how guards of S_t outside the bounds of $P_m \square P_n$ are mapped to white or grey guards.

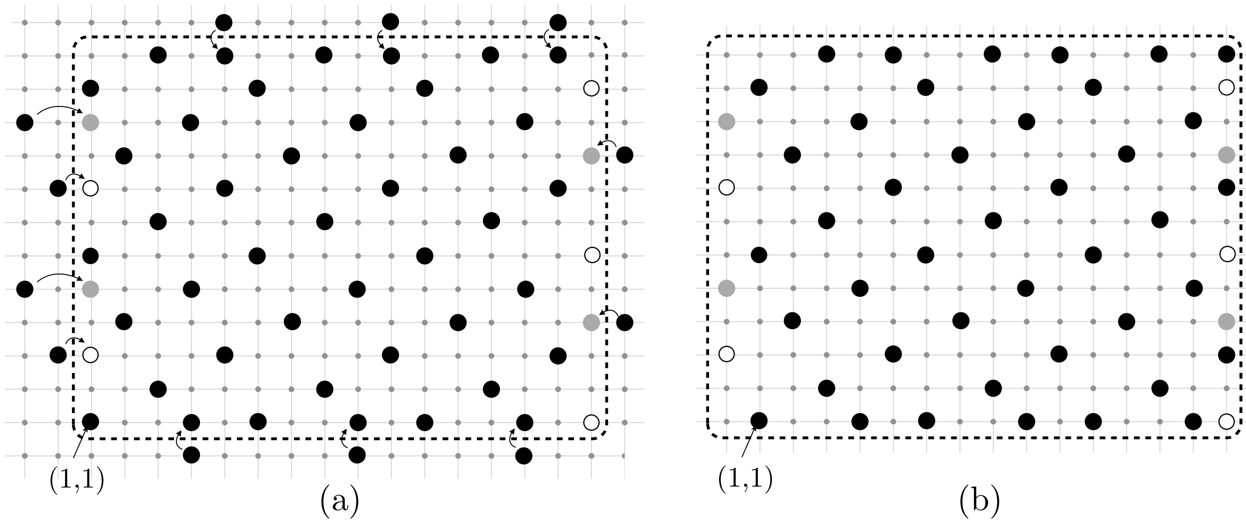


Figure 11: Examples of placing core (black), white and grey guards in $P_{16} \square P_{12}$.

Two vertices u, v are said to be *horizontally adjacent* if u is a neighbor to the left or right of v, in the usual drawing of $P_m \square P_n$. By the placement of guards, no black, white or grey guards are horizontally adjacent, except perhaps at the corner vertices. We leave the careful examination of the corners until the end of the proof.

To complete the initial part of the proof, we now show that given an attack at any vertex, the guards can move to a dominating set that leaves the attacked vertex without a guard. To do this, the guards will essentially be positioned in two dominating sets. The set of black guards are called *the core*. If a vertex with a black guard is attacked, then the core *shifts to the right*; that is, each black guard moves to a neighboring vertex to the right. In the resulting dominating set, we say *the core is on the right*. In general, the white and grey guards do not move unless attacked. As an example, consider the positions of guards described in Figure 11 (a); suppose any vertex of $P_{16} \square P_{12}$ with a black guard is attacked. The black guards shift to the right and their new positions are shown in Figure 11 (b). If the core is on the right and a vertex with a black guard is attacked, then the core *shifts to the left*. In Figure 11 (b), if a vertex with a black guard is attacked, then the core shift to the left, resulting in the configuration of guards displayed in Figure 11 (a).

Now consider the situation when a vertex with a white or grey guard is attacked. First, suppose a vertex with a grey guard is attacked. Without loss of generality, the grey guard g occupies a vertex in column 1 of $P_m \square P_n$. If the core is on the left, then guard g dominates the vertex at which it is located, but all neighbors are dominated by other guards. Thus, we move the grey guard to any unoccupied neighboring vertex and no other guards move (if the grey guard has no unoccupied neighbor, then per the definition of the eviction problem, no action is necessary). If the core is on the right, then we shift the core to the left and move the guard g to any unoccupied neighbor as in the previous case. In the subsequent attack, g will simply move back to its previous position.

