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Connection and separation in hypergraphs

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Abstract

In this paper we study various fundamental connectivity properties of hypergraphs from a graph-theoretic perspective, with the emphasis on cut edges, cut vertices, and blocks. We prove a number of new results involving these concepts. In particular, we describe the exact relationship between the block decomposition of a hypergraph and the block decomposition of its incidence graph.

Keywords: Hypergraph, incidence graph, walk, trail, path, cycle, connected hypergraph, cut edge, cut vertex, separating vertex, block.

1 Introduction

A data base search under "hypergraph" returns hundreds of journal articles published in the last couple of years alone, but only a handful of monographs. Among the latter, most either treat very specific problems in hypergraph theory (for example, colouring in [8] and also in [9]), or else are written with a non-mathematician audience in mind, and hence focus on applications (for example, [6]). A mathematician or mathematics student looking for a general introduction to hypergraphs is left with Berge's decades-old *Hypergraphs* [3] and *Graphs and Hypergraphs* [2], and Voloshin's much more recent *Introduction to Graphs and Hypergraphs* [9], aimed at undergraduate students. The best survey on hypergraphs that we could find, albeit already quite out of date, is Duchet's chapter [7] in the *Handbook on Combinatorics.* In particular, it describes the distinct paths that lead to the study of hypergraphs from graph theory, optimization theory, and extremal combinatorics, explaining the fragmented terminology and disjointed nature of the results. Berge's work, for example, though an impressive collection of results, shows a distinct bias for hypergraphs arising from extremal set theory and optimization theory, and as such is rather unappealing to graph theorists, in general.

The numerous journal publications, on the other hand, treat a great variety of specific problems on hypergraphs. Graph theorists find various ways of generalizing concepts from graph theory, often without justifying their own approach or comparing it with others. The same term in hypergraphs (for example, cycle) may have a variety of different meanings. Sometimes, authors implicitly assume that results for graphs extend to hypergraphs. A coherent theory of hypergraphs, as we know it for graphs, is sorely lacking.

This article can serve as an introduction to hypergraphs from a graph-theoretic perspective, with a focus on basic connectivity. To prepare the ground for the more involved results on block decomposition of hypergraphs, we needed to carefully and systematically examine the fundamental connectivity properties of hypergraphs, attempting to extend basic results such as those found in the first two chapters of a graph theory textbook. We are strongly biased in our approach by the second author's graph-theoretic perspective, as well as in our admiration for Bondy and Murty's graph theory "bible" [5] and its earlier incarnation [4]. While we expect that some of these observations have been made before, to the best of our knowledge they have never been tied to a coherent theory of connection in hypergraphs and published in a widely accessible form.

Our paper is organized as follows. In Section 2 we present the fundamental concepts involving hypergraphs, as well as some immediate observations. Section 3 forms the bulk of the work: from graphs to hypergraphs, we generalize the concepts of various types of walks, connection, cut edges and cut vertices, and blocks, and prove a number of new results involving these concepts.

A longer version of this paper is available on ArXiv [1].

2 Fundamental concepts

2.1 Hypergraphs and subhypergraphs

We shall begin with some basic definitions pertaining to hypergraphs. The graph-theoretic terms used in this article are either analogous to the hypergraph terms defined here, or else are standard and can be found in [5].

Definition 2.1. A hypegraph H is an ordered pair (V, E), where V and E are disjoint finite sets such that $V \neq \emptyset$, together with a function $\psi : E \to 2^V$, called the *incidence function*. The elements of $V = V(H)$ are called *vertices*, and the elements of $E = E(H)$ are called *edges*. The number of vertices $|V|$ and number of edges $|E|$ are called the *order* and *size* of the hypergraph, respectively. Often we denote $n = |V|$ and $m = |E|$. A hypergraph with a single vertex is called *trivial*, and a hypergraph with no edges is called *empty*.

Two edges $e, e' \in E$ are said to be *parallel* if $\psi(e) = \psi(e')$, and the number of edges parallel to edge e (including e) is called the *multiplicity* of e. A hypergraph H is called *simple* if no edge has multiplicity greater than 1; that is, if ψ is injective.

As it is customary for graphs, the incidence function may be omitted when no ambiguity can arise (in particular, when the hypergraph is simple, or when we do not need to distinguish between distinct parallel edges). An edge e is then identified with the subset $\psi(e)$ of V, and for $v \in V$ and $e \in E$, we conveniently write $v \in e$ or $v \notin e$ instead of $v \in \psi(e)$ or $v \notin \psi(e)$, respectively. Moreover, E is then treated as a multiset, and we use double braces to emphasize this fact when needed. Thus, for example, $\{1, 2\} = \{\!\{1, 2\}\!\}$ but $\{1, 1, 2\} = \{1, 2\} \neq \{\!\{1, 1, 2\}\!\}$.

Definition 2.2. Let $H = (V, E)$ be a hypergraph. If $v, w \in V$ are distinct vertices and there exists $e \in E$ such that $v, w \in e$, then v and w are said to be *adjacent* in H (via edge e). Similarly, if $e, f \in E$ are distinct (but possibly parallel) edges and $v \in V$ is such that $v \in e \cap f$, then e and f are said to be *adjacent* in H (via vertex v).

Each ordered pair (v, e) such that $v \in V$, $e \in E$, and $v \in e$ is called a *flag* of H; the (multi)set of flags is denoted by $F(H)$. If (v, e) is a flag of H, then we say that vertex v is *incident* with edge e.

The *degree* of a vertex $v \in V$ (denoted by $\deg_H(v)$ or simply $\deg(v)$ if no ambiguity can arise) is the number of edges $e \in E$ such that $v \in e$. A vertex of degree 0 is called *isolated*, and a vertex of degree 1 is called *pendant*. A hypergraph H is *regular of degree r* (or *r-regular*) if every vertex of H has degree r.

The maximum (minimum) cardinality $|e|$ of any edge $e \in E$ is called the *rank* (*corank*, respectively) of H. A hypergraph H is *uniform of rank r* (or *r-uniform*) if $|e| = r$ for all $e \in E$. An edge $e \in E$ is called a *singleton edge* if $|e| = 1$, and *empty* if $|e| = 0$.

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The concepts of isomorphism and incidence matrix for hypergraphs are straightforward generalizations from graphs and designs; see [1] for more details.

The following types of subhypergraphs will be used in this paper.

Definition 2.3. Let $H = (V, E)$ be a hypergraph.

1. A hypergraph $H' = (V', E')$ is called a *subhypergraph* of H if $V' \subseteq V$ and either $E' = \emptyset$ or the incidence matrix of H', after a suitable permutation of its rows and columns, is a submatrix of the incidence matrix of H. Thus, every edge $e' \in E'$ is of the form $e \cap V'$ for some $e \in E$, and the corresponding mapping from E' to E is injective.

2. A subhypergraph $H' = (V', E')$ of H with $E' = \{\!\{e \cap V' : e \in E, e \cap V' \neq \emptyset\}\!\}$ is said to be *induced by* V'.

3. If $|V| \geq 2$ and $v \in V$, then $H \backslash v$ will denote the subhypergraph of H induced by $V - \{v\}$, also called a *vertex-deleted subhypergraph* of H.

4. A hypergraph $H' = (V', E')$ is called a *hypersubgraph* of H if $V' \subseteq V$ and $E' \subseteq E$.

5. A hypersubgraph $H' = (V', E')$ of H is said to be *induced by* V', denoted by $H[V']$, if $E' = \{\!\{e \in E : e \subseteq V', e \neq \emptyset\}\!\}$.

6. A hypersubgraph $H' = (V', E')$ of H is said to be *induced by* E', denoted by $H[E']$, if $V' = \cup_{e \in E'} e$.

7. For $E' \subseteq E$ and $e \in E$, we write shortly $H - E'$ and $H - e$ for the hypersubgraphs $(V, E - E')$ and $(V, E - \{\!\{e\}\!\})$, respectively. The hypersubgraph $H - e$ may also be called an *edge-deleted hypersubgraph*.

Note that the above definitions of subhypergraphs and vertex-subset-induced subhypergraphs are consistent with [7]. A more detailed discussion of these terms can be found in [1].

Observe that, informally speaking, the vertex-deleted subhypergraph $H \backslash v$ is obtained from H by removing vertex v from V and from all edges of H, and then discarding the empty edges.

It is easy to see that every hypersubgraph of $H = (V, E)$ is also a subhypergraph of H, but not conversely. However, not every hypersubgraph of H induced by $V' \subseteq V$ is a subhypergraph of H induced by V'.

Observe also that if H is a 2-uniform hypergraph (and hence a loopless graph), its hypersubgraphs, vertex-subset-induced hypersubgraphs, edge-subset-induced hypersubgraphs, and edge-deleted hypersubgraphs are precisely its subgraphs, vertex-subset-induced subgraphs, edge-subset-induced subgraphs, and edge-deleted subgraphs (in the graph-theoretic sense), respectively. However, its vertex-deleted subgraphs are obtained by deleting all singleton edges from its vertex-deleted subhypergraphs.

The union and intersection of hypergraphs is again defined analogously to graphs, and if a hypergraph H is an edge-disjoint union of hypergraphs H_1 and H_2, then $\{H_1, H_2\}$ is a *decomposition* of H, and we write $H = H_1 \oplus H_2$.

The *dual* of a non-empty hypergraph H is a hypergraph H^T whose incidence matrix is the transpose of the incidence matrix of H. To obtain the dual $H^T = (E^T, V^T)$ of a hypergraph $H = (V, E)$, we label the edges of H as $e_1, \ldots, e_m$ (with distinct parallel edges receiving distinct labels). Then let $E^T = \{e_1, \ldots, e_m\}$ and $V^T = \{\!\{v^T : v \in V\}\!\}$, where $v^T = \{e \in E^T : v \in e\}$ for all $v \in V$. Observe that $(v, e) \in F(H)$ if and only if $(e, v^T) \in F(H^T)$. Hence $(H^T)^T = H$.

Lemma 2.4. *Let $H = (V, E)$ be a non-empty hypergraph with the dual $H^T = (E^T, V^T)$, and let $v \in V$ and $e \in E$. Then:*

1. $\deg_H(v) = |v^T|$.
2. *v is an isolated vertex (pendant vertex) in H if and only if v^T is an empty edge (singleton edge, respectively) in H^T.*
3. *If $|V| \geq 2$, H has no empty edges, and $\{v\} \notin E$, then $(H \backslash v)^T = H^T - v^T$.*
4. *If $|E| \geq 2$, H has no isolated vertices, and e contains no pendant vertices, then $(H - e)^T = H^T \backslash e$.*

Proof. The first two statements of the lemma follow straight from the definition of vertex degree.

To see the third statement, assume that $|V| \geq 2$, H has no empty edges, and $\{v\} \notin E$. Now $H \backslash v$ is obtained from H by deleting vertex v, deleting all flags containing v from $F(H)$, and discarding all resulting empty edges. Hence $(H \backslash v)^T$ is obtained from H^T by deleting edge v^T, deleting all flags containing v^T from $F(H^T)$, and discarding all resulting isolated vertices. However, any such isolated vertex would in H^T correspond either to an isolated vertex or a pendant vertex incident only with the edge v^T. This would imply existence of an empty edge or an edge $\{v\}$ in H, a contradiction. Hence $(H \backslash v)^T$ was obtained from H^T just by deleting edge v^T and all flags containing v^T; that is, $(H \backslash v)^T = H^T - v^T$.

To prove the fourth statement, assume that $|E| \geq 2$, H has no isolated vertices, and e contains no pendant vertices. Recall that $H - e$ is obtained from H, and similarly $(H - e)^T$ from H^T, by deleting e and all flags containing e. This operation on H^T is exactly vertex deletion provided that $(H - e)^T$ has no empty edges. Now an empty edge in $(H - e)^T$ corresponds to an isolated vertex in $H - e$, and hence in H, it corresponds either to an isolated vertex or a pendant vertex incident with e. However, by assumption, H does not have such vertices. We conclude that $(H - e)^T = H^T \backslash e$ as claimed. □

2.2 Graphs associated with a hypergraph

A hypergraph is, of course, an incidence structure, and hence can be represented with an incidence graph (to be defined below). This representation retains complete information about the hypergraph, and thus allows us to translate problems about hypergraphs into problems about graphs — a much better explored territory.

Definition 2.5. Let $H = (V, E)$ be a hypergraph with incidence function ψ. The *incidence graph* $\mathcal{G}(H)$ of H is the graph $\mathcal{G}(H) = (V_G, E_G)$ with $V_G = V \cup E$ and $E_G = \{ve : v \in V, e \in E, v \in \psi(e)\}$.

Observe that the incidence graph $\mathcal{G}(H)$ of a hypergraph $H = (V, E)$ with $E \neq \emptyset$ is a bipartite simple graph with bipartition $\{V, E\}$. We shall call a vertex x of $\mathcal{G}(H)$ a *v-vertex* if $x \in V$, and an *e-vertex* if $x \in E$. Note that the edge set of $\mathcal{G}(H)$ can be identified with the flag (multi)set $F(H)$; that is, $E_G = \{ve : (v, e) \in F(H)\}$.

The following is an easy observation, hence the proof is left to the reader.

Lemma 2.6. *Let $H = (V, E)$ be a non-empty hypergraph and $H^T = (E^T, V^T)$ its dual. The incidence graphs $\mathcal{G}(H)$ and $\mathcal{G}(H^T)$ are isomorphic with an isomorphism $\varphi : V \cup E \to E^T \cup V^T$ defined by $\varphi(e) = e$ for all $e \in E$, and $\varphi(v) = v^T$ for all $v \in V$.*

Next, we outline the relationship between subhypergraphs of a hypergraph and the subgraphs of its incidence graph. The proof of this lemma is straightforward and hence omitted.

Lemma 2.7. *Let $H = (V, E)$ be a hypergraph and $H' = (V', E')$ a subhypergraph of H. Then:*

1. *$\mathcal{G}(H')$ is the subgraph of $\mathcal{G}(H)$ induced by the vertex set $V' \cup E'$.*
2. *If H' is a hypersubgraph of H, then in addition, $\deg_{\mathcal{G}(H')}(e) = \deg_{\mathcal{G}(H)}(e) = |e|$ for all $e \in E'$.*

Conversely, take a subgraph G' of $\mathcal{G}(H)$. Then:

1. *$V(G') = V' \cup E'$ for some $V' \subseteq V$ and $E' \subseteq E$, and $E(G') \subseteq \{ve : v \in V', e \in E', v \in e\}$.*
2. *G' is the incidence graph of a subhypergraph of H if and only if $V' \neq \emptyset$ and for all $e \in E'$ we have $\{ve : v \in e \cap V'\} \subseteq E(G')$.*
3. *G' is the incidence graph of a hypersubgraph of H if and only if $V' \neq \emptyset$ and $\deg_{G'}(e) = \deg_{\mathcal{G}(H)}(e) = |e|$ for all $e \in E'$.*

In the following lemma, we determine the incidence graphs of vertex-deleted subhypergraphs and edge-deleted hypersubgraphs.

Lemma 2.8. *Let $H = (V, E)$ be a hypergraph. Then:*

1. *For all $e \in E$, we have $\mathcal{G}(H - e) = \mathcal{G}(H) \backslash e$.*
2. *If $|V| \geq 2$, H has no empty edges, and $v \in V$ is such that $\{v\} \notin E$, then $\mathcal{G}(H \backslash v) = \mathcal{G}(H) \backslash v$.*

Proof.

1. Recall that $H - e$ is obtained from H by deleting e from E, thus also destroying all flags containing e. This is equivalent to deleting e from the vertex set of $\mathcal{G}(H)$, as well as all edges of $\mathcal{G}(H)$ incident with e, which results in the vertex-deleted subgraph $\mathcal{G}(H) \backslash e$.
2. Now $H \backslash v$ is obtained from H by deleting v from V and from all edges containing v, and then discarding all resulting empty edges. However, if H has no empty edges and $\{v\} \notin E$, then there are no empty edges to discard, and so this operation is equivalent to deleting v from the vertex set of $\mathcal{G}(H)$ and deleting all edges of $\mathcal{G}(H)$ incident with v, resulting in the vertex-deleted subgraph $\mathcal{G}(H) \backslash v$. Hence $\mathcal{G}(H) \backslash v = \mathcal{G}(H \backslash v)$.

□

3 Connection in Hypergraphs

3.1 Walks, trails, paths, cycles

In this section, we would like to systematically generalize the standard graph-theoretic notions of walks, trails, paths, and cycles to hypergraphs. In this context, we need to distinguish between distinct parallel edges, hence the original definition of a hypergraph that includes the incidence function will be used.

Definition 3.1. Let $H = (V, E)$ be a hypergraph with incidence function ψ, let $u, v \in V$, and let $k \geq 0$ be an integer. A (u, v)-*walk of length* k in H is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ of vertices and edges (possibly repeated) such that $v_0, v_1, \dots, v_k \in V$, $e_1, \dots, e_k \in E$, $v_0 = u$, $v_k = v$, and for all $i = 1, 2, \dots, k$, the vertices v_{i-1} and v_i are adjacent in H via the edge e_i.

If $W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ is a walk in H, then vertices v_0 and v_k are called the *endpoints* of W, and $v_1, \dots, v_{k-1}$ are the *internal vertices* of W.

We denote the set of all edges of a walk W by $E(W)$, and the set of all its vertices by $V(W)$; that is, $V(W) = \bigcup_{e \in E(W)} e$. Furthermore, vertices $v_0, v_1, \dots, v_k$ are called the *anchor vertices* (or *anchors*) of W, and we write $V_a(W) = \{v_0, v_1, \dots, v_k\}$.

Observe that since adjacent vertices are by definition distinct, no two consecutive vertices in a walk are the same. Note that the edge set $E(W)$ of a walk W may contain distinct parallel edges.

Recall that a trail in a graph is a walk with no repeated edges. For a walk in a graph, having no repeated edges is necessary and sufficient for having no repeated flags; in a hypergraph, only sufficiency holds. This observation suggests two possible ways to define a trail.