Second, suppose a vertex with a white guard is attacked. Without loss of generality, the white guard w occupies a vertex in column 1 of $P_m \square P_n$. Consider the special case when w has at most one external private neighbor (i.e. a neighboring vertex which is dominated by no other guard). Then w moves to that neighbor, otherwise it moves to any unoccupied neighbor.[1] If the core is on the right, then the core shifts to the left, otherwise no guards other than w move. In the subsequent attack, w will simply move back to his previous position.

Now consider the general case in which w might have more than one external private neighbor. Suppose w is located at vertex $(1, j)$. If the core is on the left, then there is a black (core) guard b located at either $(2, j+1)$ or $(2, j-1)$ (this is due to the fact that w was placed at $(1, j)$ because $(0, j)$ was in the perfect dominating set on the infinite grid graph). Then two neighbors of (i, j) are dominated by guard b. As w has at most three neighbors, since it located in column 1, it has at most one external private neighbor and we have the previous situation. If the core is on the right, then the core moves to the left and we have the previous situation. In the subsequent attack, w again moves back to its previous position.

Thus, we have shown the black, white and grey guards form a dominating set that can defend against any sequence of attacks. Returning to the positions of guards in the infinite grid graph, observe that the black, white and grey guards used in $P_m \square P_n$ correspond to the vertices in a perfect dominating set of an $(m+3) \times (n+2)$ sub-grid of the infinite grid graph in which $P_m \square P_n$ has been embedded. From [4], this number of guards is

$$\left\lfloor \frac{(n+2)(m+3)}{5} \right\rfloor.$$

This completes the initial part of the proof.

We now set out to improve the bound by removing four guards, one from each corner of the grid.

First, consider the possible locations of guards in the lower-left corner of $P_m \square P_n$. We consider whether any of the vertices $(1,1)$, $(2,1)$, $(1,2)$, $(2,2)$ are in the perfect dominating set of the infinite grid graph. Note that we shall consider all these possibilities, rather than assuming a fixed initial location of a guard in the lower-left corner (say at $(1,1)$), since the general argument can then be applied to other corners (and the initial location of guards in those corners depends on n and m, as well as the location of guards in the lower left corner).

Suppose that $(1,1)$ or $(2,1)$ is in the perfect dominating set on the infinite grid graph. Then there is either a core guard located at $(1,1)$ or $(2,1)$ and the core is on the left in the initial eviction set of $P_m \square P_n$, as shown in Figure 12 (a) and (b) respectively. It easy to see that although vertices $(-1,0)$ and $(0,0)$, respectively, are in the perfect dominating set of the infinite grid graph, in either case, the corresponding guard in $P_m \square P_n$ is not required. These vertices are indicated by a star in Figure 12 (a) and (b).

[1] By design of our guard strategy, there always exists a neighbor of w with no guard. This fact is somewhat irrelevant to the correctness of the proof, since if w had no unoccupied neighbor, then the attack would require no action on the part of the defender.

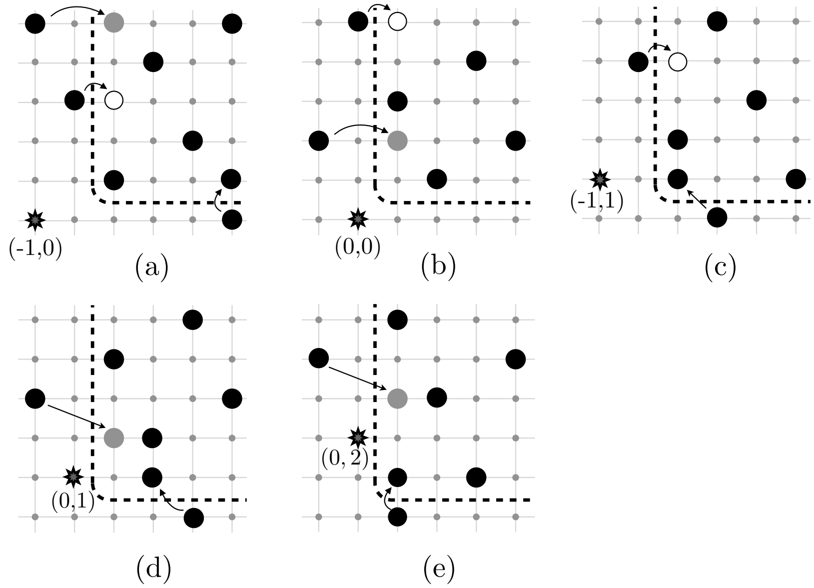


Figure 12: Lower left corner, with the core drawn on the left.

Suppose that $(1, 2)$ is in the perfect dominating set on the infinite grid graph. Then there is a core guard located at $(1, 2)$ and the core is on the left in the initial eviction set on $P_m \square P_n$, as shown in Figure 12 (c). Then instead of placing a guard at $(2, 1)$ (since $(2, 0)$ is in the perfect dominating set on the infinite grid graph), we place a guard at $(1, 1)$. In the infinite grid graph, vertex $(-1, 1)$ is in the perfect dominating set, but the corresponding guard in $P_m \square P_n$ is not required; this vertex is indicated by a star in Figure 12 (c).

Suppose that $(2, 2)$ is in the perfect dominating set on the infinite grid graph. Then there is a core guard located at $(2, 2)$ and the core is on the left in the initial eviction set of $P_m \square P_n$, as shown in Figure 12 (d). Then instead of placing a grey guard at $(1, 3)$ (since $(-1, 3)$ is in the perfect dominating set on the infinite grid graph), we place a grey guard at $(1, 2)$; additionally, there is a core guard at $(2, 1)$ instead of $(3, 1)$. In the infinite grid graph, vertex $(0, 1)$ is in the perfect dominating set, but the corresponding guard in $P_m \square P_n$ is not required; this vertex is indicated by a star in Figure 12 (d).

To conclude the possibilities for the lower left corner, suppose that none of $(1, 1)$, $(2, 1)$, $(1, 2)$, $(2, 2)$ is in the perfect dominating set on the infinite grid graph. If the core is on the left in the initial eviction set of $P_m \square P_n$, as shown in Figure 12 (e), then instead of placing a grey guard at $(1, 4)$ and a white guard at $(1, 2)$ (since $(-1, 4)$ and $(0, 2)$ are in the perfect dominating set on the infinite grid graph), we place a grey guard at $(1, 3)$. In the infinite grid graph, vertex $(1, 0)$ is in the perfect dominating set, but the corresponding guard in $P_m \square P_n$ is not required; this vertex is indicated by a star in Figure 12 (e).

In Figure 12 (a)-(e), all vertices of $P_m \square P_n$ are dominated by black, white and grey guards. If the core moves to the right or a grey or white vertex is attacked, it is easy to see that all vertices of $P_m \square P_n$ remain dominated. A symmetric argument (or a rotation of 180^o) can be used to show that one guard is not required for the upper right corner and has therefore been omitted.

DOI: 10.20429/tag.2017.040102

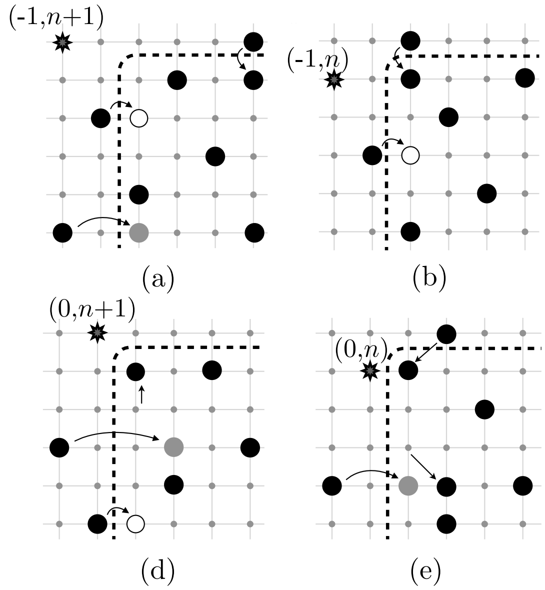


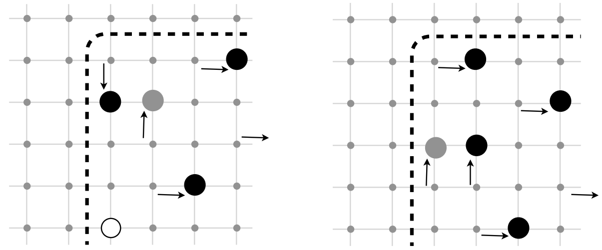
Figure 13: Upper left corner, with the core drawn on the left.