Definition 3.2. Let $W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ be a walk in a hypergraph $H = (V, E)$ with incidence function ψ.

1. If the anchor flags $(v_0, e_1), (v_1, e_1), (v_1, e_2), \dots, (v_{k-1}, e_k), (v_k, e_k)$ are pairwise distinct, then W is called a *trail*.

2. If the edges $e_1, \dots, e_k$ are pairwise distinct, then W is called a *strict trail*.

3. If the anchor flags $(v_0, e_1), (v_1, e_1), (v_1, e_2), \dots, (v_{k-1}, e_k), (v_k, e_k)$ and the vertices $v_0, v_1, \dots, v_k$ are pairwise distinct (but the edges need not be), then W is called a *pseudo path*.

4. If both the vertices $v_0, v_1, \dots, v_k$ and the edges $e_1, \dots, e_k$ are pairwise distinct, then W is called a *path*.

We emphasize that in the above definitions, "distinct" should be understood in the strict sense; that is, parallel edges need not be distinct.

We extend the above definitions to closed walks in the usual way.

Definition 3.3. Let $W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ be a walk in a hypergraph $H = (V, E)$ with incidence function ψ. If $k \geq 2$ and $v_0 = v_k$, then W is called a *closed walk*. Moreover:

1. If W is a trail (strict trail), then it is called a *closed trail* (*closed strict trail*, respectively).

2. If W is a closed trail and the vertices $v_0, v_1, \dots, v_{k-1}$ are pairwise distinct (but the edges need not be), then W is called a *pseudo cycle*.

3. If the vertices $v_0, v_1, \dots, v_{k-1}$ and the edges $e_1, \dots, e_k$ are pairwise distinct, then W is called a *cycle*.

From the above definitions, the following observations are immediate.

Lemma 3.4. *Let W be a walk in a hypergraph H. Then:*

1. *If W is a trail, then no two consecutive edges in W are the same (including the last and the first edge if W is a closed trail).*

2. *If W is a (closed) strict trail, then it is a (closed) trail.*

3. *If W is a pseudo path (pseudo cycle), then it is a trail (closed trail, respectively), but not necessarily a strict trail (closed strict trail, respectively).*

4. *If W is a path (cycle), then it is both a pseudo path (pseudo cycle, respectively) and a strict trail (closed strict trail, respectively).*

We mention that several special types of hypergraph cycles have been defined and studied in the literature, for example, *loose cycles* and *tight cycles*. Our definition coincides with the one in [2] and [7]; in fact, our cycles are sometimes called *Berge cycles*.

In a graph, a path or cycle can be identified with the corresponding subgraph (also called path or cycle, respectively). This is not the case in hypergraphs. First, we note that there are (at least) two ways to define a subhypergraph associated with a path or cycle. We define these more generally for walks.

Definition 3.5. Let W be a walk in a hypergraph $H = (V, E)$. Define the hypersubgraph $\mathcal{H}(W)$ and a subhypergraph $\mathcal{H}'(W)$ of H associated with the walk W as follows:

$$\mathcal{H}(W) = (V(W), E(W))$$

and

$$\mathcal{H}'(W) = (V_a(W), \{\!\{ e \cap V_a(W) : e \in E(W) \}\!\}).$$

That is, $\mathcal{H}'(W)$ is the subhypergraph of $\mathcal{H}(W)$ induced by the set of anchor vertices $V_a(W)$.

Second, we observe that, even when W is a path or a cycle, not much can be said about the degrees of the vertices in the associated subhypergraphs $\mathcal{H}(W)$ and $\mathcal{H}'(W)$. Thus, unlike in graphs, we can not use a path (cycle) W (as a sequence of vertices and edges) and its associated subhypergraphs $\mathcal{H}(W)$ and $\mathcal{H}'(W)$ interchangeably.

The following lemma will justify the terminology introduced in this section.

Lemma 3.6. *Let $H = (V, E)$ be a hypergraph and $G = \mathcal{G}(H)$ its incidence graph. Let $v_i \in V$ for $i = 0, 1, \ldots, k$, and $e_i \in E$ for $i = 1, \ldots, k$, and let $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ be an alternating sequence of vertices and edges of H. Denote the corresponding sequence of vertices in G by W_G. Then the following hold:*

1. *W is a (closed) walk in H if and only if W_G is a (closed) walk in G with no two consecutive v-vertices the same.*

2. *W is a trail (path, cycle) in H if and only if W_G is a trail (path, cycle, respectively) in G.*

3. *W is a strict trail in H if and only if W_G is a trail in G that visits every $e \in E$ at most once.*

4. *W is a pseudo path (pseudo cycle) in H if and only if W_G is a trail (closed trail, respectively) in G that visits every $v \in V$ at most once.*

Proof. 1. If W is a walk in H, then any two consecutive elements of the sequence W are incident in H, and hence the corresponding vertices are adjacent in G. Thus W_G is a walk in G. Moreover, no two consecutive vertices in W are the same, whence not two consecutive v-vertices in W_G are the same. The converse is shown similarly. Clearly W is closed if and only if W_G is.

Observe that the anchor vertices and the edges of W correspond to the v-vertices and e-vertices of W_G, respectively, and the anchor flags of W correspond to the edges of W_G.

2. If W is a trail in H, then W is a walk with no repeated anchor flags; hence W_G is a walk in G with no repeated edges, that is, a trail. Conversely, if W_G is a trail in G, then it is a walk with no repeated edges, and hence no two identical consecutive v-vertices. It follows that W is a walk in H with no repeated anchor flags, that is, a trail.

   Similarly, if W is a path (cycle) in H, then W is a walk with no repeated edges and no repeated vertices (except the endpoints for a cycle). Hence W_G is a walk in G with no repeated vertices (except the endpoints for a cycle), that is, a path (cycle, respectively). The converse is shown similarly.

3. If W is a strict trail in H, then it is a trail with no repeated edges. Hence W_G is a trail in G with no repeated e-vertices. The converse is shown similarly.

4. If W is a pseudo path (pseudo cycle) in H, then it is a trail with no repeated vertices (except the endpoints for a pseudo cycle). Hence W_G is a trail in G with no repeated v-vertices (except the endpoints for a pseudo cycle). The converse is similar.

$\square$

The next observations are easy to see, hence the proof is omitted.

Lemma 3.7. *Let $H = (V, E)$ be a non-empty hypergraph and $H^T = (E^T, V^T)$ its dual. Let $v_i \in V$ for $i = 0, 1, \dots, k-1$, and $e_i \in E$ for $i = 0, 1, \dots, k-1$, and let $W = v_0 e_0 v_1 e_1 v_2 \dots v_{k-1} e_{k-1} v_0$ be a closed walk in H. Denote $W^T = e_0 v_1^T e_1 v_2^T \dots v_{k-1}^T e_{k-1} v_0^T e_0$, where for each vertex v_i of H, the symbol v_i^T denotes the corresponding edge in H^T. Then the following hold:*

1. *If $e_i \neq e_{i+1}$ for all $i \in \mathbb{Z}_k$, then W^T is a closed walk in H^T.*
2. *If W is a closed trail (cycle) in H, then W^T is a closed trail (cycle, respectively) in H^T.*
3. *If W is a strict closed trail in H, then W^T is a pseudo cycle in H^T.*
4. *If W is a pseudo cycle in H, then W^T is a strict closed trail in H^T.*

3.2 Connected hypergraphs

Connected hypergraphs are defined analogously to connected graphs, using existence of walks (or equivalently, existence of paths) between every pair of vertices. The main result of this section is the observation that a hypergraph (without empty edges) is connected if and only if its incidence graph is connected. The reader will observe that existence of empty edges in a hypergraph does not affect its connectivity; however, it does affect the connectivity of the incidence graph.

Definition 3.8. Let $H = (V, E)$ be a hypergraph. Vertices $u, v \in V$ are said to be *connected* in H if there exists a (u, v)-walk in H. The hypergraph H is said to be *connected* if every pair of distinct vertices are connected in H.

Lemma 3.9. *Let $H = (V, E)$ be a hypergraph, and $u, v \in V$. There exists a (u, v)-walk in H if and only if there exists a (u, v)-path.*

Proof. Suppose H has a (u, v)-walk. By Lemma 3.6, it corresponds to a (u, v)-walk in the incidence graph $\mathcal{G}(H)$, and by a classical result in graph theory, existence of a (u, v)-walk in a graph guarantees existence of a (u, v)-path. Finally, by Lemma 3.6, a (u, v)-path in $\mathcal{G}(H)$ (since $u, v \in V$) corresponds to a (u, v)-path in H.

The converse obviously holds by definition. □

It is clear that vertex connection in a hypergraph $H = (V, E)$ is an equivalence relation on the set V. Hence the following definition makes sense.

Definition 3.10. Let $H = (V, E)$ be a hypergraph, and let $V' \subseteq V$ be an equivalence class with respect to vertex connection. The hypersubgraph of H induced by V' is called a *connected component* of H. We denote the number of connected components of H by $\mathrm{c}(H)$.