Now consider the upper left corner. First, suppose $(2, n)$ is in the perfect dominating set of the infinite grid graph. Recall that this implies there is a core guard at $(2, n)$ and the core is on the left in the initial eviction set of $P_m \,\square\, P_n$ as shown in Figure 13 (a). Although $(-1, n+1)$ is in the perfect dominating set of the infinite grid graph, the corresponding guard in $P_m \,\square\, P_n$ is not required. This vertex is indicated by a star in Figure 13 (a).

Second, suppose $(2, n-1)$ is in the perfect dominating set of the infinite grid graph. Then $(1, n+1)$ is also in the perfect dominating set of the infinite grid graph. Recall that this implies there is a core guard at $(1, n)$ and the core is on the left in the initial eviction set of $P_m \,\square\, P_n$ as shown in Figure 13 (b). Although $(-1, n)$ is in the perfect dominating set of the infinite grid graph, the corresponding guard in $P_m \,\square\, P_n$ is not required. This vertex is indicated by a star in Figure 13 (b).

Third, suppose $(1, n)$ is in the perfect dominating set of the infinite grid graph. Then there is a core guard at $(1, n)$ and the core is on the left in the initial eviction set on $P_m \,\square\, P_n$, as shown in Figure 13 (c). Then instead of placing a guard at $(3, n)$ (since $(3, n+1)$ is in the perfect dominating set on the infinite grid graph), we place a guard at $(2, n)$. In the infinite grid graph, vertex $(1, n)$ is in the perfect dominating set, but the corresponding guard in $P_m \,\square\, P_n$ is not required; this vertex is indicated by a star in Figure 13 (c).

Fourth, suppose $(1, n-1)$ is in the perfect dominating set on the infinite grid graph. Then there is a core guard located at $(1, n-1)$ and the core is on the left in the initial eviction set of $P_m \,\square\, P_n$. In this case, instead of placing a core guard at $(1, n-1)$ and a grey guard at $(1, n-2)$, we place a core guard at $(1, n)$ and a grey guard at $(2, n-2)$ as shown in Figure 13 (d). In the infinite grid graph, vertex $(0, n+1)$ is in the perfect dominating set, but the corresponding guard in $P_m \,\square\, P_n$ is not required; this vertex is indicated by a star in Figure 13 (d). From the figure, it is clear that the guards form a dominating set when the core is on the left. If the core moves to the right in response to an attack, then the core guard at $(1, n)$ moves to $(1, n-1)$ and the grey guard at $(2, n-2)$ moves to $(2, n-1)$ (and all other core guards move to the right) and consequently all vertices are dominated. Figure 14 (i) illustrates the movement of the guards (with arrows). If the core moves to the left (or that grey vertex is attacked), then the grey guard returns to its previous vertex.



(i) (ii)

Figure 14: The movement of guards in response to an attack.

To conclude the possibilities for the upper left corner, suppose that none of $(2, n)$, $(2, n-1)$, $(1, n)$, $(1, n-1)$ is in the perfect dominating set on the infinite grid graph. If the core is on the left in the initial eviction set, as shown in Figure 13 (e), then instead of placing a core guard at $(2, n)$ (since $(2, n+1)$ is in the perfect dominating set on the infinite grid graph), we place a core guard at $(1, n)$; additionally, instead of placing a core guard at $(1, n-2)$, we place a core guard at $(2, n-3)$. In the infinite grid graph, vertex $(0, n)$ is in the perfect dominating set, but the corresponding guard in $P_m \square P_n$ is not required; this vertex is indicated by a star in Figure 13 (e). From the figure, it is clear that the guards form a dominating set when the core is on the left. If the core moves to the right in response to an attack, then the grey guard at $(1, n-3)$ moves to $(1, n-2)$ and the core guard at $(2, n-3)$ moves to $(2, n-2)$ (and all other core guards move to the right) and consequently all vertices are dominated. Figure 14 (ii) illustrates the movement of the guards (with arrows). If the core moves to the left, the guards at $(1, n-2)$ and $(2, n-2)$ returns to their previous vertex.