Observe that, by the definition of a vertex-subset-induced hypersubgraph, the connected components of a hypergraph have no empty edges. Alternatively, the connected components of H can be defined as the maximal connected hypersubgraphs of H that have no empty edges. It is easy to see that for a hypergraph $H = (V, E)$ with the multiset of empty edges denoted E_0, the hypersubgraph $H - E_0$ decomposes into the connected components of H.

Theorem 3.11. *Let $H = (V, E)$ be a hypergraph without empty edges. Then H is connected if and only if its incidence graph $G = \mathcal{G}(H)$ is connected.*

Proof. Assume H is connected. Take any two vertices x, y of G. If x and y are both v-vertices, then there exists an (x, y)-walk in H, and hence, by Lemma 3.6, an (x, y)-walk in G. If x is an e-vertex and y is a v-vertex in G , then x is a non-empty edge in H. Choose any $v \in x$. Since H is connected, it possesses a (v, y)-walk W. Then xW is an (x, y)-walk in G. The remaining case $x, y \in E$ is handled similarly. We conclude that G is connected.

Assume G is connected. Take any two vertices u, v of H. Then there exists (u, v)-path in G, and hence by Lemma 3.6, a (u, v)-path in H. Therefore H is connected. □

Corollary 3.12. *Let H be a hypergraph and $G = \mathcal{G}(H)$ its incidence graph. Then:*

1. *If H' is a connected component of H, then $\mathcal{G}(H')$ is a connected component of G.*
2. *If G' is a connected component of G with at least one v-vertex, then there exists a connected component H' of H such that $G' = \mathcal{G}(H')$.*
3. *If H has no empty edges, then there is a one-to-one correspondence between connected components of H and connected components of its incidence graph.*

Proof.

1. Let H' be a connected component of H, and let $G' = \mathcal{G}(H')$. Since H' has no empty edges by definition, G' is connected by Theorem 3.11. Let G'' be the connected component of G containing G' as a subgraph. Then G'' contains v-vertices and $\deg_{G''}(e) = \deg_G(e)$ for all e-vertices e of G'', and so by Lemma 2.7, $G'' = \mathcal{G}(H'')$ for some hypersubgraph H'' of H. Since G'' is connected and the incidence graph of a hypergraph, it has no isolated e-vertices. Hence H'' has no empty edges, and so by Theorem 3.11, H'' is connected since G'' is. Now H' is a maximal connected hypersubgraph of H without empty edges, and a hypersubgraph of a connected hypersubgraph H'' without empty edges; it must be that $H'' = H'$. Consequently, $G' = G''$ and so G' is indeed a connected component of G.
2. Let G' be a connected component of G with at least one v-vertex. Then $\deg_{G'}(e) = \deg_G(e)$ for all e-vertices e of G', and so by Lemma 2.7, $G' = \mathcal{G}(H')$ for some hypersubgraph H' of H. Since G' is connected and the incidence graph of a hypergraph, it has no isolated e-vertices; hence H' has no empty edges. Thus, by Theorem 3.11, H' is connected since G' is. Let H'' be the connected component of H containing H', and $G'' = \mathcal{G}(H'')$. Again by Theorem 3.11, G'' is connected, and hence $G'' = G'$ by the maximality of G'. It follows that $H' = H''$, so indeed $G' = \mathcal{G}(H')$, where H' is a connected component of H.
3. Since H has no empty edges, every connected component of G has at least one v-vertex. The conclusion now follows directly from the first two statements of the corollary.

□

Corollary 3.13. *Let H be a hypergraph without empty edges and $G = \mathcal{G}(H)$ its incidence graph. Then:*

1. $\mathrm{c}(H) = \mathrm{c}(G)$.

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2. *If H is non-empty and has no isolated vertices, and H^T is its dual, then* $\mathrm{c}(H) = \mathrm{c}(H^T)$.

Proof. 1. Since H has no empty edges, by Corollary 3.12 there is a one-to-one correspondence between the connected components of H and G. Therefore, $\mathrm{c}(H) = \mathrm{c}(G)$.

2. Assume H is non-empty and has no isolated vertices. Then H^T is well defined and has no empty edges, and so $\mathrm{c}(H^T) = \mathrm{c}(\mathcal{G}(H^T))$ by the first statement. Since by Lemma 2.6 a hypergraph and its dual have isomorphic incidence graphs, it follows that $\mathrm{c}(H^T) = \mathrm{c}(\mathcal{G}(H^T)) = \mathrm{c}(G) = \mathrm{c}(H)$.

□

3.3 Cut edges and cut vertices

In this section, we define cut edges and cut vertices in a hypergraph analogously to those in a graph. The existence of cut edges and cut vertices is one of the first measures of strength of connectivity of a connected graph. In hypergraphs, however, we must consider two distinct types of cut edges.

Definition 3.14. A *cut edge* in a hypergraph $H = (V, E)$ is an edge $e \in E$ such that $\mathrm{c}(H - e) > \mathrm{c}(H)$.

Lemma 3.15. *Let e be a cut edge in a hypergraph $H = (V, E)$. Then*

$$\mathrm{c}(H) < \mathrm{c}(H - e) \leq \mathrm{c}(H) + |e| - 1.$$

Proof. The inequality on the left follows straight from the definiton of a cut edge. To see the inequality on the right, first observe that e is not empty. Let $H_1, \ldots, H_k$ be the connected components of $H - e$ whose vertex sets intersect e. Since e has at least one vertex in common with each $V(H_i)$, we have $|e| \geq k$. Hence $\mathrm{c}(H - e) = \mathrm{c}(H) + k - 1 \leq \mathrm{c}(H) + |e| - 1$. □

Definition 3.16. A cut edge e of a hypergraph H is called *strong* if $\mathrm{c}(H - e) = \mathrm{c}(H) + |e| - 1$, and *weak* otherwise.

Observe that a cut edge has cardinality at least two, and that any cut edge of cardinality two (and hence any cut edge in a simple graph) is necessarily strong.

Recall that an edge of a graph is a cut edge if and only if appears in no cycle. We shall now show that an analogous statement holds for hypergraphs if we replace "cut edge" with "strong cut edge".

Theorem 3.17. *Let e be an edge in a connected hypergraph $H = (V, E)$. The following are equivalent:*

1. *e is a strong cut edge, that is,* $\mathrm{c}(H - e) = |e|$.
2. *e contains exactly one vertex from each connected component of $H - e$.*
3. *e lies in no cycle of H.*

Proof. (1) $\Rightarrow$ (2): Let e be a strong cut edge of H. Since H is connected, the edge e must have at least one vertex in each connected component of $H-e$. Since there are $|e|$ connected components of $H-e$, the edge e must have exactly one vertex in each of them.

(2) $\Rightarrow$ (1): Assume e contains exactly one vertex from each connected component of $H-e$. Then clearly $\mathrm{c}(H-e)=|e|$.

(2) $\Rightarrow$ (3): Assume e contains exactly one vertex from each connected component of $H-e$, and suppose e lies in a cycle $C=v_0e_1v_1e_2v_2\dots v_{k-1}ev_0$ of H. Then $v_0e_1v_1e_2v_2\dots v_{k-1}$ is a path in $H-e$, and so v_0 and v_{k-1} are two vertices of e in the same connected component of $H-e$, a contradiction. Hence e lies in no cycle of H.

(3) $\Rightarrow$ (2): Assume e lies in no cycle of H. Since H is connected, the edge e must contain at least one vertex from each connected component of $H-e$. Suppose e contains two vertices u and v in the same connected component H' of $H-e$. Then H' contains a (u,v)-path P, and $Pveu$ is a cycle in H that contains e, a contradiction. Hence e possesses exactly one vertex from each connected component of $H-e$. □

The above theorem can be easily generalized to all (possibly disconnected) hypergraphs as follows.

Corollary 3.18. *Let e be an edge in a hypergraph $H=(V,E)$. The following are equivalent:*

1. *e is a strong cut edge, that is,* $\mathrm{c}(H-e)=\mathrm{c}(H)+|e|-1$.
2. *e contains exactly one vertex from each connected component of $H-e$ that it intersects.*
3. *e lies in no cycle of H.*

We know that an even graph has no cut edges; in other words, every edge of an even graph (that is, a graph with no odd-degree vertices) lies in a cycle. This statement is false for hypergraphs, as the example below demonstrates. In the following two theorems, however, we present two generalizations to hypergraphs that do hold.

Counterexample 3.19. For every even $n\geq 2$, define a hypergraph $H=(V,E)$ as follows. Let $V=\{v_i : i=1,\dots,2n\}$ and $E=\{e_i : i=1,\dots,2n\}$, and let $F(H)=\{(v_i,e_j) : i,j=1,\dots,n\}\cup\{(v_i,e_j) : i,j=n+1,\dots,2n\}\cup\{(v_1,e_{n+1})\}-\{(v_1,e_1)\}$. Then every vertex in H has degree n, which is even, but e_{n+1} is a cut edge in H.

Theorem 3.20. *Let $H=(V,E)$ be a k-uniform hypergraph such that $\deg_H(u)\equiv 0 \pmod{k}$ for every vertex u of H. Then H has no cut edges.*

Proof. Suppose e is a cut edge of H, and let $H_1=(V_1,E_1)$ be a connected component of $H-e$ that contains a vertex of e. Furthermore, let $r=|e\cap V_1|$. Then $1\leq r\leq k-1$, and so $\sum_{v\in V_1}\deg_{H_1}(v)\equiv k|V_1|-r\not\equiv 0 \pmod{k}$. However, $\sum_{v\in V_1}\deg_{H_1}(v)=\sum_{f\in E_1}|f|=k|E_1|$, a contradiction. Hence H cannot have cut edges. □

Theorem 3.21. *Let $H=(V,E)$ be a hypergraph such that the degree of each vertex and the cardinality of each edge are even. If e is a cut edge of H, then every connected component of $H-e$ contains an even number of vertices of e. In particular, H has no strong cut edges.*

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Proof. Suppose e is a cut edge of H, and let $H_1 = (V_1, E_1)$ be any connected component of $H - e$. Furthermore, let $r = |e \cap V_1|$. Then $\sum_{v \in V_1} \deg_{H_1}(v) = (\sum_{v \in V_1} \deg_H(v)) - r = \sum_{f \in E_1} |f|$. Since $\sum_{v \in V_1} \deg_H(v)$ and $\sum_{f \in E_1} |f|$ are both even, so is r. Thus e intersects every connected component in an even number of vertices, and hence by Corollary 3.18 cannot be a strong cut edge. □

We now turn our attention to cut vertices. Recall that the vertex-deleted subhypergraph $H \backslash v$ is obtained from H by deleting v from the vertex set, as well as from all edges containing v, and then discarding any resulting empty edges.

Definition 3.22. A *cut vertex* in a hypergraph $H = (V, E)$ with $|V| \geq 2$ is a vertex $v \in V$ such that $\mathrm{c}(H \backslash v) > \mathrm{c}(H)$.

Before we can prove a result similar to Lemma 3.15 for cut vertices, we need to examine the relationship between cut vertices and cut edges of a hypergraph and its dual, as well as the relationship between cut vertices and cut edges of a hypergraph and cut vertices of its incidence graph.

Theorem 3.23. *Let $H = (V, E)$ be a hypergraph without empty edges, and $G = \mathcal{G}(H)$ be its incidence graph.*

1. *Take any $e \in E$. Then e is a cut edge of H if and only if it is a cut vertex of G.*

2. *Let $|V| \geq 2$ and take any $v \in V$ such that $\{v\} \notin E$. Then v is a cut vertex of H if and only if it is a cut vertex of G.*

Proof.
1. By Lemma 2.8, we have $\mathcal{G}(H - e) = G \backslash e$. Since H, and hence $H - e$, has no empty edges, Corollary 3.13 tells us that $\mathrm{c}(H) = \mathrm{c}(G)$ and $\mathrm{c}(H - e) = \mathrm{c}(\mathcal{G}(H - e))$. Hence $\mathrm{c}(H - e) = \mathrm{c}(G \backslash e)$. Thus $\mathrm{c}(H - e) - \mathrm{c}(H) = \mathrm{c}(G \backslash e) - \mathrm{c}(G)$, and it follows that e is a cut edge of H if and only if it is a cut vertex of G.

2. Since H has no empty edges and $\{v\} \notin E$, Lemma 2.8 shows that $\mathcal{G}(H \backslash v) = G \backslash v$. Since H and $H \backslash v$ have no empty edges, Corollary 3.13 gives $\mathrm{c}(H) = \mathrm{c}(G)$ and $\mathrm{c}(H \backslash v) = \mathrm{c}(\mathcal{G}(H \backslash v))$, respectively. Hence $\mathrm{c}(H \backslash v) - \mathrm{c}(H) = \mathrm{c}(G \backslash v) - \mathrm{c}(G)$, and v is a cut vertex of H if and only if it is a cut vertex of G.

□

In the next corollary, recall that we denote the dual of a hypergraph $H = (V, E)$ by $H^T = (E^T, V^T)$, where E^T is the set of labels for the edges in E, $V^T = \{v^T : v \in V\}$, and $v^T = \{e \in E^T : v \in e\}$ for all $v \in V$.

Corollary 3.24. *Let $H = (V, E)$ be a non-empty hypergraph with neither empty edges nor isolated vertices, and let H^T be its dual.*

1. *Let $|E| \geq 2$ and let $e \in E$ be an edge without pendant vertices. Then e is a cut edge of H if and only if e is a cut vertex of H^T.*

2. *Let $|V| \geq 2$ and let $v \in V$ be such that $\{v\} \notin E$. Then v is a cut vertex of H if and only if v^T is a cut edge of H^T.*

Proof. 1. First, since H has no empty edges, by Theorem 3.23, e is a cut edge of H if and only if it is a cut e-vertex of $\mathcal{G}(H)$, and hence if and only if e is a cut v-vertex of $\mathcal{G}(H^T)$. On the other hand, since e contains no pendant vertices of H, we have that $\{e\} \notin V^T$. Also, H^T has no empty edges since H has no isolated vertices. Hence by Theorem 3.23, e is a cut vertex of H^T if and only if e is a cut v-vertex of $\mathcal{G}(H^T)$. The result follows.

2. By Theorem 3.23, since H has no empty edges and $\{v\} \notin E$, vertex v is a cut vertex of H if and only if v is a cut v-vertex of $\mathcal{G}(H)$, and hence if and only if v^T is a cut e-vertex of $\mathcal{G}(H^T)$. Again by Theorem 3.23, since H^T has no empty edges, this is the case if and only if v^T is a cut edge of H^T.

□

Corollary 3.25. *Let $H = (V, E)$ be a hypergraph with $|V| \geq 2$, $|E| \geq 1$, and with neither empty edges nor isolated vertices. Furthermore, let v be a cut vertex such that $\{v\} \notin E$. Then $\mathrm{c}(H\backslash v) \leq \mathrm{c}(H) + \deg_H(v) - 1$.*

Proof. Consider the dual H^T of H. Since v is a cut vertex of H and $\{v\} \notin E$, by Corollary 3.24, the edge v^T of H^T is a cut edge, and hence $\mathrm{c}(H^T - v^T) \leq \mathrm{c}(H^T) + |v^T| - 1$ by Lemma 3.15. By Corollary 3.13 we have $\mathrm{c}(H^T) = \mathrm{c}(H)$, and by Lemma 2.4, we have $|v^T| = \deg_H(v)$. It remains to show that $\mathrm{c}(H^T - v^T) = \mathrm{c}(H\backslash v)$. Using Corollary 3.13 and Lemma 2.8, we have

$$\mathrm{c}(H^T - v^T) = \mathrm{c}(\mathcal{G}(H^T - v^T)) = \mathrm{c}(\mathcal{G}(H^T)\backslash v^T) = \mathrm{c}(\mathcal{G}(H)\backslash v)) = \mathrm{c}(\mathcal{G}(H\backslash v)) = \mathrm{c}(H\backslash v)$$

since $H^T - v^T$ has no empty edges, since $\mathcal{G}(H^T - v^T) = \mathcal{G}(H^T)\backslash v^T$, and since $\mathcal{G}(H^T)\backslash v^T$ is isomorphic to $\mathcal{G}(H)\backslash v$, which in turn is equal to $\mathcal{G}(H\backslash v)$ because $\{v\} \notin E$.

We conclude that $\mathrm{c}(H\backslash v) \leq \mathrm{c}(H) + \deg_H(v) - 1$. □

A graph with a cut edge and at least three vertices necessarily possesses a cut vertex. Here is the analogue for hypergraphs.

Theorem 3.26. *Let $H = (V, E)$ be a hypergraph with a cut edge e such that for some non-trivial connected component H' of $H - e$, we have $|e \cap V(H')| = 1$. Then H has a cut vertex.*

Proof. We may assume H is connected. Let H' and H'' be two connected components of $H - e$, with H' non-trivial and $e \cap V(H') = \{u\}$. Take any $x \in V(H') - \{u\}$ and $y \in V(H'')$. Since e is a cut edge, every (x, y)-path P in H must contain the edge e, and since u is the only vertex of e in $V(H')$, any such path P must also contain u as an anchor vertex. Hence x and y are disconnected in $H\backslash u$, and u is a cut vertex of H. □

Corollary 3.27. *Let $H = (V, E)$ be a connected hypergraph with a strong cut edge e such that $|e| < |V|$. Then H has a cut vertex.*

Proof. Let $H_1, \ldots, H_k$ be the connected components of $H - e$. By Theorem 3.17, the edge e contains exactly one vertex from each H_i (for $i = 1, \ldots, k$), and so $k = |e| < |V|$. Hence $|V(H_i)| \geq 2$ for at least one connected component H_i, and $|e \cap V(H_i)| = 1$ since e is a strong cut edge. It follows by Theorem 3.26 that H has a cut vertex. □

3.4 Blocks and non-separable hypergraphs

Throughout this section, we shall assume that our hypergraphs are connected and have no empty edges. We begin by extending the notion of a cut vertex as follows.

Definition 3.28. Let $H = (V, E)$ be a connected hypergraph without empty edges. A vertex $v \in V$ is a *separating vertex* for H if H decomposes into two non-empty connected hypersubgraphs with just vertex v in common. That is, $H = H_1 \oplus H_2$, where H_1 and H_2 are two non-empty connected hypersubgraphs of H with $V(H_1) \cap V(H_2) = \{v\}$.