In Figure 13 (a)-(e), all vertices of $P_m \square P_n$ are dominated by black, white and grey guards. If the core moves to the right or a grey or white vertex is attacked, it is easy to see that all vertices of $P_m \square P_n$ remain dominated.

A symmetric argument (or a rotation of 180^o) can be used to show that one guard is not required for the lower right corner and has therefore been omitted.

As one guard has been eliminated for each corner, the eviction number of $P_m \square P_n$ is at most $\lfloor \frac{(n+2)(m+3)}{5} \rfloor - 4$. ■

We also provide upper bounds for $e_{\mathrm{m}}^{\infty}(P_5 \square P_n)$, $e_{\mathrm{m}}^{\infty}(P_6 \square P_n)$ and $e_{\mathrm{m}}^{\infty}(P_7 \square P_n)$.

Theorem 23 *For $n \geq 14$,*

$$\begin{aligned} e_{\mathrm{m}}^{\infty}(P_5 \square P_n) &\leq \left\lceil \frac{4n+3}{3} \right\rceil \\ e_{\mathrm{m}}^{\infty}(P_6 \square P_n) &\leq \left\lceil \frac{3n+1}{2} \right\rceil \\ e_{\mathrm{m}}^{\infty}(P_7 \square P_n) &\leq \left\lceil \frac{7n+5}{4} \right\rceil . \end{aligned}$$

https://digitalcommons.georgiasouthern.edu/tag/vol4/iss1/2
DOI: 10.20429/tag.2017.040102

Proof: Let $n \geq 14$. By (1)-(2), $e_{\mathrm{m}}^{\infty}(P_1 \square P_n) \leq \gamma(P_n) + 1 = \lceil \frac{n}{3} \rceil + 1$ and by Lemma 2 and [13], $e_{\mathrm{m}}^{\infty}(P_4 \square P_n) = n$. By considering the disjoint subgraphs $P_1 \square P_n$ and $P_4 \square P_n$, we conclude $e_{\mathrm{m}}^{\infty}(P_5 \square P_n) \leq e_{\mathrm{m}}^{\infty}(P_1 \square P_n) + e_{\mathrm{m}}^{\infty}(P_4 \square P_n) \leq \lceil \frac{4n+3}{3} \rceil$ as desired.

By Theorem 1, $e_{\mathrm{m}}^{\infty}(P_2 \square P_n) = \lceil \frac{n+1}{2} \rceil$. By considering the disjoint subgraphs $P_2 \square P_n$ and $P_4 \square P_n$, we conclude $e_{\mathrm{m}}^{\infty}(P_6 \square P_n) \leq e_{\mathrm{m}}^{\infty}(P_2 \square P_n) + e_{\mathrm{m}}^{\infty}(P_4 \square P_n) \leq \lceil \frac{3n+1}{2} \rceil$ as desired.

From the results of Section 3 and [13], $e_{\mathrm{m}}^{\infty}(P_3 \square P_n) \leq \gamma(P_3 \square P_n) + 1 = \lceil \frac{3n+1}{4} \rceil + 1$. By considering the disjoint subgraphs $P_3 \square P_n$ and $P_4 \square P_n$, we conclude $e_{\mathrm{m}}^{\infty}(P_7 \square P_n) \leq e_{\mathrm{m}}^{\infty}(P_3 \square P_n) + e_{\mathrm{m}}^{\infty}(P_4 \square P_n) \leq \lceil \frac{7n+21}{4} \rceil$ as desired. ■

5 Open Questions

We make the following conjecture.

Conjecture 24 *There exist $m > 1$ and $n > 1$ such that $DD_{\mathrm{m}}(P_m \square P_n) \neq e_{\mathrm{m}}^{\infty}(P_m \square P_n)$.*

When $n = 5$, it is easy to verify that $DD_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 1$. The same holds for $n = 9$: consider a dominating set with vertices in row 2 column 1, row 1 column 3, row 3 column 3, and row 2 columns 5 through 9. See Figure 15 which shows this dominating set (part (a)) and a disjoint one such that there is a perfect matching between the two (part (b)). Note that the vertex in row 2 column 5 of the dominating set in Figure 15 (a) is matched with the vertex in row 2 column 4 of the dominating set in Figure 15 (b). This pattern does not extend when n gets large and $n \equiv 1 \pmod 4$, since $\gamma(P_3 \square P_{n+4}) = \gamma(P_3 \square P_n) + 3$ when $n \equiv 1 \pmod 4$. However, when n gets large and $n \equiv 1 \pmod 4$, perhaps it is the case that $DD_{\mathrm{m}}(P_3 \square P_n) > \gamma(P_3 \square P_n) + 1$.

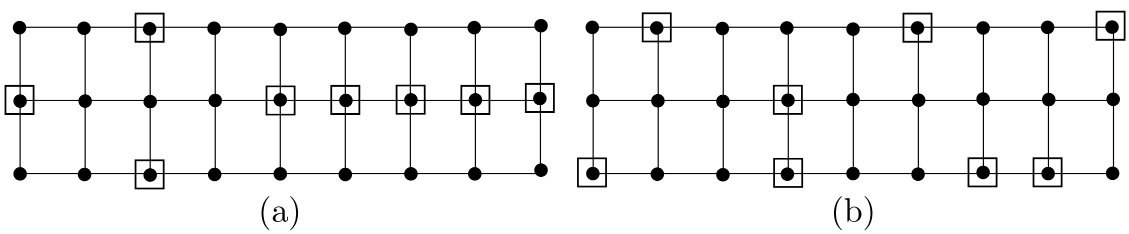


Figure 15: Dominating sets of $P_3 \square P_9$.

Question 25 *Is it true that $DD_{\mathrm{m}}(P_3 \square P_n) = \gamma(P_3 \square P_n) + 2$ when $n \equiv 1 \pmod 4$ and $n > 9$?*

It would be of interest to bound $DD_{\mathrm{m}}(P_m \square P_n)$ for large m, n.

Question 26 *Can the bound in Theorem 22 be improved?*

Question 27 *Is it true that $DD_{\mathrm{m}}(P_5 \square P_n) = \gamma(P_5 \square P_n) = e_{\mathrm{m}}^{\infty}(P_5 \square P_n)$ when $n \geq 5$?*

In [5], it was determined that $\gamma(P_6 \square P_6) = 10$. Combined with Theorem 23, we find $\gamma(P_6 \square P_6) = DD_{\mathrm{m}}(P_6 \square P_6) = e_{\mathrm{m}}^{\infty}(P_6 \square P_6) = 10$.

Question 28 *Is $DD_{\mathrm{m}}(P_6 \square P_n) = \gamma(P_6 \square P_n) = e_{\mathrm{m}}^{\infty}(P_6 \square P_n)$ for all $n \geq 6$?*

Appendix

In [13], the authors state the following lemma.

Lemma 29 *$\gamma(P_4 \,\Box\, P_n) \leq n$ for all $n = 4, 7, 8$ and $n \geq 10$.*

While the result is true, the proof is flawed. The authors label the vertices of the four copies of P_n as $w_1, w_2, \ldots, w_n$; $x_1, x_2, \ldots, x_n$; $y_1, y_2, \ldots, y_n$; $z_1, z_2, \ldots, z_n$ where (w_i, x_i), (x_i, y_i), (y_i, z_i) are edges in $P_4 \,\Box\, P_n$. The authors break the proof into three cases to prove the result. In the first case, the authors suppose $n = 4 + 3k$ for some integer $k \geq 0$. They let

$$D_1 = \{w_3, x_1, y_4, z_2\}, \qquad D_2 = \{w_{6t}, x_{6t}, z_{6t+1} \mid t = 0, \ldots, \lceil k/2 \rceil\},$$

$$D_3 = \{w_{6t+3}, y_{6t+4}, z_{6t+2} \mid t = 0, \ldots, \lfloor k/2 \rfloor\}$$

and state that $D_1 \cup D_2 \cup D_3$ forms a dominating set. We note that the set $D_1 \cup D_2 \cup D_3$ actually contains $n+1$, rather than n vertices, contains x_0 and w_0 which do not exist, and actually does not form a dominating set. For example, $N[z_5] = \{z_4, z_5, z_6, y_5\}$ but $N[z_5] \notin D_1 \cup D_2 \cup D_3$.