Theorem 3.29. *Let $H = (V, E)$ be a connected hypergraph without empty edges, with $|V| \geq 2$ and $v \in V$.*

1. *If v is a cut vertex of H, then v is a separating vertex of H.*

2. *If v is a separating vertex of H and $\{v\} \notin E$, then v is a cut vertex of H.*

Proof. 1. Assume v is a cut vertex of H, let V_1 be the vertex set of one connected component of $H\backslash v$, and let $V_2 = V(H\backslash v) - V_1$. Furthermore, let H_1 and H_2 be the subhypergraphs induced by the sets $V_1 \cup \{v\}$ and $V_2 \cup \{v\}$, respectively, so that $E(H_i) = \{\!\{e \cap (V_i \cup \{v\}) : e \in E, e \cap (V_i \cup \{v\}) \neq \emptyset\}\!\}$ for $i = 1, 2$. Clearly $V(H_1) \cap V(H_2) = \{v\}$. We show that H_1 and H_2 are in fact hypersubgraphs of H with just vertex v in common.

Take any edge $e \in E$ and suppose $e \cap V_i \neq \emptyset$ for both $i = 1, 2$. Let $e' = e \cap (V_1 \cup V_2)$. Then e' is an edge of $H\backslash v$ with vertices in both V_1 and V_2, contradicting the fact that V_1 is a connected component of $H\backslash v$. Hence either $e \subseteq V(H_1)$ or $e \subseteq V(H_2)$, and hence either $e \in E(H_1)$ or $e \in E(H_2)$, showing that H decomposes into hypersubgraphs H_1 and H_2 with just vertex v in common.

To see that each H_i is connected, note that every vertex $x \in V_i$ is connected to v in H, and hence also in H_i. Since H_1 and H_2 are non-trivial and connected, they must be non-empty.

Thus v is a separating vertex for H.

2. Assume v is a separating vertex of H such that $\{v\} \notin E$. Let H_1 and H_2 be non-empty connected hypersubgraphs of H with just vertex v in common such that $H = H_1 \oplus H_2$. Hence either $e \in E(H_1)$ or $e \in E(H_2)$ for all $e \in E$. For each $i = 1, 2$, since hypergraph H_i is non-empty and connected without edges of the form $\{v\}$, there exists a vertex $v_i \in V(H_i) - \{v\}$ connected to v in H_i. We can now see that vertices v_1 and v_2 are connected in H but not in $H\backslash v$, since every (v_1, v_2)-path in H must contain v as an anchor vertex. It follows that $H\backslash v$ is disconnected, and so v is a cut vertex of H.

□

Observe that the additional condition in the second statement of the theorem cannot be omitted: a vertex incident with a singleton edge and at least one more edge (which, as we show below, is necessarily a separating vertex) need not be a cut vertex. A simple example is a hypergraph $H = (V, E)$ with $V = \{u, v\}$ and $E = \{e_1, e_2\}$ for $e_1 = \{v\}$ and $e_2 = \{u, v\}$. Then v is a separating vertex of H since $H = H_1 \oplus H_2$ for $H_1 = (\{v\}, \{e_1\})$ and $H_2 = (\{u, v\}, \{e_2\})$, so v is a separating vertex. However, v is not a cut vertex since $H\backslash v = (\{u\}, \{\{u\}\})$ is connected.

Lemma 3.30. *Let $H = (V, E)$ be a connected hypergraph without empty edges, with $|E| \geq 2$, and with $v \in V$ such that $\{v\} \in E$. Then v is a separating vertex for H.*

Proof. Since H is connected and has at least two (non-empty) edges, it must have at least two edges incident with v. Let $e_1 = \{v\}$ and e_2 be another edge incident with v. Furthermore, let $H_1 = (\{v\}, \{e_1\})$ and $H_2 = (V, E - \{e_1\})$. Then H_1 and H_2 are two non-empty connected hypersubgraphs of H with just vertex v in common such that $H = H_1 \oplus H_2$. Hence v is a separating vertex for H. □

Recall that in a graph without loops, separating vertices are precisely the cut vertices. Hence these two terms are equivalent for the incidence graph of a hypergraph. Next, we determine the correspondence between separating vertices of a hypergraph and separating vertices (cut vertices) of its incidence graph.

Theorem 3.31. *Let $H = (V, E)$ be a connected hypergraph without empty edges, and $G = \mathcal{G}(H)$ be its incidence graph. Take any $v \in V$. Then v is a separating vertex of H if and only if it is a separating vertex (cut vertex) of G.*

Proof. If $|V| \geq 2$ and $\{v\} \notin E$, then by Theorem 3.29, v is a separating vertex of H if and only if it is a cut vertex of H and therefore, by Theorem 3.23, if and only if it is a cut vertex (separating vertex) of G.

Assume $e = \{v\} \in E$. If v is a separating vertex of H, then it must be incident with another edge e'. Hence in the graph $G \backslash v$, vertex e is an isolated vertex and e' lies in another connected component, showing that v is a cut vertex for G. Conversely, if v is a cut vertex of G, then G must contain e-vertices adjacent to v other than e, and hence H contains edges incident with v other than e. Hence, by Lemma 3.30, v is a separating vertex of H.

The remaining case is that $|V| = 1$ and $\{v\} \notin E$. Then H must be empty, G is a trivial graph, and v is a separating vertex for neither. □

Corollary 3.32. *Let $H = (V, E)$ be a connected non-empty hypergraph with neither empty edges nor isolated vertices, and let H^T be its dual. Let $v \in V$ and $e \in E$, and let v^T and e be the corresponding edge and vertex, respectively, in H^T. Then:*

1. *v is a separating vertex of H if and only if v^T is a cut edge of H^T.*
2. *e is a cut edge of H if and only if it is a separating vertex of H^T.*

Proof. Observe that by Corollary 3.13, H^T is connected since H is. Clearly, it is also non-empty with neither empty edges nor isolated vertices.

1. By Theorem 3.31, v is a separating vertex of H if and only if it is a cut vertex of its incidence graph $\mathcal{G}(H)$, and by Theorem 3.23, v^T is a cut edge of H^T if and only if it is a cut vertex of $\mathcal{G}(H^T)$. Since $\mathcal{G}(H)$ and $\mathcal{G}(H^T)$ are isomorphic with an isomorphism mapping v to v^T, the result follows.
2. Interchanging the roles of H and H^T, this statement follows from the previous one.

□

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We shall now define blocks of a hypergraph, and in the rest of this section, investigate their properties.

Definition 3.33. A connected hypergraph without empty edges that has no separating vertices is called *non-separable*. A *block* of a hypergraph H is a maximal non-separable hypersubgraph of H.

Lemma 3.34. *Let H be a connected hypergraph without empty edges and B an empty block of H. Then $H = B$, and H is empty and trivial.*

Proof. Since B is empty and connected, it contains a single vertex, say v. If H is non-empty, then it contains an edge e incident with v. But then $(e, \{e\})$ is a non-separable hypersubgraph of H that properly contains the block B, a contradiction. Hence H is empty. Since it is connected, it must also be trivial (that is, $V = \{v\}$). Consequently, $H = B$. □

In a graph, every cycle is contained within a block. What follows is the analogous result for hypergraphs.

Lemma 3.35. *Let H be a hypergraph without empty edges, C a cycle in H, and $\mathcal{H}(C)$ and $\mathcal{H}'(C)$ the hypersubgraph and subhypergraph, respectively, of H associated with C (see Definition 3.5). Then $\mathcal{H}(C)$ and $\mathcal{H}'(C)$ are non-separable.*

Proof. As in Definition 3.5, let $V(C)$, $V_a(C)$, and $E(C)$ be the sets of vertices, anchors, and edges of the cycle C, respectively. Recall that $\mathcal{H}(C) = (V(C), E(C))$ and $\mathcal{H}'(C) = (V_a(C), \{\!\{e \cap V_a(C) : e \in E(C)\}\!\})$.

To see that $\mathcal{H}(C)$ is non-separable, first observe that it is connected. Let G_C be the incidence graph of $\mathcal{H}(C)$. Then G_C consists of a cycle C_G with v-vertices and e-vertices alternating, and with additional v-vertices (corresponding to vertices of C that are not anchors) adjacent to some of the e-vertices of the cycle. Suppose $v \in V$ is a separating vertex of $\mathcal{H}(C)$. By Theorem 3.31, v is then a cut v-vertex of G_C. Because G_C is bipartite, every connected component of $G_C \backslash v$ must contain e-vertices. However, $G_C \backslash v$ contains the cycle C_G if v is not an anchor, and the path $C_G \backslash v$ if v is an anchor, both containing all e-vertices of G_C. Thus $G_C \backslash v$ must have a single connected component, and G_C has no cut vertices, a contradiction. Hence $\mathcal{H}(C)$ is non-separable.

Similarly it can be shown that $\mathcal{H}'(C)$ is non-separable. (Note that the incidence graph of $\mathcal{H}'(C)$ possesses a Hamilton cycle.) □

We are now ready to show that a hypergraph decomposes into its blocks just as a graph does.

Theorem 3.36. *Let $H = (V, E)$ be a connected hypergraph without empty edges. Then:*

1. *The intersection of any two distinct blocks of H contains no edges and at most one vertex.*
2. *The blocks of H form a decomposition of H.*
3. *The hypersubgraph $\mathcal{H}(C)$ associated with any cycle C of H is contained within a block of H.*

Proof. 1. Suppose B_1 and B_2 are distinct blocks of H that share more than just a single vertex. First assume that B_1 and B_2 have at least two vertices in common, and let $B = B_1 \cup B_2$. We'll show B is a non-separable hypergraph. First, B is connected since B_1 and B_2 are connected with intersecting vertex sets. Take any $v \in V(B)$. Can v be a separating vertex of B? Since B_1 and B_2 are non-separable, v is not a separating vertex in either block, and hence by Theorem 3.29, v is not a cut vertex in either block, and $B_1 \backslash v$ and $B_2 \backslash v$ are connected. Since $B \backslash v = (B_1 \backslash v) \cup (B_2 \backslash v)$, and $B_1 \backslash v$ and $B_2 \backslash v$ are connected with at least one common vertex, it follows that $B \backslash v$ is connected. Hence v is not a cut vertex of B. If v is a separating vertex of B, then by Theorem 3.29, we must have $e \in E(B)$ for $e = \{v\}$. Hence, without loss of generality, $e \in E(B_1)$. But then, by Lemma 3.30, v is a separating vertex of B_1, because B_1 is connected with at least two vertices and hence at least one more edge incident with v — a contradiction. Hence B is a non-separable hypersubgraph of H, and since B_1 and B_2 are maximal non-separable hypersubgraphs of H, we must have $B_1 = B_2 = B$, a contradiction.

Hence B_1 and B_2 have at most one common vertex. Suppose they have a common edge e. Then e must be a singleton edge, say $e = \{v\}$. If B_1 or B_2 contains another edge, then by Lemma 3.30, v is a separating vertex for this block, a contradiction. Hence $B_1 = B_2 = (\{v\}, \{e\})$, again a contradiction. We conclude that B_1 and B_2 have no common edges and at most one common vertex.

2. If H has an isolated vertex v, then $V = \{v\}$ and $E = \emptyset$, so H is a block. Hence assume every vertex of H is incident with an edge. Observe that any $e \in E$ induces a hypersubgraph $(e, \{e\})$ of H, which is non-separable and hence is a hypersubgraph of a block of H. Thus every edge and every vertex of H is contained in a block. Since by the first statement of the theorem no two blocks share an edge, every edge of H is contained in exactly one block, and H is an edge-disjoint union of its blocks.

3. By Lemma 3.35, the hypersubgraph $\mathcal{H}(C)$ of a cycle C is non-separable, and hence a hypersubgraph of a block of H.

□

The next lemma will be used several times.

Lemma 3.37. *Let H' be a connected hypersubgraph of a connected hypergraph H without empty edges, and $v \in V(H')$. If H' contains edges of two blocks of H that intersect in vertex v, then v is a separating vertex of H'.*

Proof. Let B_1 and B_2 be distinct blocks of H intersecting in vertex v such that H' contains an edge from each of them. Note that B_1 and B_2 must both be non-empty, since otherwise $B_1 = B_2 = H$ is empty by Lemma 3.34. If B_1 is trivial, then $\{v\} \in E(B_1) \cap E(H')$, and v is a separating vertex of H' by Lemma 3.30. Hence assume B_1 and B_2 are both non-trivial. Since H' is connected, we may assume there exist a vertex x adjacent to v in $B_1 \cap H'$ via edge e_1, and a vertex y adjacent to v in $B_2 \cap H'$ via edge e_2. Suppose there exists an (x, y)-path P in $H' \backslash v$. Then Pye_2ve_1x is a cycle in H' containing vertices v, x, and y. By Statement (3) of Theorem 3.36, these three vertices lie in a common block B, and by Statement (1) of the same result, $B_1 = B = B_2$, a contradiction. Hence x and y must lie in distinct connected components of $H'\backslash v$. It follows that v is a cut vertex of H', and hence a separating vertex of H' by Theorem 3.29. □

Theorem 3.38. *Let $H = (V, E)$ be a connected hypergraph without empty edges, and $v \in V$. Then v is a separating vertex of H if and only if it lies in more than one block.*

Proof. Assume v is a separating vertex of H. Then $H = H_1 \oplus H_2$, where H_1 and H_2 are non-empty connected hypersubgraphs with just vertex v in common. Hence there exist $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ such that $v \in e_1 \cap e_2$. By Statement (2) of Theorem 3.36, there exist blocks B_1 and B_2 of H such that $e_1 \in E(B_1)$ and $e_2 \in E(B_2)$.

Observe that $B_1 \cap H_1$ is connected: since B_1 is connected, and H_1 and H_2 intersect only in the vertex v, every vertex in $B_1 \cap H_1$ is connected to v in $B_1 \cap H_1$. Similarly, $B_1 \cap H_2$ is connected.

Suppose that $B_1 = B_2$. Then $B_1 = (B_1 \cap H_1) \oplus (B_1 \cap H_2)$ with $B_1 \cap H_1$ and $B_1 \cap H_2$ connected, non-empty, and intersecting only in vertex v — a contradiction, because B_1 is non-separable. Hence B_1 and B_2 must be distinct blocks of H containing vertex v.

Conversely, assume that v lies in the intersection of distinct blocks B_1 and B_2 of H. By Lemma 3.34, B_1 and B_2 are non-empty. Then H itself is a connected hypersubgraph of H containing edges from two blocks of H that intersect in v. It follows from Lemma 3.37 that v is a separating vertex of H. □

Theorems 3.36 and 3.38 show that a block graph of a hypergraph can be defined just as for graphs. Namely, let H be a connected hypergraph without empty edges, S the set of its separating vertices, and $\mathcal{B}$ the collection of its blocks. Then the *block graph* of H is the bipartite graph with vertex bipartition $\{S, \mathcal{B}\}$ and edge set $\{vB : v \in S, B \in \mathcal{B}, v \in V(B)\}$. From the third statement of Theorem 3.36 it then follows that the block graph of H is a tree.

Next, we show that blocks of a hypergraph correspond to maximal clusters of blocks of its incidence graph, to be defined below.

Definition 3.39. Let $H = (V, E)$ be a connected hypergraph without empty edges, and $G = \mathcal{G}(H)$ its incidence graph. A *cluster of blocks* of G is a connected union of blocks of G, no two of which share a v-vertex.

Theorem 3.40. *Let $H = (V, E)$ be a connected hypergraph without empty edges and H' its hypersubgraph, and let $G = \mathcal{G}(H)$ and $G' = \mathcal{G}(H')$ be their incidence graphs, respectively. Then H' is a block of H if and only if G' is a maximal cluster of blocks of G.*

Proof. Assume H' is a block of H. We first show that $G' = \mathcal{G}(H')$ is a cluster of blocks of G. Let C be the union of all blocks of G that have a common edge with G'. Observe that since H' is connected and has no empty edges, G' is connected by Theorem 3.11, and consequently C is connected. Suppose that two distinct blocks of C, say B_1 and B_2, share a v-vertex of G. Since G' contains an edge from both B_1 and B_2, v is a separating vertex of G' by Lemma 3.37. However, by Theorem 3.31, v is then a separating vertex of the block H' of H, a contradiction.

Hence no two distinct blocks in C intersect in a v-vertex, and C is a cluster of blocks of G. Let C^* be a maximal cluster of blocks of G containing C. Then C^* is connected, and has no separating v-vertices by Theorem 3.38. Since C^* is maximal, no e-vertex of C^* can be contained in a block not in C^*. Consequently, for every e-vertex e of C^*, all edges of the form ev (for $v \in V$) are contained in C^*. Hence, by Lemma 2.7, C^* is the incidence graph of a hypersubgraph H^* of H. Now H^* is connected and has no separating vertices since C^* is connected and has no separating v-vertices. Moreover, H^* contains the block H'. We conclude that $H^* = H'$ and $C^* = G'$. It follows that G' is a maximal cluster of blocks of G.

Conversely, let G' be a maximal cluster of blocks of G. Then for every e-vertex e of G', all edges of G of the form ev (for $v \in V$ such that $v \in e$) must be in G', so by Lemma 2.7, $G' = \mathcal{G}(H')$ for some hypersubgraph H' of H. Since G' is connected and has no separating v-vertices, H' is connected and non-separable. Hence H' is contained in a block B of H. By the previous paragraph, $\mathcal{G}(B)$ is a maximal cluster of blocks of G, and it also contains the maximal cluster G'. We conclude that $\mathcal{G}(B) = G'$, that is, G' is the incidence graph of a block of H. □

The next corollary is immediate.

Corollary 3.41. *Let $H = (V, E)$ be a connected hypergraph without empty edges, and $G = \mathcal{G}(H)$ its incidence graph. Then H is non-separable if and only if G is a cluster of blocks of G.*

To complete the discussion on the blocks of the incidence graph of a hypergraph, we show the following.