In the second and third cases, the authors provide dominating sets of sizes $n+2$ and $n+1$, rather than of size n. We now provide a corrected proof of Lemma 29.

Proof: Label the vertices of $P_4 \,\Box\, P_n$ as above. We break the proof into three cases:

Case 1: Suppose $n = 3k + 4$ for some integer $k \geq 0$. Let

$$D_1 = \{w_3, x_1, y_4, z_2\}, \qquad D_2 = \{w_{6t-1}, x_{6t+1}, z_{6t} \mid t = 1, \ldots, \lceil k/2 \rceil\},$$

$$D_3 = \{w_{6t+3}, y_{6t+4}, z_{6t+2} \mid t = 1, \ldots, \lfloor k/2 \rfloor\}.$$

We claim $D_1 \cup D_2 \cup D_3$ forms a dominating set of size n. Note $|D_1| = 4$ and for every unit increase of k, the value of n increases by 3 and the size of $D_1 \cup D_2 \cup D_3$ increases by 3.

Next, suppose $N[z_{6t+i}] = \{z_{6t+i-1}, z_{6t+i}, z_{6t+i+1}, y_{6t+i}\}$ is not dominated for some $i \in \{0, 1, 2, 3, 4, 5\}$, $t \in \{0, 1, \ldots, \lfloor k/2 \rfloor\}$ and $1 < 6t + i < 3k + 4$. Then from D_2, $6t \neq 6t + i \Rightarrow i \neq 0$ and from D_3, $6t + 2$ is not equal to $6t + i$, $6t + i + 1$, $6t + i - 1$, so $i \neq 1, 2, 3$. From D_3, $y_{6t+4} \neq y_{6t+i} \Rightarrow i \neq 4$. This leaves $i = 5$. However, z_{6t+5} is adjacent to $z_{6(t+1)} \in D_2$ and $t + 1 \leq \lceil k/2 \rceil$ while $t \leq \lfloor k/2 \rfloor$. This yields a contradiction. A very similar argument shows y_{6t+i}, x_{6t+i}, w_{6t+i} are dominated.

Case 2: Suppose $n = 3k + 8$ for some integer $k \geq 0$. Let

$$D_1 = \{w_3, w_7, x_1, x_5, y_4, y_8, z_2, z_6\}, \qquad D_2 = \{w_{6t+3}, x_{6t+5}, z_{6t+2} \mid t = 1, 2, \ldots, \lceil k/2 \rceil\},$$

$$D_3 = \{w_{6t+7}, y_{6t+8}, z_{6t+6} \mid t = 1, 2, \ldots, \lfloor k/2 \rfloor\}.$$

We claim $D_1 \cup D_2 \cup D_3$ forms a dominating set of size n. Note $|D_1| = 8$ and for every unit increase in k, the value of n increases by 3 and the size of $D_1 \cup D_2 \cup D_3$ increases by 3. To show all vertices are dominated, an argument similar to that in Case 1 can be used.

Case 3: Suppose $n = 3k + 12$ for some integer $k \geq 0$. Let

$$D_1 = \{w_3, w_7, w_{11}, x_1, x_5, x_9, y_4, y_8, y_{12}, z_2, z_6, z_{10}\},$$

$$D_2 = \{w_{6t+7}, x_{6t+9}, z_{6t+8} \mid t = 1, 2, \ldots, \lceil k/2 \rceil\},$$

$$D_3 = \{w_{6t+11}, y_{6t+12}, z_{6t+10} \mid t = 1, 2, \ldots, \lfloor k/2 \rfloor.$$

We claim $D_1 \cup D_2 \cup D_3$ forms a dominating set of size n. Note $|D_1| = 12$ and for every unit increase in k, the value of n increases by 3 and the size of $D_1 \cup D_2 \cup D_3$ increases by 3. To show all vertices are dominated, an argument similar to that in Case 1 can be used. ■

An example of the dominating set on $P_4 \square P_n$ for $n = 3k + 12$ is illustrated in Figure 16; the vertices of D_1 are indicated by circles, the vertices of D_2 are indicated by squares, and the vertices of D_3 are indicated by triangles.

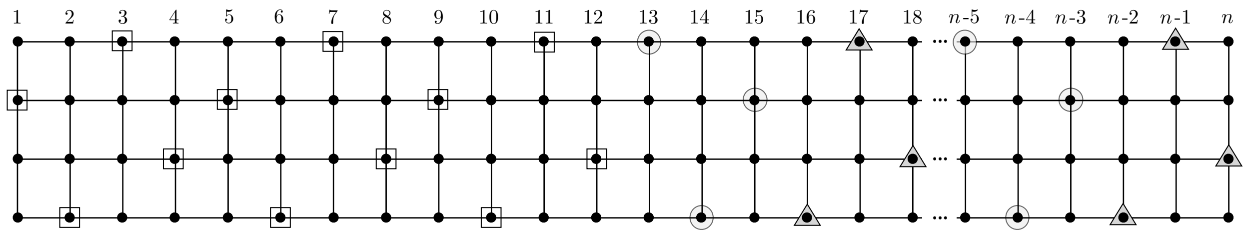


Figure 16: The dominating set $D_1 \cup D_2 \cup D_3$ for $n = 3k + 12$.

References

[1] Alanko, S., S. Crevals, A. Isopoussu, P. Östergård, V. Pettersson, Computing the domination number of grid graphs, *Electronic J. of Combin.* **18** (2011), paper #P141.

[2] Burger, A.P., E.J. Cockayne, W.R. Grundlingh, C.M. Mynhardt, J.H. van Vuuren, W. Winterbach, Infinite order domination in graphs, *J. Combin. Math. Combin. Comput.* **50** (2004), 179–194.

[3] Caro, Y. and W.F. Klostermeyer, Eternal independent sets in graphs, *Theory and Applications of Graphs* **3** (2016).

[4] Chang, T.Y., Domination numbers of grid graphs, PhD Thesis, University of South Florida, 1992.

[5] Chang, T.Y., W.E. Clark, E.O. Hare, Domination number of complete grid graphs, I, *Ars Combin.* **38** (1985), 97–111.

[6] Cockayne, E.J., E. O. Hare, S. T. Hedetniemi, T. V. Wimer, Bounds for the domination number of grid graphs, *Congressus Numerantium* **47** (1985), 217–228.

[7] Finbow, S., S. Gaspers, M.E. Messinger, P. Ottaway, A note on the eternal dominating set problem, submitted.

[8] Finbow, S., M.E. Messinger, M. van Bommel, Eternal domination on $3 \times n$ grids, *Aust. J. of Combin.* **61** (2015) 156–174.

[9] Goddard, W., S.M. Hedetniemi, S.T. Hedetniemi, Eternal security in graphs, *J. Combin. Math. Combin. Comput.* **52** (2005), 169–180.

[10] Goldwasser, J.L., W.F. Klostermeyer, C.M. Mynhardt, Eternal protection in grid graphs, *Util. Math.* **91** (2013) 47–64.

[11] Gonçalves, D., A. Pinlou, M. Rao, S. Thomassé, The domination number of grids, *SIAM J. of Discrete Math.* **25** (2011) 1443–1453.

[12] Hedetniemi, S.M., S. T. Hedetniemi, R. C. Laskar, L. Markus and P. J. Slater, Disjoint dominating sets in graphs, *Proc. ICDM 2006, Ramanujan Mathematics Society Lecture Notes Series* 7 (2008), 87100.

[13] Jacobson, M.S., L.F. Kinch, On the domination number of products of graphs: I, *Ars Combin.* **18** (1983) 33–44.

[14] Klostermeyer, W.F., M. Lawrence, G. MacGillivray, Dynamic Dominating Sets: the Eviction Model for Eternal Domination, *J. Combin. Math. Combin. Comput.* **97** (2016), 247–269

[15] Klostermeyer W.F., and C. M. Mynhardt, A dynamic domination problem in trees, W. Klostermeyer and C. M. Mynhardt, *Transactions on Combinatorics* **4** (2015) 15–31.

DOI: 10.20429/tag.2017.040102

[16] Klostermeyer, W.F., and C.M. Mynhardt, Protecting a graph with mobile guards, *Appl. Anal. Discrete Math.* **10** (2016), pp. 1-29.