Theorem 3.42. *Let $H = (V, E)$ be a non-separable hypergraph with at least two edges of cardinality greater than 1. Let $G = \mathcal{G}(H)$ be its incidence graph and x a cut vertex of G. Then $x \in E$ and x is a weak cut edge of H.*

Proof. If $x \in V$, then x is a separating vertex of H by Theorem 3.31, a contradiction. Hence $x \in E$, and x is a cut edge of H by Theorem 3.23. Suppose x is a strong cut edge. If $|x| < |V|$, then H has a cut vertex by Corollary 3.27, amd hence a separating vertex by Theorem 3.29, a contradiction. Hence $|x| = |V|$, and by Theorem 3.17, $H - x$ has exactly $|x|$ connected components, implying that x is the only edge of H of cardinality greater than 1, a contradiction. Hence x must be a weak cut edge of H. □

In the last four theorems we attempt to generalize the following classic result from graph theory.

Theorem 3.43. *[5]*

1. *A connected graph is non-separable if and only if any two of its edges lie on a common cycle.*

2. *A connected graph with at least three vertices has no cut vertex if and only if any two of its vertices lie on a common cycle.*

Theorem 3.44. *Let $H = (V, E)$ be a non-separable hypergraph with $|V| \geq 2$ and $|E| \geq 2$, and let $G = \mathcal{G}(H)$ be its incidence graph. Assume in addition that $V \notin E$ and that H has no weak cut edges. Then any two distinct vertices of H and any two distinct edges of H lie on a common cycle.*

Proof. Suppose that G has a separating vertex x. If $x \in V$, then by Theorem 3.31, x is a separating vertex of H, a contradiction. Thus $x \in E$, and x is a cut edge of H by Theorem 3.23. By assumption, x is a strong cut edge and $|x| < |V|$. Hence H has a cut vertex, and hence a separating vertex, by Corollary 3.27 and Theorem 3.29, respectively — a contradiction.

Hence G has no cut vertex, and by Theorem 3.43, any two vertices of G lie on a common cycle. It then follows from Lemma 3.6 that any two vertices (and any two edges) of H lie on a common cycle. □

Theorem 3.45. *Let $H = (V, E)$ be a connected hypergraph with $|V| \geq 2$, without edges of cardinality less than 2, and without vertices of degree less than 2. Then the following are equivalent:*

1. *H has no separating vertices and no cut edges.*
2. *Every pair of elements from $V \cup E$ lie on a common cycle.*
3. *Every pair of vertices lie on a common cycle.*
4. *Every pair of edges lie on a common cycle.*

Proof. Let $G = \mathcal{G}(H)$ be the incidence graph of H.

(1) $\Rightarrow$ (2): Since H has no separating vertices and no cut edges, G has no cut vertices by Theorems 3.31 and 3.23. Hence by Theorem 3.43, since $|V(G)| \geq 3$, every pair of vertices of G lie on a common cycle in G, and therefore every pair of elements from $V \cup E$ lie on a common cycle in H.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Since every pair of vertices of H lie on a common cycle in H, every pair of v-vertices of G lie on a common cycle in G. Consequently, by Theorem 3.36, all v-vertices of G are contained in the same block B, and if G has any other blocks, then they are isomorphic to K_2. Let B_1 be one of these "trivial" blocks, and let e be its e-vertex. Then $\deg_G(e) = 1$ — a contradiction, since H has no singleton edges. It follows that G has no "trivial" blocks, and hence no cut vertices. Therefore every pair of e-vertices of G lie on a common cycle in G, and every pair of edges of H lie on a common cycle in H.

(4) $\Rightarrow$ (1): Since every pair of edges of H lie on a common cycle in H, every pair of e-vertices of G lie on a common cycle in G. Consequently, all e-vertices of G are contained in the same block B, and if G has any other blocks, then they are isomorphic to K_2. Let B_1 be one of these "trivial" blocks, and let v be its v-vertex. Then $\deg_G(v) = 1$ — a contradiction, since H has no pendant vertices. It follows that G has no "trivial" blocks, and hence no cut vertices. Therefore H has no separating vertices and no cut edges by Theorems 3.31 and 3.23, respectively. □

Theorem 3.46. *Let $H = (V, E)$ be a connected hypergraph with $|V| \geq 2$, without edges of cardinality less than 2, and without vertices of degree less than 2. Then the following are equivalent:*

1. *H has no cut edges.*

2. *Every pair of elements from $V \cup E$ lie on a common strict closed trail.*

3. *Every pair of vertices lie on a common strict closed trail.*

4. *Every pair of edges lie on a common strict closed trail.*

Proof. Let $G = \mathcal{G}(H)$ be the incidence graph of H.

(1) $\Rightarrow$ (2): Since H has no cut edges, G has no cut e-vertices by Theorem 3.23. Take any two elements x_0 and x_k of $V \cup E$. We construct a strict closed trail in H containing x_0 and x_k as follows. Let B_1 and B_k be blocks of G containing x_0 and x_k, respectively, and let $P = B_1 x_1 B_2 \ldots B_{k-1} x_{k-1} B_k$ be the unique (B_1, B_k)-path in the block tree of G. Here, of course, $B_1, \ldots, B_k$ are blocks of G, $x_1, \ldots, x_{k-1}$ are separating (cut) vertices of G, and each separating vertex x_i (necessarily a v-vertex) is shared between blocks B_i and B_{i+1}. (We may assume that vertex x_0 does not lie in block B_2, and x_k does not lie in B_{k-1}, otherwise the path P may be shortened accordingly.) By Theorem 3.43, each pair of vertices x_{i-1} and x_i, for $i = 1, \ldots, k$, lie on a common cycle C_i within block B_i. Note that these cycles $C_1, \ldots, C_k$ are pairwise edge-disjoint and intersect only in the v-vertices $x_1, \ldots, x_{k-1}$. Let $T = C_1 \oplus \ldots \oplus C_k$. Then T is a closed trail in G containing x_0 and x_k that does not repeat any e-vertices. (We count the first and last vertex of a closed trail — which are identical — as one occurrence of this vertex.) We conclude that every pair of vertices of G lie on a common closed trail in G that traverses each e-vertex at most once. Therefore, by Lemma 3.6, every pair of elements from $V \cup E$ lie on a common strict closed trail in H.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Since every pair of vertices of H lie on a common strict closed trail in H, every pair of v-vertices of G lie on a common closed trail in G that visits each e-vertex at most once. Suppose G has a cut e-vertex e. Let let v_1 and v_2 be two v-vertices in distinct connected components of $G \backslash e$. Since e is a cut vertex, v_1 and v_2 are disconnected in $G \backslash e$. On the other hand, by assumption, v_1 and v_2 lie on a closed trail T that traverses e at most once. Hence $T \backslash e$ contains a (v_1, v_2)-path of $G \backslash e$, a contradiction. Consequently, G has no cut e-vertices, which implies (as seen in the previous paragraph) that any two vertices — and hence any two e-vertices — lie on a common closed trail in G that does not repeat any e-vertices. Therefore every pair of edges of H lie on a common strict closed trail in H.

(4) $\Rightarrow$ (1): Since every pair of edges of H lie on a common strict closed trail in H, every pair of e-vertices of G lie on a common closed trail in G that has no repeated e-vertices. Suppose G has a cut e-vertex e. Since H has no vertex of degree less than 2, $G \backslash e$ has no trivial connected components; that is, each connected component of $G \backslash e$ contains an e-vertex. Let e_1 and e_2 be two e-vertices from distinct connected components of $G \backslash e$. Then e_1 and e_2 are disconnected in $G \backslash e$. On the other hand, by assumption, e_1 and e_2 lie on a closed trail T that traverses e at most once. Hence $T \backslash e$ contains an (e_1, e_2)-path of $G \backslash e$, a contradiction. It follows that G has no cut e-vertices, and H has no cut edges by Theorem 3.23. □

We conclude with the dual version of the previous theorem.

Corollary 3.47. *Let $H = (V, E)$ be a connected hypergraph with $|E| \geq 2$, without edges of cardinality less than 2, and without vertices of degree less than 2. Then the following are equivalent:*

1. *H has no separating vertices.*

2. *Every pair of elements from $V \cup E$ lie on a common pseudo cycle.*

3. *Every pair of edges lie on a common pseudo cycle.*

4. *Every pair of vertices lie on a common pseudo cycle.*

Proof. Let H^T be the dual of H, and observe that (by Corollary 3.13 and since H must have at least 2 vertices) H^T satisfies the assumptions of Theorem 3.46. Since separating vertices of H correspond precisely to cut edges of H^T by Corollary 3.32, and pseudo cycles of H correspond to strict closed trails of H^T by Lemma 3.7, the corollary follows easily from Theorem 3.46. □

4 Conclusion

In this paper, we generalized several concepts related to connection in graphs to hypergraphs. While some of these concepts generalize naturally in a unique way, or behave in hypergraphs similarly to graphs, other concepts lend themselves to more than one natural generalization, or reveal surprising new properties. Many more concepts from graph theory remain unexplored for hypergraphs, and we hope that our work will stimulate more research in this area.

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